Exceptional Jordan Matrix Models, Octonionic p-branes and Star product Deformations

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Abstract

A brief review of the essentials behind the construction of a Chern-Simons-like brane action from the Large N limit of Exceptional Jordan Matrix Models paves the way to the construction of actions for membranes and p-branes moving in octonionic-spacetime backgrounds endowed with octonionic-valued metrics. The main result is that action of a membrane moving in spacetime backgrounds endowed with an octonionic-valued metric is not invariant under the usual diffeomorphisms of its world volume coordinates $\sigma^a \rightarrow \sigma'^a(\sigma^b)$, but instead it is invariant under the rigid $E_{6(-26)}$ transformations which preserve the volume (cubic) form. The star-product deformations of octonionic p-branes follow. In particular, we focus on the octonionic membrane along with the phase space quantization methods developed by [26] within the context of Nonassociative Quantum Mechanics. We finalize with some concluding remarks on Double and Exceptional Field theories, Nonassociative Gravity and A_∞, L_∞ algebras.

Keywords: Jordan, Division Algebras, Branes, Matrix Models, Star products, Nonassociative Geometry.

1 Introduction : The Large N limit of Exceptional Jordan Matrix Models and Chern-Simons Membranes

Exceptional, Jordan, Division, Clifford and Noncommutative algebras are deeply related and essential tools in many aspects in Physics, see for instance [1], [2], [3],

[5]. [6], [8]. Sometime ago, Ohwashi [9] constructed Exceptional Jordan Matrix Models based on the compact E_6 group and involving a *double* number of the required physical degrees of freedom due to a complex-valued action [9]. This led Ohwashi to interpret his complex action as representing an interacting pair of mirror universes within the compact E_6 matrix model and equipped with a $Sp(4,\mathbf{H})/Z_2$ symmetry based on the quaternionic valued symplectic group. The interacting picture resembles that of the bi-Chern-Simons gravity models. A nonassociative formulation of bosonic strings in $D = 26$ using Jordan algebras was presented a while back by [25].

In this introduction we briefly review how the large N limit of the Exceptional Matrix Models proposed by [9] leads to a novel version of Chern-Simons branes beyond those formulated by Zaikov [11]. Ohwashi [9] defined his E_6 Matrix model based on the algebra $\mathbf{J^c} \otimes \mathbf{G}$, with $\mathbf{J}^c$ being the complex Exceptional Jordan algebra of degree three $J_3[\mathbf{C} \otimes \mathbf{O}]$, and $\mathbf{G}$ is the $u(N)$ Lie algebra corresponding to the $U(N)$ group with structure constants f_{ABC} : $[T_A, T_B] = f_{ABC} T_C$. $T_A, A = 1, 2, \cdots, N^2$ are the $u(N)$ generators. The matrix $\mathcal{M}^A T_A$ elements of the $\mathbf{J^c} \otimes \mathbf{G}$ algebra are of the form

$$\begin{pmatrix} \mathcal{A}_1^A T_A & \Phi_3^A T_A & \bar{\Phi}_2^A T_A \\ \bar{\Phi}_3^A T_A & \mathcal{A}_2^A T_A & \Phi_1^A T_A \\ \Phi_2^A T_A & \bar{\Phi}_1^A T_A & \mathcal{A}_3^A T_A \end{pmatrix} \tag{1.1}$$

where $\mathcal{A}_I, (I = 1, 2, 3)$ are complex-valued numbers and Φ_I are elements of the *complex* Graves-Cayley octonion algebra comprised of the complex octonions $\mathbf{C} \otimes \mathbf{O}$:

$$\mathbf{X} \ \equiv \ X_o e_o \ + \ X_m e_m \ = \ (a_o + i b_o) e_o \ + \ (a_m + i b_m) e_m, \quad m = 1, 2, 3, ..., 7 \tag{1.2}$$

The bar operation $\bar{\Phi}$ denotes the octonionic-conjugation

$$(a_o + i b_o) e_o \ - \ (a_m + i b_m) e_m \tag{1.3}$$

that must not be confused with complex conjugation

$$(a_o - i b_o) e_o \ + \ (a_m - i b_m) e_m \tag{1.4}$$

The Noncommutative and Nonassociative algebra of octonions is determined from the relations

$$e_o^2 \ = \ e_o, \ \ e_o \, e_m \ = \ e_m \, e_o \ = \ e_m, \ \ e_m \, e_n \ = \ - \, \delta_{mn} \, e_o + \sigma_{mnp} \, e_p, \ m, n, p = 1, 2, 3, ...7. \tag{1.5}$$

where the fully antisymmetric structure constants σ_{mnp} are taken to be 1 for the combinations $(123), (516), (624), (435), (471), (673), (672)$. The quadratic form is defined by

$$(\ \mathbf{X}, \mathbf{X} \ ) \ = \ \mathbf{Re}(\bar{\mathbf{X}} \ \mathbf{X}) \ = (X_o \ X_o \ + X_m \ X_m) \ =$$

$$(a_o + ib_o)^2 \;+\; \sum_{m=1}^{m=7} \;(a_m + ib_m)^2 \;\in \mathbf{C} \tag{1.6}$$

The real part of $\mathbf{X}$ is $X_o = a_o + ib_o$ and must not be confused with the real parts of the complex entries defining the complex octonion.

The non-vanishing associator is defined by

$$\{\mathbf{X}, \mathbf{Y}, \mathbf{Z}\} = (\mathbf{XY})\mathbf{Z} - \mathbf{X}(\mathbf{YZ}) \tag{1.7a}$$

In particular, the associator of the imaginary units is

$$\{e_l, e_m, e_n\} \;=\; d_{lmnp}\; e_p, \quad d_{lmnp} \;=\; \epsilon_{lmnprst}\; \sigma^{rst} \tag{1.7b}$$

The Hermitian product is defined in terms of the ordinary complex conjugate $*$ and the quadratic form (1.6) as

$$< \;\mathbf{X}, \mathbf{Y}\; > \;\equiv\; (\; \mathbf{X}^*, \;\mathbf{Y}\;) \;=\; (a_o - ib_o)\;(c_o + id_o) \;+\; \sum_{m=1}^{m=7} \;(a_m - ib_m)\;(c_m + id_m) \tag{1.8}$$

$$< \;\mathbf{X}, \mathbf{X}\; > \;\equiv\; (\; \mathbf{X}^*, \;\mathbf{X}\;) \;=\; (a_o - ib_o)\;(a_o + ib_o) + \sum_{m=1}^{m=7} \;(a_m - ib_m)\;(a_m + ib_m) \;=$$

$$a_o^2 \;+\; b_o^2 \;+\; \sum_{m=1}^{m=7} \;(a_m^2 \;+\; b_m^2) \tag{1.9}$$

The action of Ohwashi was based on the cubic form

$$S = (\; \rho^2(\mathcal{M}^{[A}), \; \rho(\mathcal{M}^{B}), \; \mathcal{M}^{C]} \;)\; f_{ABC} \quad (X, Y, Z) = tr\;(\; X \,\cdot\, (Y \times_F Z)\;) \tag{1.10}$$

$\rho, \; \rho^3 = 1$ is the cycle mapping (based on the *triality* symmetry of $SO(8)$) that takes the index $I \to I + 1$, modulo 3. It is essential to introduce the cycle mapping in (1.10) otherwise the expression would have been identically equal to zero due to the fact that the cubic form is symmetric in its three entries while f_{ABC} is antisymmetric . The product $Y \times_F Z$ is the *symmetric* Freudenthal product

$$Y \times_F Z = Y \cdot Z - \frac{1}{2} tr\;(Y)\; Z - \frac{1}{2} tr\;(Z)\; Y + \frac{1}{2} tr\;(Y)\; tr\;(Z) - \frac{1}{2} tr\;(Y \cdot Z)\; \mathbf{1}. \tag{1.11}$$

and $X \cdot Y$ is the commutative but non-associative Jordan product given by the anti-commutator $\frac{1}{2}(XY + YX)$ obeying the Jordan identity $(X \cdot Y) \cdot X^2 = X \cdot (Y \cdot X^2)$. The cubic form (1.10) is very different from the trilinear form trace $(X \cdot (Y \cdot Z))$ used by Smolin [10] to construct the F_4 matrix model based on the Exceptional $J_3[O]$ algebra rather than the complex Exceptional $J_3[C \times O]$ algebra. The action of Ohwashi is complex-valued while that of Smolin is real-valued. The explicit evaluation of the expression (1.10) can be found in [9] where he includes a detailed appendix with numerous important formulae that are indispensable to be able to write down all the explicit terms of the cubic form. The Ohwashi action [9] after very lengthy algebra becomes

$$S = Trace\ [\ \epsilon^{IJK}(\ \mathbf{A}_I\ [\mathbf{A}_J,\ \mathbf{A}_K] + \eta^{ij}\ \mathbf{\Phi}_{0I}\ [\mathbf{\Phi}_{iJ},\ \mathbf{\Phi}_{jK}] +$$

$$\sigma^{ijk}\ \mathbf{\Phi}_{iI}\ [\mathbf{\Phi}_{jJ},\ \mathbf{\Phi}_{kK}]\ + \eta^{ij}\ \mathbf{A}_I\ [\mathbf{\Phi}_{iJ},\ \mathbf{\Phi}_{jK}]\ + \mathbf{A}_I\ [\mathbf{\Phi}_{0J},\ \mathbf{\Phi}_{0K}] +$$

$$\mathbf{\Phi}_{0I}\ [\mathbf{\Phi}_{0J},\ \mathbf{\Phi}_{0K}]\ ) + \sigma^{ijk}\ \sum_{I=1}^{I=3} \mathbf{\Phi}_{iI}\ [\mathbf{\Phi}_{jI},\ \mathbf{\Phi}_{kI}]\ ],\ \ i,j,k = 1,2,3,\cdots,7;\ \ I,J,K = 1,2,3 \tag{1.12}$$

where the matrix-valued quantities are denoted by the bold-face letters $\mathbf{A}_I = \mathcal{A}_I^A T_A;\ \mathbf{\Phi}_{iJ} = \Phi_{iJ}^A T_A$, etc...

The large N limit of the E_6 Exceptional Matrix Model action described by eq-(1.12) was obtained in [12]. This limit is given by the following Chern-Simons-like brane action

$$S = \int_V\ d^3\sigma\ \epsilon^{abc}\ [\ \epsilon^{IJK}(\ \partial_a\mathcal{A}_I\ \partial_b\mathcal{A}_J\ \partial_c\mathcal{A}_K + \eta^{ij}\ \partial_a\Phi_{0I}\ \partial_b\Phi_{iJ}\ \partial_c\Phi_{jK}\ +$$

$$\sigma^{ijk}\ \partial_a\Phi_{iI}\ \partial_b\Phi_{jJ}\ \partial_c\Phi_{kK}\ \ + \eta^{ij}\ \partial_a\mathcal{A}_I\ \partial_b\Phi_{iJ}\ \partial_c\Phi_{jK}\ \ + \partial_a\mathcal{A}_I\ \partial_b\Phi_{0J}\ \partial_c\Phi_{0K}\ +$$

$$\partial_a\Phi_{0I}\ \partial_b\Phi_{0J}\ \partial_c\Phi_{0K}\ )\ +\ \sigma^{ijk}\ \sum_{I=1}^{I=3}\partial_a\Phi_{iI}\ \partial_b\Phi_{jI}\ \partial_c\Phi_{kI}\ ],\ \ a,b,c = 1,2,3. \tag{1.13}$$

The *bulk* 3-dim action (1.13) is such that its 2-dim *boundary* action is given by the large N limit of eq-(1.12). The action (1.13) furnishes a *novel* Chern-Simons membrane model (not to be confused with the Zaikov's Chern-Simons membrane [11]).

The crux of this large $N \to \infty$ limit correspondence relies on the fact that $N \times N$ matrices $\mathbf{M} \to X(\sigma^1, \sigma^2, \sigma^3)$ become the 27 membrane complex coordinates in the continuum limit. The complex dimension of the $\mathbf{J}^c$ algebra is 27. The $trace_{N\times N} \to \int d^3\sigma$ becomes an integral ; the *commutators* $\to$ *brackets* and the Jordan algebra non-associator $[X,Y,Z] = X\cdot(Y\cdot Z) - (X\cdot Y)\cdot Z$ has a correspondence with the Nambu-Poisson brackets $\{X(\sigma^a), Y(\sigma^a), Z(\sigma^a)\}$ as discussed by [5]. Similar results can be obtained in the large N limit of the F_4 Matrix Models of [10], with the only difference that one must use the *trilinear* form based on Jordan products instead of the cubic form (based on the Jordan and Freudenthal product).

The action (1.13) in condensed form can be written as

$$S = \int_V\ d^3\sigma\ \epsilon^{abc}\ (\ \partial_a\mathbf{J},\ \partial_b\ \rho(\mathbf{J}),\ \partial_c\ \rho^2(\mathbf{J})\ ). \tag{1.14}$$

The above action can also be recast in terms of Nambu-Poisson brackets as

$$\int d^3\sigma\ [\ \epsilon^{IJK}(\ \{\ \mathcal{A}_I,\ \mathcal{A}_J,\ \mathcal{A}_K\ \}\ +\ \eta^{ij}\ \{\ \Phi_{0I},\ \Phi_{iJ},\ \Phi_{jK}\ \}\ +$$

$$\sigma^{ijk} \{ \Phi_{iI}, \Phi_{jJ}, \Phi_{kK} \} + \eta^{ij} \{ \mathcal{A}_I, \Phi_{iJ}, \Phi_{jK} \} + \{ \mathcal{A}_I, \Phi_{0J}, \Phi_{0K} \} +$$

$$\{ \Phi_{0I}, \Phi_{0J}, \Phi_{0K} \}) + \sigma^{ijk} \sum_{I=1}^{I=3} \{ \Phi_{iI}, \Phi_{jI}, \Phi_{kI}\}]. \quad (1.15)$$

The integrand is a total derivative that can be integrated over a two-dim boundary domain $\Sigma \equiv \partial V$ giving

$$S = \int_{\partial V} [d^2\Sigma]_a \epsilon^{abc} [\epsilon^{IJK}(\mathcal{A}_I \partial_b \mathcal{A}_J \partial_c \mathcal{A}_K + \eta^{ij} \Phi_{0I} \partial_b \Phi_{iJ} \partial_c \Phi_{jK} +$$

$$\sigma^{ijk} \Phi_{iI} \partial_b \Phi_{jJ} \partial_c \Phi_{kK} + \eta^{ij} \mathcal{A}_I \partial_b \Phi_{iJ} \partial_c \Phi_{jK} + \mathcal{A}_I \partial_b \Phi_{0J} \partial_c \Phi_{0K} +$$

$$\Phi_{0I} \partial_b \Phi_{0J} \partial_c \Phi_{0K}) + \sigma^{ijk} \sum_{I=1}^{I=3} \Phi_{iI} \partial_b \Phi_{jI} \partial_c \Phi_{kI}]. \quad (1.16)$$

To sum up, the novel Chern-Simons action (1.16) is the large N limit of the Ohwashi action (1.12) and is associated with the 2-dim boundary of an open 3-dim region, the world volume of an open membrane, and is the candidate action for a non-perturbative *bosonic* formulation of M theory in $D = 27$ dimensions [12] first proposed by Horowitz and Susskind [14].

Having reviewed the essentials behind the construction of the Chern-Simons-like brane actions, the outline of this work goes as follows. In section **2** we present the construction of membranes and p-branes in octonionic-spacetime backgrounds endowed with octonionic-valued metrics. The main result of this section is that action of a membrane moving in spacetime backgrounds endowed with an octonionic-valued metric is not invariant under the usual diffeomorphisms of its world volume coordinates $\sigma^a \to \sigma'^a(\sigma^b)$, but instead it is invariant under the rigid $E_{6(-26)}$ transformations which preserve the volume (cubic) form. Section **3** is devoted to the star-product deformations of octonionic membranes and quantized Nambu-Poisson Brackets. In particular, we focus on the octonionic membrane, along with the phase space quantization methods developed by [26] within the context of Nonassociative Quantum Mechanics. This work is an extension of the previous work [18] on Octonionic Gravity, Exceptional Jordan Strings and Nonassociative Ternary Gauge Field Theories. We finalize with some concluding remarks on Double and Exceptional Field theories, Nonassociative Gravity and A_∞, L_∞ algebras.

2 Octonionic p-branes

2.1 Membranes in Octonionic-Metric Backgrounds

A complexification of ordinary gravity (not to be confused with Hermitian-Kahler geometry) has been known for a long time. Complex gravity requires that $g_{\mu\nu} = g_{(\mu\nu)} + i g_{[\mu\nu]}$ so that now one has $g_{\nu\mu} = (g_{\mu\nu})^*$, which implies that the diagonal components of the metric $g_{z_1 z_1} = g_{z_2 z_2} = g_{\bar{z}_1 \bar{z}_1} = g_{\bar{z}_2 \bar{z}_2}$ must be real.

A treatment of a non-Riemannan geometry based on a complex tangent space and involving a symmetric $g_{(\mu\nu)}$ plus antisymmetric $g_{[\mu\nu]}$ metric component was first proposed by Einstein-Strauss [21] (and later on by [23]) in their unified theory of Electromagentism with gravity by identifying the EM field strength $F_{\mu\nu}$ with the antisymmetric metric $g_{[\mu\nu]}$ component.

Borchsenius [22] formulated the quaternionic extension of Einstein-Strauss unified theory of gravitation with EM by incorporating appropriately the $SU(2)$ Yang-Mills field strength into the degrees of a freedom of a quaternionc-valued metric. Oliveira and Marques [24] later on provided the Octonionic Gravitational extension of Borchsenius theory involving two interacting $SU(2)$ Yang-Mills fields and where the exceptional group G_2 was realized naturally as the automorphism group of the octonions. In [19] we formulated a $R \otimes C \otimes H \otimes O$-valued gravitational theory as a plausible candidate for a grand unified field theory based on the composition algebra involving the four division algebras R, C, H, O.

In the first part of this section we shall describe membranes moving in spacetime target backgrounds endowed with an octonionic-valued Hemitian metric $\mathbf{g}_{\mu\nu} = g^o_{(\mu\nu)} e_o + g^i_{[\mu\nu]} e_i; (\mathbf{g}_{\mu\nu})^\dagger = \mathbf{g}_{\mu\nu}$. The real part is symmetric in its indices, while the imaginary components are antisymmetric. As a result of embedding the membrane's world volume into the target spacetime background, the membrane's world-volume Hermitian metric $\mathbf{g}_{ab}$ is given by the pullback of the D-dim background octonionic-hermitian metric $\mathbf{g}_{\mu\nu}$

$$\mathbf{g}_{ab} \ = \ g^o_{(ab)} e_o + g^i_{[ab]} e_i \ = \ \partial_a X^\mu \ \partial_b X^\nu \ \mathbf{g}_{\mu\nu} \ = \ \partial_a X^\mu \ \partial_b X^\nu \ (\ g^o_{(\mu\nu)} e_o + g^i_{[\mu\nu]} e_i \ ) \tag{2.1}$$

with $\mu, \nu = 1, 2, \cdots, D; \quad a, b = 1, 2, 3; \quad i = 1, 2, \cdots, 7$. The D spacetime coordinates $X^\mu(\sigma^a)$ are real-valued, and the infinitesimal line interval $ds^2 = \mathbf{g}_{\mu\nu} dX^\mu dX^\nu$ is also real-valued since $(g^i_{[\mu\nu]} e_i) dX^\mu dX^\nu = 0$.

The explicit matrix elements of the metric $\mathbf{g}_{ab}$ are

$$\mathbf{g}_{ab} \ = \ \begin{pmatrix} g_{11} & g^o_{(12)} \ + \ g^i_{[12]} e_i & g^o_{(13)} \ + \ g^i_{[13]} e_i \\ g^o_{(12)} \ - \ g^i_{[12]} e_i & g_{22} & g^o_{(23)} \ + \ g^i_{[23]} e_i \\ g^o_{(13)} \ - \ g^i_{[13]} e_i & g^o_{(23)} \ - \ g^i_{[23]} e_i & g_{33} \end{pmatrix} \tag{2.2}$$

Our proposal for the Dirac-Nambu-Goto-like action associated with a membrane moving in a D-dim background endowed with an octonionic-valued metric $\mathbf{g}_{\mu\nu}$ is

$$S_{DNG} \ = \ - \ T_2 \int d^3\sigma \ \sqrt{|Det_F(\mathbf{g}_{ab})|} \tag{2.3}$$

T_2 is the membrane tension, and the minus sign in (2.3) is chosen based on the ordinary actions associated to membranes moving in spacetime backgrounds endowed with real-valued metrics with signature $(-,+,+,+,\cdots)$. Since the determinant of a matrix with octonionic-valued entries is *not* defined, in the very special case when the 3×3 octonionic matrix is Hermitian, one can borrow the construction of the well-defined Freudenthal determinant $Det_F\mathbf{X}$ of the 3×3 Jordan matrices $\mathbf{X}$ which is given by the cubic form $Det(\mathbf{X}) = \frac{1}{3}(\mathbf{X},\mathbf{X},\mathbf{X})$ in eq-(1-10).

Thus, after setting the following correspondence among the entries of the Jordan matrix (1.1) and those of the metric $\mathbf{g}_{ab}$ in eq-(2.2)

$$g_{11} \leftrightarrow A^1, \quad g_{22} \leftrightarrow A^2, \quad g_{33} \leftrightarrow A^3$$
$$g_{23} \leftrightarrow \Phi^1, \quad g_{31} \leftrightarrow \Phi^2, \quad g_{12} \leftrightarrow \Phi^3, \tag{2.4}$$

one can then write down the Freudenthal determinant

$$Det_F(\mathbf{g}_{ab}) \;=\; g_{11}\, g_{22}\, g_{33} - g_{11}\, \mathbf{g}_{23}\, \bar{\mathbf{g}}_{23} - g_{22}\, \mathbf{g}_{31}\, \bar{\mathbf{g}}_{31} - g_{33}\, \mathbf{g}_{12}\, \bar{\mathbf{g}}_{12} + 2\,\mathbf{Re}\{\, \mathbf{g_{23}}\, \mathbf{g_{31}} \mathbf{g_{12}}\} \tag{2.5}$$

The bilinear products in (2.5) are $\mathbf{g}_{23}\bar{\mathbf{g}}_{23} = \bar{\mathbf{g}}_{23}\mathbf{g}_{23} = (g^o_{(23)})^2 \;+\; (g^i_{[23]})^2, \cdots$. And the real part of the triple product is given by

$$\mathbf{Re}\{\mathbf{g_{23}}\; \mathbf{g_{31}}\; \mathbf{g_{12}}\} \;=$$
$$g^o_{23}\, g^o_{31}\, g^o_{12} - \delta_{ij}(g^o_{23}\, g^i_{31}\, g^j_{12} + g^i_{23}\, g^o_{31}\, g^j_{12} + g^i_{23}\, g^j_{31}\, g^o_{12}) - \sigma_{ijk}\, g^i_{23}\, g^j_{31}\, g^k_{12} \tag{2.6}$$

The relation respecting the cyclicity property

$$\mathbf{Re}((\mathbf{xy})\;\mathbf{z}) \;= \mathbf{Re}(\mathbf{x}\;(\mathbf{yz})) \;=\; \mathbf{Re}(\mathbf{xyz}) \;=\; \mathbf{Re}(\mathbf{zxy}) \;=\; \mathbf{Re}(\mathbf{yzx}) \;=$$
$$x_o\; y_o\; z_o \;-\; x_o\; y_i\; z_i \;-\; x_i\; y_o\; z_i \;-\; x_i\; y_i\; z_o \;-\; x_i\; y_j\; z_k\; \sigma_{ijk} \tag{2.7a}$$

is what allows us to unambiguously evaluate the real part of the triple product, despite the nonassociativity. The diagonal metric components g_{11}, g_{22}, g_{33} are real-valued, and the off-diagonal components are octonionic-valued

$$\mathbf{g}_{12} \;=\; g^o_{(12)} e_o \;+\; g^i_{[12]} e_i, \quad \mathbf{g}_{13} \;=\; g^o_{(13)} e_o \;+\; g^i_{[13]} e_i, \quad \mathbf{g}_{23} \;=\; g^o_{(23)} e_o \;+\; g^i_{[23]} e_i \tag{2.7b}$$

Consequently, due to the complicated expression of $Det_F(\mathbf{g}_{ab})$ (2.5), the membrane action (2.3) involving the Freudenthal determinant (2.5) is far more *complicated* than ordinary actions involving determinants of real-valued metrics.

One of the most salient features of the action (2.3), based on the *cubic* form (1.10), is that it is invariant under $E_{6(-26)}$ transformations. As shown by Yokota [7], see also [9], the Freudenthal determinant $Det_F\mathbf{X}; \mathbf{X} \in \mathbf{J} = J_3(\mathbf{O})$, defined by the cubic form (1.10), is invariant under the rigid $E_{6(-26)}$ transformations which are implemented via the following isometries $\alpha : \mathbf{J} \to \mathbf{J}$,

$$\mathbf{X} \to \alpha\mathbf{X},\;\; \alpha \in Iso_R(\mathbf{J},\mathbf{J})|\; \frac{1}{3}(\alpha\mathbf{X},\alpha\mathbf{X},\alpha\mathbf{X}) \;=\; \frac{1}{3}(\mathbf{X},\mathbf{X},\mathbf{X}) \;=\; Det_F(\mathbf{X}) \tag{2.8}$$

Consequently, the action S_{DNG} in eq-(2.3) is $E_{6(-26)}$-invariant. The compact E_6, and complexified E_6^c, belong to the isometries $Iso_C(\mathbf{J}^c, \mathbf{J}^c)$ of the *complex* Exceptional Jordan algebra $\mathbf{J}^c = J_3(\mathbf{C} \otimes \mathbf{O})$ [7] instead of the Exceptional Jordan algebra $\mathbf{J} = J_3(\mathbf{O})$. Whereas F_4 is the group of isometries $Iso_R(\mathbf{J}, \mathbf{J})$ which preserve the *trilinear* form $tr(\alpha\mathbf{X}, \alpha\mathbf{Y}, \alpha\mathbf{Z}) = tr(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$. The trilinear form trace $(\mathbf{X} \cdot (\mathbf{Y} \cdot \mathbf{Z}))$ differs from the cubic form trace $(\mathbf{X} \cdot (\mathbf{Y} \times_F \mathbf{Z}))$.

The main conclusion is that the action (2.3) for a membrane moving in spacetime backgrounds endowed with an octonionic metric is not invariant under the usual diffeomorphisms of its world volume coordinates , $\sigma^a \to \sigma'^a(\sigma^b)$, but instead it is invariant under the rigid $E_{6(-26)}$ transformations which preserve the volume (cubic) form (1.10).

2.2 Branes in Octonionic Spacetime Backgrounds

Next we shall construct actions for p-branes moving in octonionic spacetime backgrounds $\mathbf{Z}^\mu(\sigma^a) = Z_o^\mu(\sigma^a)e_o + Z_i^\mu(\sigma^a)e_i; a = 0, 1, 2, \cdots, p$, and endowed with octonionic-valued metrics $\mathbf{g}_{\mu\nu}$. Given an spacetime interval defined as

$$(ds)^2 \;=\; \mathbf{Re}\;(\;d\mathbf{Z}^\mu\;\mathbf{g}_{\mu\nu}\;d\mathbf{Z}^\nu\;) \tag{2.9}$$

the *real* part of the pullback of the spacetime metric onto the $p+1$-dim world-volume yields the embedding metric $h_{ab} = \mathbf{Re}(\partial_a\mathbf{Z}^\mu\mathbf{g}_{\mu\nu}\partial_b\mathbf{Z}^\nu)$. We found earlier that the *real* part of a triple octonionic product (2.7) is unambiguously defined despite the nonassociativity. The real parts of a *quartic*, and higher products, are *not*.

In the most general case, the octonionic metric $\mathbf{g}_{\mu\nu}$ does *not* need to be Hermitian; i.e. it does not need to have the form $\mathbf{g}_{\mu\nu} = g^o_{(\mu\nu)}e_o + g^i_{[\mu\nu]}e_i$. The reason being that by taking the real part of the triple products in eq-(2.9) one ensures that $(ds)^2$ is real-valued.

If the octonionic-valued metric $\mathbf{g}_{\mu\nu}$ is chosen to be Hermitian $(\mathbf{g}_{\mu\nu})^\dagger = \mathbf{g}_{\mu\nu}$, and $\bar{\mathbf{g}}_{\mu\nu} = \mathbf{g}_{\bar{\mu}\bar{\nu}}$, after a careful inspection, one arrives at the following relations

$$\begin{gathered} \mathbf{g}_{\mu\nu} \;=\; g^o_{(\mu\nu)}e_o \;+\; g^i_{[\mu\nu]}e_i \\ g^o_{\mu\nu} = g^o_{\nu\mu} = g^o_{\bar{\mu}\bar{\nu}} = g^o_{\bar{\nu}\bar{\mu}} \\ g^i_{\mu\nu} = -g^i_{\nu\mu} = -g^i_{\bar{\mu}\bar{\nu}} = g^i_{\bar{\nu}\bar{\mu}} \end{gathered} \tag{2.10}$$

Due to these relations among the components of $\mathbf{g}_{\mu\nu}$ and $\mathbf{g}_{\bar{\mu}\bar{\nu}}$, it is not necessary to include the terms $d\mathbf{Z}^{\bar{\mu}}\;\mathbf{g}_{\bar{\mu}\bar{\nu}}\;d\mathbf{Z}^{\bar{\nu}}$ in eq-(2.9).

By the same token, one may also include an interval of the form

$$(ds)^2 \;=\; \mathbf{Re}\;(\;d\mathbf{Z}^\mu\;\mathbf{g}_{\mu\bar{\nu}}\;d\mathbf{Z}^{\bar{\nu}}\;) \tag{2.11}$$

If the octonionic-valued metric is chosen to be Hermitian : $\mathbf{g}_{\mu\bar{\nu}} = g^o_{(\mu\bar{\nu})}e_o + g^i_{[\mu\bar{\nu}]}e_i$, and $\bar{\mathbf{g}}_{\mu\bar{\nu}} = \mathbf{g}_{\bar{\mu}\nu}$, after a careful inspection it leads to the following Hermiticity conditions

$$g^o_{\mu\bar\nu} \ = \ g^o_{\bar\nu\mu} \ = \ g^o_{\bar\mu\nu} \ = \ g^o_{\nu\bar\mu}$$

$$g^i_{\mu\bar\nu} \ = \ - \ g^i_{\bar\nu\mu} \ = \ - \ g^i_{\bar\mu\nu} \ = \ g^i_{\nu\bar\mu}$$

Once again, due to these relations among the components of $\mathbf{g}_{\mu\bar\nu}$ and $\mathbf{g}_{\bar\mu\nu}$, it is not necessary to include the terms $d\mathbf{Z}^{\bar\mu} \ \mathbf{g}_{\bar\mu\nu} \ d\mathbf{Z}^\nu$ in eq-(2.11). In both cases the real components of the metric is symmetric in its indices, while the imaginary components are antisymmetric.

To sum up, when $\mathbf{g}_{\mu\nu}$ and the spacetime coordinates $\mathbf{Z}^\mu = Z^\mu_o e_o + Z^\mu_i e_i$ are both octonionic-valued , one can construct a more general p-brane action of the form

$$S_{DNG} \ = \ - \ T_p \int d^{p+1}\sigma \ \sqrt{|det \ h_{ab}|} \ =$$

$$- \ T_p \int d^{p+1}\sigma \ \sqrt{|det \ \mathbf{Re} \ (\ \partial_a \mathbf{Z}^\mu \ \mathbf{g}_{\mu\nu} \ \partial_b \mathbf{Z}^\nu \ )|} \qquad (2.12)$$

where T_p is the p-brane tension of physical dimension $(mass)^{p+1}$, and the span of the p-brane indices are $a, b = 0, 1, \cdots, p$. Once again, it is the key relation

$$\mathbf{Re}(\ (\mathbf{xy}) \ \mathbf{z} \ ) \ = \mathbf{Re}(\ \mathbf{x} \ (\mathbf{yz}) \ ) \ = \ \mathbf{Re}(\ \mathbf{xyz} \ ) \ =$$

$$x_o \ y_o \ z_o \ - \ x_o \ y_i \ z_i \ - \ x_i \ y_o \ z_i \ - \ x_i \ y_i \ z_o \ - \ x_i \ y_j \ z_k \ \sigma_{ijk} \qquad (2.13)$$

which allows us to uniquely evaluate the real part of the triple product $\mathbf{Re} \ (\partial_a \mathbf{Z}^\mu \ \mathbf{g}_{\mu\nu} \ \partial_b \mathbf{Z}^\nu \ )$ despite the *nonassociativity* of the octonions.

Thus, the real part of the pullback of the octonionic target space Hermitian metric $\mathbf{g}_{\mu\nu}$ is explicitly given by

$$h_{ab} \ = \ \partial_a Z^\mu_o \ g^o_{\mu\nu} \ \partial_b Z^\nu_o \ - \ \partial_a Z^\mu_o \ g^i_{\mu\nu} \ \partial_b Z^\nu_i \ - \ \partial_a Z^\mu_i \ g^o_{\mu\nu} \ \partial_b Z^\nu_i$$

$$- \ \partial_a Z^\mu_i \ g^i_{\mu\nu} \ \partial_b Z^\nu_o \ - \ \sigma_{ijk} \ \partial_a Z^\mu_i \ g^j_{\mu\nu} \ \partial_b Z^\nu_k \qquad (2.14)$$

with $i, j = 1, 2, \cdots, 7$, and repeated indices are summed over. The determinant of the above expression for h_{ab} is very complicated since h_{ab} is comprised of the sum of many different terms. Inserting this complicated expression for the $det(h_{ab})$ into eq-(2.12) furnishes the DNG action for a p-brane moving in an octonionic spacetime background and endowed with an octonionic-valued Hermitian metric.

A similar action can be constructed based on the metric $\mathbf{g}_{\mu\bar\nu}$

$$S'_{DNG} \ = \ - \ T_p \int d^{p+1}\sigma \ \sqrt{|det \ \mathbf{Re} \ (\ \partial_a \mathbf{Z}^\mu \ \mathbf{g}_{\mu\bar\nu} \ \partial_b \mathbf{Z}^{\bar\nu} \ )|}, \quad a, b = 0, 1, \cdots, p \qquad (2.15)$$

And in the most general case, one can combine both metrics $\mathbf{g}_{\mu\nu}, \mathbf{g}_{\mu\bar\nu}$ into the more general action

$$S''_{DNG} \ = \ - \ T_p \ \int d^{p+1}\sigma \ \sqrt{|det \ \mathbf{Re} \ (\ \partial_a \mathbf{Z}^\mu \ \mathbf{g}_{\mu\nu} \ \partial_b \mathbf{Z}^\nu + \partial_a \mathbf{Z}^\mu \ \mathbf{g}_{\mu\bar{\nu}} \ \partial_b \mathbf{Z}^{\bar{\nu}} \ )|} \tag{2.16}$$

When the metric is real ($\mathbf{g}_{\mu\nu} \rightarrow g_{\mu\nu}$), and the spacetime coordinates are real ($\mathbf{Z}^\mu \rightarrow X^\mu$) one recovers for the determinant of h_{ab} the usual expression given by the sums of the squares of Nambu-Poisson-brackets [29]

$$h_{ab} \ = \ \partial_a X^\mu \ g_{\mu\nu} \ \partial_b X^\nu \ \Rightarrow$$

$$det(h_{ab}) \ = \ \{X^{\mu_1}, X^{\mu_2}, \cdots, X^{\mu_{p+1}}\} \, \{X^{\nu_1}, X^{\nu_2}, \cdots, X^{\nu_{p+1}}\} \, g_{\mu_1\nu_1} \, g_{\mu_2\nu_2} \ \cdots \ g_{\mu_{p+1}\nu_{p+1}} \tag{2.17a}$$

where the Nambu-Poisson brackets are defined as

$$\{X^{\mu_1}, X^{\mu_2}, \cdots, X^{\mu_{p+1}}\} \ \equiv \ \epsilon^{a_1 a_2 \cdots a_{p+1}} \ \partial_{a_1} X^{\mu_1} \ \partial_{a_2} X^{\mu_2} \ \cdots \ \partial_{a_{p+1}} X^{\mu_{p+1}} \tag{2.17b}$$

In general, in a curved background one has $g_{\mu\nu} = g_{\mu\nu}(X^\rho)$. Because the embedding spacetime coordinates $X^\rho(\sigma^1, \sigma^2, \cdots, \sigma^{p+1})$ are functions of the p-brane's $p+1$-dimensional world-volume coordinates, one cannot pull the metric factors inside the Nambu-Poisson brackets in eq-(2.17). Only when the background metric is independent of the X^ρ coordinates (it is flat) that one can pull the metric factors inside the brackets leading to

$$det(h_{ab}) \ = \ \{X^{\mu_1}, X^{\mu_2}, \cdots X^{\mu_{p+1}}\} \ \{X_{\mu_1}, X_{\mu_2}, \cdots, X_{\mu_{p+1}}\} \tag{2.18}$$

and the DNG action becomes

$$S_{DNG} \ = \ - \ T_p \ \int d^{p+1}\sigma \ \sqrt{|det \ (\ \partial_a X^\mu \ \partial_b X^\nu \ \eta_{\mu\nu} \ )|} \ =$$

$$- \ T_p \ \int d^{p+1}\sigma \ \sqrt{(\ \{X^{\mu_1}, X^{\mu_2}, \cdots, X^{\mu_{p+1}}\}_{NPB} \ )^2} \tag{2.19}$$

A Polyakov-Howe-Tucker octonionic p-brane action S_p based on the metric $\mathbf{g}_{\mu\nu}$ is of the form

$$S_p \ = \ - \ \frac{T_p}{2} \ \int d^{p+1}\sigma \ \sqrt{|det(h_{ab})|} \ h^{ab} \ \mathbf{Re} \ (\ \partial_a \mathbf{Z}^\mu \ \mathbf{g}_{\mu\nu} \ \partial_b \mathbf{Z}^\nu \ ) \ +$$

$$(p-1) \ \int d^{p+1}\sigma \ \sqrt{|det(h_{ab})|} \tag{2.20}$$

where $a, b = 0, 1, \cdots, p$ and h_{ab} is an auxiliary real-valued world-volume metric. Eliminating h_{ab} via its equations of motion and inserting it back into the action (2.20) yields the DNG action (2.12). A similar action can be constructed based on the metric $\mathbf{g}_{\mu\bar{\nu}}$

$$S'_p \ = \ - \ \frac{T_p}{2} \ \int d^{p+1}\sigma \ \sqrt{|det(h_{ab})|} \ h^{ab} \ \mathbf{Re} \ (\ \partial_a \mathbf{Z}^\mu \ \mathbf{g}_{\mu\bar{\nu}} \ \partial_b \mathbf{Z}^{\bar{\nu}} \ ) \ +$$

$$(p-1) \int d^{p+1}\sigma \ \sqrt{|det(h_{ab})|} \qquad (2.21)$$

And a more general action combining both metrics $\mathbf{g}_{\mu\nu}; \mathbf{g}_{\mu\bar{\nu}}$ is of the form

$$S''_p \ = \ - \frac{T_p}{2} \int d^{p+1}\sigma \ \sqrt{|det(h_{ab})|} \ h^{ab} \ \mathbf{Re} \ (\ \partial_a \mathbf{Z}^\mu \ \mathbf{g}_{\mu\nu} \ \partial_b \mathbf{Z}^\nu + \partial_a \mathbf{Z}^\mu \ \mathbf{g}_{\mu\bar{\nu}} \ \partial_b \mathbf{Z}^{\bar{\nu}} \ ) +$$

$$(p-1) \int d^{p+1}\sigma \ \sqrt{|det(h_{ab})|} \qquad (2.22)$$

To get a picture of what an octonionic spacetime background endowed with an octonionic metric looks like, let us concentrate in the very special case of diagonal metrics. Namely $\mathbf{g}_{\mu\nu} = 0$ when $\mu \neq \nu$, and such that the nonzero diagonal components are all real-valued

$$\mathbf{g}_{\mu\mu} = g^o_{(\mu\mu)} e_o \ + \ g^i_{[\mu\mu]} e_i \ = \ g^o_{(\mu\mu)} e_o, \quad \mu = 1, 2, \cdots, D \qquad (2.23)$$

there is no sum over μ in eq-(2.23). Hence, the interval $(ds)^2$ (2-10) becomes

$$(ds)^2 \ = \ (\ dZ^1_o \ g^o_{11} \ dZ^1_o - dZ^1_i \ g^o_{11} \ dZ^1_i \ ) + (\ dZ^2_o \ g^o_{22} \ dZ^2_o - dZ^2_i \ g^o_{22} \ dZ^2_i \ ) + \cdots +$$

$$(\ dZ^D_o \ g^o_{DD} \ dZ^D_o \ - \ dZ^D_i \ g^o_{DD} \ dZ^D_i \ ) \qquad (2.24)$$

One may then identify the coordinates

$$Z^1_o \leftrightarrow t^{(1)}, \quad Z^2_o \leftrightarrow t^{(2)}, \quad \cdots, \quad Z^D_o \leftrightarrow t^{(D)} \qquad (2.25)$$

with D temporal directions $t^{(1)}, t^{(2)}, \cdots, t^{(D)}$. And the coordinates

$$Z^1_i \leftrightarrow x^{(1)}_i, \quad Z^2_i \leftrightarrow x^{(2)}_i, \quad \cdots, \quad Z^D_i \leftrightarrow x^{(D)}_i; \quad i \ = \ 1, 2, \cdots, 7 \qquad (2.26)$$

can be identified with $7 \times D$ spatial coordinates. By setting

$$g^o_{11} < 0, \quad g^o_{22} < 0, \quad g^o_{33} < 0, \quad g^o_{DD} < 0 \qquad (2.27)$$

the interval $(ds)^2$ (2-24) can be written as the direct sum of D eight-dimensional spacetimes intervals $(ds)^2_8$, each one of signature $(-, +, +, +, \cdots, +)$,

$$(ds)^2_8 \ = \ dt^{(1)} \ g^o_{11} \ dt^{(1)} \ - g^o_{11} \ dx^{(1)}_i \ \delta_{ij} \ dx^{(1)}_j, \quad g^o_{11} < 0 \qquad (2.28a)$$

$$(ds)^2_8 \ = \ dt^{(2)} \ g^o_{22} \ dt^{(2)} \ - g^o_{22} \ dx^{(2)}_i \ \delta_{ij} \ dx^{(2)}_j, \quad g^o_{22} < 0, \quad \cdots\cdots \qquad (2.28b)$$

$$(ds)^2_8 \ = \ dt^{(D)} \ g^o_{DD} \ dt^{(D)} \ - g^o_{DD} \ dx^{(D)}_i \ \delta_{ij} \ dx^{(D)}_j, \quad g^o_{DD} < 0 \qquad (2.28c)$$

This direct sum of D eight-dimensional spacetimes intervals has the appearance of an 8-fold periodicity : $\mathbf{O}^D \leftrightarrow M^8 \oplus M^8 \oplus M^8 \cdots \oplus M^8$. On the other hand, there is also the correspondence $\mathbf{O}^2 \leftrightarrow M^{(14,2)} \leftrightarrow SO(14, 2)$, conformal group in $D = 14$. $\mathbf{O}^3 \leftrightarrow M^{(21,3)} \leftrightarrow SO(21,3)$. $\mathbf{O}^4 \leftrightarrow M^{(28,4)} \leftrightarrow SO(28,4)$, quasi-conformal group in $D = 28$.

The most renowned case, when *all* the coordinates of $\mathbf{Z}^\mu$ are *spatial*, is the Wilson's construction of the 24-dim Leech lattice based on $\mathbf{O}^3$ [15]. Dixon [6] has also offered a different construction of the 24-dim Leech lattice based on the *ternary* products of $\mathbf{O}$. Infinite extensions of the Exceptional algebras based on the notion of an 8-fold Exceptional Periodicity can be found in [20].

To finalize this subsection, we should add that extreme caution must be taken when one wishes to find the inverse of the octonionic metrics $\mathbf{g}_{ab}, \mathbf{g}_{\mu\nu}$. The inverse of the sum of two matrices with real or complex entries is given by the Sherman-Morrison-Woodbery formula

$$\mathbf{M} \ = \ (\mathbf{A} + \mathbf{B})^{-1} \ = \ \mathbf{A}^{-1} \ - \ (\ \mathbf{A} + \mathbf{A}\mathbf{B}^{-1}\mathbf{A} \ )^{-1} \ = \ M^{(ab)} \ + \ M^{[ab]} \quad (2.29)$$

Lets asssume $\mathbf{A}$ is symmetric and $\mathbf{B}$ is antisymmetric and that one extracts the symmetric and antisymmetric pieces of the right-hand side of eq-(2.29). In doing so, one would obtain the symmetric matrix $M^{(ab)} = \frac{1}{2}(\mathbf{M} + \tilde{\mathbf{M}})$ and the antisymmetric matrix $M^{[ab]} = \frac{1}{2}(\mathbf{M} - \tilde{\mathbf{M}})$. By inspection one can verify that the matrix $M^{(ab)} \neq \mathbf{A}^{-1}$, and $M^{[ab]} \neq \mathbf{B}^{-1}$. On the contrary, the matrices $M^{(ab)}$ and $M^{[ab]}$ turn out to be complicated functions of *both* matrices $\mathbf{A}, \mathbf{B}$. Furthermore, the formula (2.29) is not valid for octonionic-valued matrices due to the noncommutativity and nonassociativity of octonions.

3 Star Deformations of Octonionic Membranes, and Quantized Nambu-Poisson Brackets

The quantization of the membrane has been an extremely difficult task for many reasons. It is our belief that a *nonassociative* extension of QM [26] may hold the key. The latter approach differs from the old methods based on the geometry of the Moufang plane and Octonionic Quantum Mechanics [28]. In this section we shall present two different approaches in the construction of star product deformations of the DNG action. A quantization of branes as a conglomeration of quantum fields has been proposed by [13].

3.1 The Star Product Deformed Octonionic Membrane

Rather than focusing on the bivector $\Theta^{ab}\partial_a \wedge \partial_b$ in Poisson manifolds which is involved in the standard star product of two functions $f(\sigma^a), g(\sigma^a)$

$$(f \star g)(\sigma^a) \ = \ \mu_A \left(\ exp \ [\ \frac{i\hbar}{2} \ \Theta^{ab}\partial_a \wedge \partial_b \ ] \ (f \otimes g) \ \right), \quad a,b = 1,2,3 \quad (3.1a)$$

with $\mu_A(f \otimes g) = fg$, let us begin with the trivector $R^{abc}\partial_a \wedge \partial_b \wedge \partial_c$ and construct the following star-deformed product of three functions $f(\sigma^a), g(\sigma^a), h(\sigma^a)$

$$(f \star g \star h)(\sigma^a) \ = \ \mu_A \left(\ exp \ [\ \frac{\hbar^2}{3!} \ R^{abc}\partial_a \wedge \partial_b \wedge \partial_c \ ] \ (f \otimes g \otimes h) \ \right) \qquad (3.1b)$$

with $\mu_A(f \otimes g \otimes h) = fgh$, and $a, b = 1, 2, 3$.

Before proceeding, the immediate question which arises is where does the trivector $R^{abc}\partial_a \wedge \partial_b \wedge \partial_c$ come from ? In string-theory there is a 2-form $\mathbf{B}$, the Kalb-Ramond field, whose field strength $\mathbf{R} = \mathbf{dB}$ is a 3-form. The trivector R^{abc} components are directly related to closed superstring compactifications with background $\mathbf{R}$-fluxes. In our geometric setup, by recurring to the imaginary (antisymmetric) components of the octonionic metric $g^i_{[ab]}, i = 1, 2, \cdots, 7$, one can define the rank-2 antisymmetric tensor as

$$B_{ab}(\sigma) \ \equiv \ \sum_{i=1}^{i=7} \ g^i_{[ab]}(\sigma) \qquad (3.2)$$

and leading to the rank-3 antisymmetric tensor $R_{abc} \equiv \partial_{[c}B_{ab]}$. By raising indices with the inverse of the real part of the metric $(g^o_{(ab)})^{-1} = g_o^{(ab)}$ one can then obtain the desired trivector $R^{abc}\partial_a \wedge \partial_b \wedge \partial_c$. In the most general case the latter trivector is not necessarily constant.

In the previous section, we discussed how the nontrivial inverse of the sum of two matrices with real or complex entries is given by the Sherman-Morrison-Woodbery formula (2.29), and which is no longer valid for octonionc matrices. For this reason, we must raise indices with the inverse of the real part of the octonionic metric $(g^o_{(ab)})^{-1}$. Whereas, the imaginary parts of the octonionic metric are used to construct the Kalb-Ramond-like field B_{ab}, and which in turn, furnishes the rank-3 antisymmetric tensor R_{abc} leading to the trivector in eq-(3.1b) after raising indices.

Being equipped with the star-deformed triple product of functions (3.1b) one can evaluate the $\star$-deformed Freudenthal determinant $Det_{*F}(\mathbf{g}_{ab})$ giving

$$Det_{*F}(\mathbf{g}_{ab}) \ = \ g_{11} \ \star g_{22} \ \star g_{33} \ - \ g_{11} \ \star \mathbf{g}_{23} \ \star \bar{\mathbf{g}}_{23} \ - \ g_{22} \ \star \mathbf{g}_{31} \ \star \bar{\mathbf{g}}_{31} \ -$$

$$g_{33} \ \star \mathbf{g}_{12} \ \star \ \bar{\mathbf{g}}_{12} \ + \ 2 \ \mathbf{Re}\{\mathbf{g_{23}} \ \star \ \mathbf{g_{31}} \ \star \ \mathbf{g_{12}}\} \qquad (3.3)$$

and, finally, one may write the sought-after $\star$-product deformed Octonionic Membrane action in a very condensed manner as

$$S_{*DNG} \ = \ - \ T_2 \int \ d^3\sigma \ \sqrt{|Det_{*F}(\mathbf{g}_{ab})|} \qquad (3.4)$$

This result (3.4) based on the triple product differs from the results found in the next subsection. One should note also that we have the star-deformed triple products of the metric components displayed explicitly in eq-(3.3), but *not* those involving the star-deformed metric components per se : $\mathbf{g}_{*ab} = \partial_a X^\mu(\sigma) \star \partial_b X^\nu(\sigma) \star \mathbf{g}_{\mu\nu}(X(\sigma))$. This would have complicated matters considerably. The deformed action (3.4) is no longer $E_{6(-26)}$-invariant.

3.2 Phase Space Quantization of Ordinary Membranes and Nonassociative Quantum Mechanics

Having obtained the star-product-deformed octonionic membrane action (3.4), in this subsection we shall concentrate on ordinary membranes; namely membranes moving in ordinary spacetime backgrounds equipped with real-valued metrics, and subsequently add a constant background R-flux and then follow very closely the standard deformation quantization on the cotangent bundle presented by the authors in [26].

Let us consider a manifold M of dimension d with trivial cotangent bundle $T^*M = M \times (R^d)^*$ and coordinates $x^I = (x^i, p_i)$, where $I = 1, ..., 2d; x^i \in M, p_i \in (R^d)^*$, and $i = 1, ..., d$. Given a *constant* trivector $R = \frac{1}{3!} R^{ijk} \partial_i \wedge \partial_j \wedge \partial_k$; the algebra of functions $A = C^\infty(M)$ on M, and $\mu_A(f \otimes g) = fg$ the pointwise multiplication of functions, the noncommutative and *nonassociative* star product of two functions of the phase space coordinates $f(x^i, p_i), g(x^i, p_i)$ is [26]

$$(f \star g)(x,p) \; = \; \mu_A \left(\; exp \; [\; \frac{i\hbar}{2} (R^{ijk} p_k \partial_i \otimes \partial_j \; + \; \partial_i \otimes \tilde{\partial}^i \; - \; \tilde{\partial}^i \otimes \partial_i) \;] \; (f \otimes g) \; \right) \tag{3.5}$$

with $\partial_i = \partial_{x^i}; \tilde{\partial}^i = \partial_{p_i}$. In $\hbar = c = 1$ units, the physical dimension of R^{ijk} is $(length)^3$, and $[x^i] = length$, $[p_i] = (length)^{-1}$.

The integral formula for the *nonassociative* star product is [26]

$$(f \star g)(x,p) \; = \; (\frac{1}{\pi\hbar})^{2d} \int d^{2d}z \; \int d^{2d}w \; f(x+z) \; g(x+w) \; e^{-\frac{2}{\hbar} z^I \Theta^{-1}_{IJ} w^J} \tag{3.6}$$

with

$$\Theta^{-1}_{IJ} \; = \; \begin{pmatrix} 0 & -\delta^i_j \\ \delta^i_j & R^{ijk} p_k \end{pmatrix}, \quad I, J = 1, 2, 3, \cdots, 2d \tag{3.7}$$

Nambu [29] suggested to consider *nonassociative* algebras for the quantization of his bracket. In the nonassociative case, the authors [26] remarked that one has to specify which operators are multiplied first. If one chooses the first pair one may write the Nambu-Heisenberg bracket as

$$[A, \; B, \; C]_{NH} = \; [A, \; B] \; C \; + \; [C, \; A] \; B \; + \; [B, \; C] \; A \tag{3.8}$$

where $[A, B]C = (AB)C - (BA)C$. The nonassociative star product (3.5), evaluated on a triple of coordinate functions, gives

$$[x^i,\ x^j,\ x^k]_{*NH}\ =\ i\hbar\ \left(\ R^{ijl} p_l \star x^k\ +\ R^{jkl} p_l \star x^i\ +\ R^{kil} p_l \star x^j\ \right) \tag{3.9}$$

The opposite Nambu-Heisenberg bracket defined as

$$[A,\ B,\ C]'_{NH}\ =\ C\ [B,\ A]\ +\ B\ [A,\ C]\ +\ A\ [C,\ B] \tag{3.10}$$

is in general *no* longer equal to *minus* the original Nambu-Heisenberg bracket due to the *nonassociativity* of the star product. Their sum gives the Jacobiator

$$[[A,\ B,\ C]]\ \equiv\ [\ [A,\ B],\ C\ ]\ +\ [\ [C,\ A]\ ,B\ ]\ +\ [\ [B,\ C],\ A\ ]\ =$$

$$[A\ ,B\ ,C]_{NH}\ +\ [A\ ,B\ ,C]'_{NH}\ \neq\ 0 \tag{3.11}$$

For the nonassociative star product (3.5), evaluated on a triple of coordinate functions, one obtains the non-zero Jacobiator [26]

$$[[x^i,\ x^j,\ x^k]]_*\ =\ i\hbar\ \left(\ R^{ijl}\ [p_l, x^k]_*\ +\ R^{jkl}\ [p_l, x^i]_*\ +\ R^{kil}\ [p_l, x^j]_*\ \right)\ =\ 3\hbar^2 R^{ijk} \tag{3.12}$$

where $[f,g]_* = f \star g - g \star f$ for all $f, g \in C^\infty(M)$ and $i,j,k = 1,...,d$.

The authors [26] showed that the Weyl-Wigner-Moyal-Groenowold phase space formulation of quantum mechanics [27] is powerful enough to study Nonassociative Quantum Mechanics. Observables are implemented as real functions on phase space, states are represented by pseudo-probability Wigner-type density functions, and noncommutativity of operators enters via a star product of functions, which is the deformation quantization of a classical Poisson structure. States, operators, eigenvalues, uncertainty relations of area and volume operators, dynamics and transformations were rigorously studied in [26].

Having reviewed very briefly the findings of [26] let us begin with the following 6-dim action associated with the motion of a 5-brane in a flat target D-dim spacetime background

$$S_{DNG}\ =\ -\ T_5\ \int d^6\sigma\ \sqrt{\{X^{\mu_1}, X^{\mu_2}, \cdots X^{\mu_6}\}_{NPB}\ \{X_{\mu_1}, X_{\mu_2}, \cdots X_{\mu_6}\}_{NPB}}, \tag{3.13}$$

where $X^\mu(\sigma^a)$ are the embedding maps of the 6-dim world hyper volume (swept by the 5-brane) onto the flat target spacetime background; the indices $a = 1, 2, \cdots, 6$ span the six dimensions of the world hyper volume, and T_5 is the 5-brane tension. We intend to find what the star-product deformations of the right-hand side of eq-(3.13) look like and which will guide us into constructing a *phase space* quantization of the membrane.

Lets start by looking at the cotangent space of the membrane's world volume. The coordinates of the cotangent space of the 3-dim membrane's world volume $T^*\mathcal{M} \simeq \mathcal{M} \times (R^3)^*$ are $\xi^a \in \mathcal{M}, \tilde{\xi}_a \in (R^3)^*; a = 1, 2, 3$. We shall focus now on the maps from the 6-dim phase space (corresponding to the dimension of the cotangent space of the 3-dim membrane's world volume T^*M), onto the flat D-dim spacetime background described by the coordinates $X^\mu(\xi^a, \tilde{\xi}_a)$. [1]. Since the 5-brane world hyper volume is also 6-dimensional, the idea is to match the 6-dim phase space of the membrane with the 6-dim world-volume of the 5-brane.

After establishing the following correspondence

$$(x^i, p_i) \leftrightarrow \xi^a, \tilde{\xi}_a,\ \ f(x^i, p_i),\ g(x^i, p_i) \leftrightarrow X^\mu(\xi^a, \tilde{\xi}_a),\ X^\nu(\xi^a, \tilde{\xi}_a);\ \ i = 1, 2, 3;\ \ a = 1, 2, 3 \tag{3.14}$$

in all of the above equations (3.5-3.12) it allows us to build the star products of $X^\mu(\xi^a, \tilde{\xi}_a) \star X^\nu(\xi^a, \tilde{\xi}_a)$, and in turn, construct the $\star$-product deformations of the Nambu-Poisson brackets appearing in the 5-brane action (3.13). This is achieved through the following *iterative* procedure :

Starting with the $\star$-product deformation of the 2-bracket $\{X^{\mu_1}, X^{\mu_2}\}_*$

$$\{X^{\mu_1}, X^{\mu_2}\}_* \ \equiv\ X^{\mu_1} \star X^{\mu_2}\ -\ X^{\mu_2} \star X^{\mu_1} \tag{3.15}$$

where the noncommutative and *nonassociative* $\star$ product $X^{\mu_1} \star X^{\mu_2}$ is defined by eq-(3.5) *after* using the correspondence (3.14), it allows to define the following star-product deformation of the 3-brackets

$$\{X^{\mu_1}, X^{\mu_2}, X^{\mu_3}\}_* \ =\ \{X^{\mu_1}, X^{\mu_2}\}_* \star X^{\mu_3}\ +\ \{X^{\mu_3}, X^{\mu_1}\}_* \star X^{\mu_2}\ + \\ \{X^{\mu_2}, X^{\mu_3}\}_* \star X^{\mu_1} \tag{3.16}$$

Having constructed the star-3-bracket (3.16) in terms of the star-2-bracket (3.15) , the star-4-bracket is defined in terms of the star 3-brackets as follows

$$\{X^{\mu_1}, X^{\mu_2}, X^{\mu_3}, X^{\mu_4}\}_* \ =\ \{X^{\mu_1}, X^{\mu_2}, X^{\mu_3}\}_* \star X^{\mu_4} + \{X^{\mu_4}, X^{\mu_1}, X^{\mu_2}\} \star X^{\mu_3} + \\ \{X^{\mu_3}, X^{\mu_4}, X^{\mu_1}\}\ \star\ X^{\mu_2}\ +\ \{X^{\mu_2}, X^{\mu_3}, X^{\mu_4}\}\ \star\ X^{\mu_1} \tag{3.17}$$

Similarly, by iteration, the star-product deformation of the 5-brackets are defined in terms of the star-4-brackets, and in turn, the star-product deformation of the 6-brackets is defined in terms of the star-5-brackets, and so forth. After this very lengthy iterative procedure one finally has the expression for $\{X^{\mu_1}, X^{\mu_2}, X^{\mu_3}, \cdots X^{\mu_6}\}_{*NPB}$. The iteration process also provides the proper location of the multiple parenthesis involved. Since the star product is nonassociative one must specify unambiguously the location of the parenthesis.

Concluding, the $\star$-deformed DNG membrane action ends up being

$$S_{\star DNG}\ =\ -\,(T_2)^2 \int d^3\xi \wedge d^3\tilde{\xi}\ \sqrt{\{X^{\mu_1}, X^{\mu_2}, \cdots X^{\mu_6}\}_{*NPB}\ \star\ \{X_{\mu_1}, X_{\mu_2}, \cdots X_{\mu_6}\}_{*NPB}} \tag{3.18}$$

The physical units are chosen to be $[\xi^a] = [\tilde{\xi}_a] = [X^\mu] =$ length. The membrane tension T_2 has $(length)^{-3}$ units so its square $(T_2)^2$ has the same physical

[1]Note that the 6-dim phase space does not have two temporal directions because energy is not time

dimensions $(length)^{-6}$ as those of a 5-brane tension T_5. The 6-dim measure in (3.18) is associated with the 6-dim phase space measure corresponding to the 6-dim cotangent space of the 3-dim world-volume swept by the membrane. It is interesting that the 5 brane is the "electromagnetic" (EM) dual to the membrane in $D = 11$. In general, a p-brane is the EM dual to a p'-brane in D-dim if $D = p + p' + 4$.

One may include the alternative expressions to all of these brackets by *reversing* the order of all the factors like it was performed in eq-(3.10). Due to the *nonassociativity* of the star product (3.5) the results will *not* be equal to the minuses of the brackets found above. Thus there are two different star-product deformations of the membrane action due to the nonassociativity of the star product (3.5).

To conclude, a more rigorous formulation of the embedding of the membrane's 6-dim cotangent space (phase space) into a target space background is obtained via Finsler geometry. Embedding the $2(p+1)$-dim cotangent space (phase space associated to a p-brane's $p+1$-dim world-volume) into the $2D$-dim cotangent space associated with a target D-dim spacetime background leads to the following relation between the respective cotangent space intervals

$$h_{ij}(x,p)\ dx^i\ dx^j \ +\ h_{ab}(x,p)\ (dp^a + N^a_i(x,p)dx^i)\ (dp^b + N^b_j(x,p)dx^j) \ =$$

$$g_{\mu\nu}(X,P)\ dX^\mu\ dX^\nu \ +\ g_{\alpha\beta}(X,P)\ (dP^\alpha + N^\alpha_\mu(X,P)dX^\mu)\ (dP^\beta + N^\beta_\nu dX^\nu) \tag{3.19}$$

The metric $h_{ij}(x,p), h_{ab}(x,p)$, and the nonlinear connection $N^a_i(x,p)$ are the "pullbacks" of $g_{\mu\nu}(X,P), g_{\alpha\beta}(X,P)$ and the nonlinear connection $N^\alpha_\mu(X,P)$. However, it is not clear how to relate all these quantities. The ordinary embedding leads to

$$h_{ij}(x)\ dx^i\ dx^j \ =\ g_{\mu\nu}(X)\ dX^\mu\ dX^\nu \ \Rightarrow\ h_{ij} \ =\ g_{\mu\nu}\ \frac{\partial X^\mu}{\partial x^i}\ \frac{\partial X^\nu}{\partial x^j} \tag{3.20}$$

from the right-hand's expression one constructs the DNG p-brane action ($p + 1 = d; i,j = 1,2,\cdots,d$) by taking the square root of the absolute value of the determinant of h_{ij}. One can see that eq-(3.20) is very different from eq-(3.19).

This brings to our attention the very active ongoing research on Double Field Theory and Exceptional Field Theory (see [16] for a review) which is a generalization of Kaluza-Klein theory that unifies the metric and p-form gauge field degrees of freedom of Supergravity into a generalized or extended geometry, whose *additional* coordinates may be viewed as conjugate to brane winding modes. Clearly, the interval (3.19) inspired from Finsler-geometry involves additional coordinates, a doubling of coordinates. This Finsler geometric approach warrants further investigation in connection to Double, Exceptional Field theories. Finally, we should add that another very active field of research involving the n-ary structures of NPB is L_∞, A_∞ algebras. Most recently these algebras have been instrumental in the formulation of Nonassociative Gravity [17].

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