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Progress in The Proof of The Conjecture $c < rad^2(abc)$ - Case : $c = a + 1$

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Abstract In this paper, we consider the *abc* conjecture. We give some progress in the proof of the conjecture $c < rad^2(abc)$ in the case $c = a + 1$.

Keywords Elementary number theory · real functions of one variable · Number of solutions of elementary Diophantine equations.

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To the memory of my Father who taught me arithmetic
*To my wife **Wahida**, my daughter **Sinda** and my son **Mohamed Mazen***

1 Introduction and notations

Let a a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{2}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) ([4]). It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

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Conjecture 1 (***abc* Conjecture**): Let a, b, c positive integers relatively prime with $c = a + b$, then for each $\epsilon > 0$, there exists $K(\epsilon)$ such that :

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{3}$$

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ ([2]). A conjecture was proposed that $c < rad^2(abc)$ ([1]). Here we will give a proof of it for the case $c = a + 1$.

Conjecture 2 Let a, b, c positive integers relatively prime with $c = a + b$, then:

$$c < rad^2(abc) \Longrightarrow \frac{Logc}{Log(rad(abc))} < 2 \tag{4}$$

This result, I think is the key to obtain the final proof of the veracity of the *abc* conjecture.

2 A Proof of the conjecture (2) case $c = a + 1$

Let a, c positive integers, relatively prime, with $c = a + 1$ and $R = rad(ac)$, $c = \prod_{j' \in J'} c_{j'}^{\beta_{j'}}, \beta_{j'} \geq 1$.

If $c < rad(ac)$ then we obtain:

$$c < rad(ac) < rad^2(ac) \Longrightarrow \boxed{c < R^2} \tag{5}$$

and the condition (4) is verified.

If $c = rad(ac)$, then a, c are not coprime, case to reject.

In the following, we suppose that $c > rad(ac)$ and c and a are not prime numbers.

$$c = a + 1 = \mu_a rad(a) + 1 \overset{?}{<} rad^2(ac) \tag{6}$$

2.1 $\mu_a \neq 1 \, , \mu_a \leq rad(a)$

We obtain :

$$c = a + 1 < 2\mu_a.rad(a) \Rightarrow c < 2rad^2(a) \Rightarrow c < rad^2(ac) \Longrightarrow \boxed{c < R^2} \tag{7}$$

Then (6) is verified.

2.2 $\mu_c \neq 1$, $\mu_c \leq rad(c)$

We obtain :

$$c = \mu_c rad(c) \leq rad^2(c) < rad^2(ac) \Longrightarrow \boxed{c < R^2} \tag{8}$$

and the condition (6) is verified.

2.3 $\mu_a > rad(a)$ **and** $\mu_c > rad(c)$

2.3.1 ***Case:*** $\mu_a = rad^q(a), q \geq 2$, $\mu_c = rad^p(c), p \geq 2$:

In this case, we write $c = a+1$ as $rad^{p+1}(c) - rad^{q+1}(a) = 1$. Then $rad(c), rad(a)$ are solutions of the Diophantine equation: :

$$X^{p+1} - Y^{q+1} = 1 \quad \text{with } (p+1)(q+1) \geq 9 \tag{9}$$

But the solutions of the equation (9) are :$(X = \pm 3, p+1 = 2, Y = +2, q+1 = 3)$, we obtain $p = 1 < 2$, then $rad(c), rad(a)$ are not solutions of (9) and the case $\mu_a = rad^q(a), q \geq 2$, $\mu_c = rad^p(c), p \geq 2$ is to reject.

2.3.2 ***Case:*** $rad(c) < \mu_c < rad^2(c)$ ***and*** $rad(a) < \mu_a < rad^2(a)$:

We can write:

$$\left.\begin{array}{l} \mu_c < rad^2(c) \Longrightarrow c < rad^3(c) \\ \mu_a < rad^2(a) \Longrightarrow a < rad^3(a) \end{array}\right\} \Longrightarrow ac < R^3 \Longrightarrow a^2 < ac < R^3 \Longrightarrow$$

$$a < R\sqrt{R} < R^2 \Longrightarrow \boxed{c = a + 1 < R^2} \tag{10}$$

2.3.3 ***Case:*** $\mu_c > rad^2(c)$ ***or*** $\mu_a > rad^2(a)$

I- We suppose that $\mu_c > rad^2(c)$ and $rad(a) < \mu_a \leq rad^2(a)$:

I-1- Case $rad(a) < rad(c)$: In this case $a = \mu_a.rad(a) \leq rad^2(a).rad(a) < rad^2(a)rad(c) < rad^2(ac) \Longrightarrow a < R^2 \Longrightarrow \boxed{c < R^2}$.

I-2- Case $rad(c) < rad(a) < rad^2(c)$: As $a \leq rad^2(a).rad(a) < rad^2(a).rad^2(c) \Longrightarrow a < R^2 \Longrightarrow \boxed{c < R^2}$.

Example: $2^{30}.5^2.127.353^2 = 3^7.5^5.13^5.17.1831 + 1$, $rad(c) = 2.5.127.353 = 448\,310$, $rad^2(c) = 200\,981\,856\,100$,

$\mu_c = 2^{29}.5.353 = 947\,577\,159\,680 \Longrightarrow rad^2(c) < \mu_c < rad^3(c)$,

$rad(a) = 3.5.13.17.1831 = 6\,069\,765$, $rad^2(a) = 36\,842\,047\,155\,225$,

$\mu_a = 3^6.5^4.13^4 = 13\,013\,105\,625 < rad^2(a)$. It is the case : $rad(c) < \mu_c < rad^2(c)$ and $rad(a) < \mu_a \leq rad^2(a)$ with $rad(c) = 448\,310 < rad(a) =$

$6\,069\,765 < rad^2(c) = 200\,981\,856\,100$.

I-3- Case $rad^2(c) < rad(a)$:

I-3-1- We suppose que $c \leq rad^6(c)$, we obtain:

$c \leq rad^6(c) \Longrightarrow c \leq rad^2(c).rad^4(c) \Longrightarrow c < rad^2(c).(rad(a))^2 = R^2 \Longrightarrow \boxed{c < R^2}$

Example: $5^8.7^2 = 2^4 3^7.547 + 1 \Longrightarrow 19\,140\,625 = 19\,140\,624 + 1$, $rad(c) = 5.7 = 35, rad(a) = 2.3.547 = 3\,282 \Longrightarrow rad(a) > rad^2(c)$, we obtain $c = 1\,9140\,625 > rad^3(c) = 42\,875$ and $c < rad^6(c) = 1\,838\,265\,625$ and $3\,282 = rad(a) < \mu_a = 5\,832 < rad^2(a) = 10\,771\,524 \Longrightarrow a < rad^3(a) = 35\,352\,141\,768$.

I-3-2- We suppose $c > rad^6(c) \Longrightarrow \mu_c > rad^5(c)$, we suppose $\mu_a = rad^2(a) \Longrightarrow a = rad^3(a)$. Then we obtain that $x = rad(a)$ is a solution in positive integers of the equation:

$$X^3 + 1 = c = \mu_c.rad(c) \tag{11}$$

If $c = rad^n(c)$ with $n \geq 7$, we obtain an equation like (9) that gives a contradiction. In the following, we will study the cases $\mu_c = A.rad^n(c)$ with $rad(c) \nmid A, n \geq 0$. The above equation (11) can be written as :

$$(X+1)(X^2 - X + 1) = c \tag{12}$$

Let δ any divisor of c, then:

$$X + 1 = \delta \tag{13}$$

$$X^2 - X + 1 = \frac{c}{\delta} = c' = \delta^2 - 3X \tag{14}$$

We recall that $rad(a) > rad^2(c)$, it follows that δ must verifies $\delta - 1 > rad^2(c) \Longrightarrow \delta > rad^2(c) + 1$.

I-3-2-1- We suppose that $\delta = l.rad(c) \Longrightarrow lrad(c) > rad^2(c) + 1 \Longrightarrow l > \dfrac{rad^2(c)+1}{rad(c)}$. We obtain $l \geq rad(c) + 2$ so $rad(c)$ and l have the same parity. We have $\delta = l.rad(c) < c = \mu_c.rad(c) \Longrightarrow l < \mu_c$. As δ is a divisor of c, then l is a divisor of μ_c, we write $\mu_c = l.m$. From $\mu_c = l(\delta^2 - 3X)$, we obtain:

$$m = l^2 rad^2(c) - 3rad(a) \Longrightarrow 3rad(a) = l^2 rad^2(c) - m$$

A- Case $3|m \Longrightarrow m = 3m'$, $m' > 1$: As $\mu_c = ml = 3m'l \Longrightarrow 3|rad(c)$ and $(rad(c), m')$ not coprime. We obtain:

$$rad(a) = l^2 rad(c).\frac{rad(c)}{3} - m'$$

It follows that a,c are not coprime, then the contradiction.

B - Case $m = 3 \Longrightarrow \mu_c = 3l \Longrightarrow c = 3lrad(c) = 3\delta = \delta(\delta^2 - 3X) \Longrightarrow \delta^2 = 3(1 + X) = 3\delta \Longrightarrow \delta = lrad(c) = 3$, then the contradiction.

I-3-2-2- We suppose that $\delta = l.rad^2(c), l \geq 2$. In this case $rad(a) = lrad^2(c) - 1$ verifies $rad(a) > rad^2(c)$. If $lrad(c) \nmid \mu_c$ then the case to reject. We suppose that $lrad(c)|\mu_c \Longrightarrow \mu_c = m.lrad(c)$, then $\dfrac{c}{\delta} = m = \delta^2 - 3rad(a)$.

C - Case $m = 1 = c/\delta \Longrightarrow \delta^2 - 3rad(a) = 1 \Longrightarrow (\delta - 1)(\delta + 1) = 3rad(a) = rad(a)(\delta + 1) \Longrightarrow \delta = 2 = l.rad^2(c)$, then the contradiction.

D - Case $m = 3$, we obtain $3(1 + rad(a)) = \delta^2 = 3\delta \Longrightarrow \delta = 3 = lrad^2(c)$. Then the contradiction.

E - Case $m \neq 1, 3$, we obtain: $3rad(a) = l^2rad^4(c) - m \Longrightarrow rad(a)$ and $rad(c)$ are not coprime. Then the contradiction.

I-3-2-3- We suppose that $\delta = l.rad^n(c), l \geq 2$ with $n \geq 3$. From $c = \mu_c.rad(c) = lrad^n(c)(\delta^2 - 3rad(a))$, let $m = \delta^2 - 3rad(a)$.

F - As seen above (paragraphs C,D), the cases $m = 1$ and $m = 3$ give contradictions, it follows the reject of these cases.

G - Case $m \neq 1, 3$. Let q a prime that divides m, it follows $q|\mu_c \Longrightarrow q = c_{j'_0} \Longrightarrow c_{j'_0}|\delta^2 \Longrightarrow c_{j'_0}|3rad(a)$. Then $rad(a)$ and $rad(c)$ are not coprime. It follows the contradiction.

I-3-2-4- We suppose that $\delta = \prod_{j \in J_1} c_j^{\beta_j}$, $\beta_j \geq 1$ with at least one $j_0 \in J_1$ with $\beta_{j_0} \geq 2$, $rad(c) \nmid \delta$ and $\delta - 1 = \prod_{j \in J_1} c_j^{\beta_j} - 1 > rad^2(c) = \prod_{j' \in J'} c_{j'}^2$, $J_1 \subset J'$. We can write:

$$\delta = \mu_\delta.rad(\delta), \quad rad(c) = m.rad(\delta)$$

Then we obtain:

$$c = \mu_c.rad(c) = \mu_c.m.rad(\delta) = \delta(\delta^2 - 3X) = \mu_\delta.rad(\delta)(\delta^2 - 3X) \Longrightarrow$$
$$m.\mu_c = \mu_\delta(\delta^2 - 3X) \tag{15}$$

- If $\mu_c = \mu_\delta \Longrightarrow m = \delta^2 - 3X = (\mu_c.rad(\delta))^2 - 3X$. As $\delta < \delta^2 - 3X \Longrightarrow m > \delta \Longrightarrow rad(c) > m > \mu_c.rad(\delta) > rad^5(c)$ because $\mu_c > rad^5(c)$, it follows $rad(c) > rad^5(c)$. Then the contradiction.

- We suppose that $\mu_c < \mu_\delta$. As $rad(a) = \mu_\delta rad(\delta) - 1$, we obtain:

$$rad(a) > \mu_c.rad(\delta) - 1 > 0 \Longrightarrow R > c.rad(\delta) - rad(c) > 0 \Longrightarrow$$
$$c > R > c.rad(\delta) - rad(c) > 0 \Longrightarrow 1 > rad(\delta) - \frac{rad(c)}{c} > 0, \quad rad(\delta) \geq 2$$
$$\Longrightarrow \text{The contradiction} \tag{16}$$

- We suppose that $\mu_\delta < \mu_c$. In this case, from the equation (25) and as $(m, \mu_\delta) = 1$, it follows that we can write:

$$\mu_c = \mu_1.\mu_2, \quad \mu_1, \mu_2 > 1 \tag{17}$$
$$\text{so that} \quad m.\mu_1 = \delta^2 - 3X, \quad \mu_2 = \mu_\delta \tag{18}$$

But:

$$rad(a) = \delta - 1 = \mu_\delta rad(\delta) > rad^2(c) \Longrightarrow 0 > m^2 rad^2(\delta) - \mu_2 rad(\delta) + 1$$

Let $P(Z)$ the polynomial:

$$P(Z) = m^2 Z - \mu_2 Z + 1 \Longrightarrow P(rad(\delta)) < 0 \tag{19}$$

The discriminant of $P(Z)$ is:

$$\Delta = \mu_2^2 - 4m^2 \tag{20}$$

- $\Delta = 0 \Longrightarrow \mu_2 = 2m$, but $(m, \mu_2) = 1$, then the contradiction. Case to reject.

- $\Delta < 0 \Longrightarrow P(Z)$ has no real roots. From (19) it follows that $P(Z) > 0, \forall Z \in \mathbb{R}$. Then the contradiction with $P(rad(\delta)) < 0$. Case to reject.

- $\Delta > 0 \Longrightarrow \mu_2 > 2m \Longrightarrow \dfrac{\mu_2}{m} > 2$. We denote $t = \sqrt{\Delta} > 0$. The roots of $P(Z) = 0$ are Z_1, Z_2 with $Z_1 < Z_2$, given by:

$$Z_1 = \frac{\mu_2 - t}{2m^2}, \quad Z_2 = \frac{\mu_2 + t}{2m^2} \tag{21}$$

We approximate t by $\tilde{t}$:

$$t = \sqrt{\mu_2^2 - 4m^2} = \mu_2 \left(1 - \frac{4m^2}{\mu_2^2}\right)^{\frac{1}{2}} \Longrightarrow \tilde{t} = \mu_2 - \frac{2m^2}{\mu_2} > 0$$

Then, we obtain $\tilde{Z}_1, \tilde{Z}_2$ as :

$$\tilde{Z}_1 = \frac{\mu_2 - \tilde{t}}{2m^2} = \frac{1}{\mu_2}, \quad \tilde{Z}_2 = \frac{\mu_2 + \tilde{t}}{2m^2} = \frac{\mu_2}{m^2} - \frac{1}{\mu_2} \tag{22}$$

As $\mu_2^2 - 4m^2 > 0 \Longrightarrow \mu_2^2 - m^2 > 3m^2 > 0 \Longrightarrow \dfrac{\mu_2^2}{m^2} - 1 > 0$, we will give below the proof that $rad(\delta) > \tilde{Z}_2 \Longrightarrow P(rad(\delta)) > 0$, then the contradiction with $P(rad(\delta)) < 0$; we write:

$$rad(\delta) \overset{?}{>} \frac{\mu_2}{m^2} - \frac{1}{\mu_2}, \quad \mu_2 > 0 \Longrightarrow$$

$$\mu_2.rad(\delta) \overset{?}{>} \frac{\mu_2^2}{m^2} - 1$$

$$\delta \overset{?}{>} \frac{\mu_2^2 - m^2}{m^2} > \frac{3m^2}{m^2}$$

$$\text{as } \ \delta > 3 \Longrightarrow \delta > \frac{\mu_2^2}{m^2} - 1 > 3 \Longrightarrow rad(\delta) > \frac{\mu_2}{m^2} - \frac{1}{\mu_2} > \frac{3}{\mu_2} \tag{23}$$

If follows $P(rad(\delta)) > 0$ and the contradiction with the conclusion of the equation (19).

It follows that the case $c > rad^6(c)$ and $a = rad^3(a)$ is impossible.

I-3-3- We suppose $c > rad^6(c) \Longrightarrow c = rad^6(c) + h, h > 0$ and $\mu_a < rad^2(a) \Longrightarrow a + l = rad^3(a)$, $l > 0$. Then we obtain :

$$rad^6(c) + h = rad^3(a) - l + 1 \tag{24}$$

As $rad^2(c) < rad(a)$ (see I-3), we obtain the equation:

$$rad^3(a) - (rad^2(c))^3 = h + l - 1 = m > 0$$

Let $X = rad(a) - rad^2(c)$, then X is an integer root of the polynomial $H(X)$ defined as:

$$H(X) = X^3 + 3R.rad(c)X - m = 0 \tag{25}$$

To resolve the above equation, we note $X = u + v$, then we obtain the two conditions:

$$u^3 + v^3 = m, \quad u.v = -R.rad(c) < 0 \Longrightarrow u^3.v^3 = -R^3 rad^3(c)$$

It follows that u^3, v^3 are the roots of the polynomial $G(t)$ given by:

$$G(t) = t^2 - mt - R^3 rad^3(c) = 0 \tag{26}$$

The discriminant of $G(t)$ is :

$$\Delta = m^2 + 4R^3 rad^3(c) = \alpha^2, \quad \alpha > 0 \tag{27}$$

The two real roots of (26) are:

$$t_1 = u^3 = \frac{m + \alpha}{2} \tag{28}$$

$$t_2 = v^3 = \frac{m - \alpha}{2} \tag{29}$$

As $m = rad^3(a) - rad^6(c) > 0$, we obtain that $\alpha = rad^3(a) + rad^6(c) > 0$, then from the equation (27), it follows that $(\alpha = x, m = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \tag{30}$$

with $N = 4R^3rad^3(c) > 0$. From the equations (28-29), we remark that α and m verify the following equations:

$$x + y = 2u^3 = 2rad^3(a) \tag{31}$$
$$x - y = -2v^3 = 2rad^6(c) \tag{32}$$
$$\text{then} \quad x^2 - y^2 = N = 4R^3rad^3(c) \tag{33}$$

Let $Q(N)$ be the number of the solutions of (30) and $\tau(N)$ is the number of suitable factorization of N, then we announce the following result concerning the solutions of the Diophantine equation (30) (see theorem 27.3 in [3]):
- If $N \equiv 2 (\bmod 4)$, then $Q(N) = 0$.
- If $N \equiv 1$ or $N \equiv 3 (\bmod 4)$, then $Q(N) = [\tau(N)/2]$.
- If $N \equiv 0 (\bmod 4)$, then $Q(N) = [\tau(N/4)/2]$.
$[x]$ is the integral part of x for which $[x] \leq x < [x] + 1$.

Let (α', m'), $\alpha', m' \in \mathbb{N}^*$ be another pair, solution of the equation (30), then $\alpha'^2 - m'^2 = x^2 - y^2 = N = 4R^3rad^3(c)$, but $\alpha = x$ and $m = y$ verify the equation (31) given by $x + y = 2rad^3(a)$, it follows α', m' verify also $\alpha' + m' = 2rad^3(a)$, that gives $\alpha' - m' = 2rad^6(c)$, then $\alpha' = x = \alpha = rad^3(a) + rad^6(c)$ and $m' = y = m = rad^3(a) - rad^6(c)$. We have given the proof of the uniqueness of the solutions of the equation (30) with the condition $x + y = 2rad^3(a)$. As $N = 4R^3rad^3(c) \equiv 0 (\bmod 4) \Longrightarrow Q(N) = [\tau(N/4)/2] = [\tau(rad^6(c).rad^3(a))/2] > 1$. But $Q(N) = 1$, then the contradiction.

It follows that the case $\mu_a \leq rad^2(a)$ and $c > rad^6(a)$ is impossible.

II- We suppose that $rad(c) < \mu_c \leq rad^2(c)$ and $\mu_a > rad^5(a)$:

II-1- Case $rad(c) < rad(a)$: As $c \leq rad^3(c) = rad^2(c).rad(c) \Longrightarrow c < rad^2(c).rad(a) \Longrightarrow \boxed{c < R^2}$.

II-2- Case $rad(a) < rad(c) < rad^2(a)$: As $c \leq rad^3(c) = rad^2(c).rad(c) \Longrightarrow c < rad^2(c).rad^2(a) \Longrightarrow \boxed{c < R^2}$.

II-3- Case $rad^2(a) < rad(c)$:

II-3-1- We suppose que $a \leq rad^6(a) \Longrightarrow a \leq rad^2(a).rad^4(a) \Longrightarrow a < rad^2(a).(rad(c))^2 = R^2 \Longrightarrow a < R^2 \Longrightarrow 1 + a \leq R^2$, but $(c, a) = 1$, it follows $\boxed{c < R^2}$.

II-3-2- We suppose $a > rad^6(a)$ and $\mu_c \leq rad^2(c)$. Using the same method as it was explicated in the paragraphs I-3-2, I-3-3 (permuting a,c), we arrive at a contradiction. It follows that the case $\mu_c \leq rad^2(c)$ and $a > rad^6(a)$ is impossible.

2.3.4 *III - Case $\mu_c > rad^2(c)$ and $\mu_a > rad^2(a)$*

We can write $c > rad^3(c) \Longrightarrow c = rad^3(c)+h$ and $a = rad^3(a)+l$ with $h, l > 0$ positive integers.

III-1- We suppose $rad(a) < rad(c)$. We obtain the equation:

$$rad^3(c) - rad^3(a) = l - h + 1 = m > 0 \tag{34}$$

Let $X = rad(c) - rad(a)$, from the above equation, X is a real root of the polynomial:

$$P(X) = X^3 + 3RX - m = 0 \tag{35}$$

As above, to resolve (35), we put $X = u+v$, then we obtain the two conditions:

$$u^3 + v^3 = m \tag{36}$$

$$uv = -R < 0 \Longrightarrow u^3.v^3 = -R^3 \tag{37}$$

Then u^3, v^3 are the roots of the equation:

$$H(Z) = Z^2 - mZ - R^3 = 0 \tag{38}$$

The discriminant of $H(Z)$ is:

$$\Delta = m^2+4R^3 = (rad^3(c)+rad^3(a))^2 = \alpha^2, \quad \text{taking} \ \ \alpha > 0 \Rightarrow \alpha = rad^3(c)+rad^3(a) \tag{39}$$

From the equation (39), we obtain that ($\alpha = x, m = y$) is a solution of the Diophantine equation:

$$x^2 - y^2 = N \tag{40}$$

with $N = 4R^3 > 0$ and $N \equiv 0 (\bmod 4)$. Using the same method as in I-3-3-, we arrive to a contradiction.

III-2- We suppose $rad(c) < rad(a)$. We obtain the equation:

$$rad^3(a) - rad^3(c) = h - l - 1 = m > 0 \tag{41}$$

Let $X = rad(a) - rad(c)$, from the above equation, X is a real root of the polynomial:

$$P(X) = X^3 + 3RX - m = 0 \tag{42}$$

As above, to resolve (42), we put $X = u+v$, then we obtain the two conditions:

$$u^3 + v^3 = m \tag{43}$$

$$uv = -R < 0 \Longrightarrow u^3.v^3 = -R^3 \tag{44}$$

Then u^3, v^3 are the roots of the equation:

$$H(Z) = Z^2 - mZ - R^3 = 0 \tag{45}$$

The discriminant of $H(Z)$ is:

$$\Delta = m^2+4R^3 = (rad^3(c)+rad^3(a))^2 = \alpha^2, \quad \text{taking } \ \alpha > 0 \Rightarrow \alpha = rad^3(c)+rad^3(a) \tag{46}$$

From the equation (46), we obtain that $(\alpha = x, m = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \tag{47}$$

with $N = 4R^3 > 0$ and $N \equiv 0 (\bmod 4)$. Using the same method as in I-3-3-, we arrive to a contradiction.

It follows that the case $\mu_c > rad^2(c)$ and $\mu_a > rad^2(a)$ is impossible.

We can annonce the following theorem:

Theorem 1 ***(Abdelmajid Ben Hadj Salem, 2020)*** *Let a, c positive integers relatively prime with $c = a + 1$, then $c < rad^2(ac)$.*

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