Quantum Entropy Conservation and Photon Quantum Equation*

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Abstract

I try to show that the entropy of an isolated-reversible quantum system is an invariant of motion. The first step for a verification of the theoretical hypothesis is to obtain the combined entropy of the gauge bosons and of the particles, then I try to write the wave function of the photon (of which I provide the simplest quantum solutions).

The energy and entropy (for reversible processes, or time reversal symmetry) of an isolated classical system is conserved.

I think that this is true for an isolated-reversible quantum system.

The Von Neumann quantum entropy is:

$$S(t) = -\operatorname{Tr}\left[\widehat{\rho}(t)\ln(\widehat{\rho}(t))\right] = -\operatorname{Tr}\left[\widehat{U}\,\widehat{\rho}\,\widehat{U}^\dagger \ln\left(\widehat{U}\,\widehat{\rho}\,\widehat{U}^\dagger\right)\right] =$$
$$= -\operatorname{Tr}\left[\widehat{U}\,\widehat{\rho}\,\widehat{U}^\dagger \sum_n \frac{(-1)^{n-1}}{n}\left(\widehat{U}\,\widehat{\rho}\,\widehat{U}^\dagger - \widehat{U}\,\widehat{U}^\dagger\right)^n\right] = -\operatorname{Tr}\left[\sum_n \frac{(-1)^{n-1}}{n}\widehat{U}\,\widehat{\rho}\,\left(\widehat{\rho}-1\right)^n \widehat{U}^\dagger\right] =$$
$$= -\operatorname{Tr}\left[\sum_n \frac{(-1)^{n-1}}{n}\widehat{\rho}_0\,\left(\widehat{\rho}_0-1\right)^n\right] = -\operatorname{Tr}\left[\widehat{\rho}_0\ln(\widehat{\rho}_0)\right] = S(t_0)$$

so that the entropy of an isolated-reversible system, with the unitary transformation $\widehat{U}(\widehat{H}) = e^{-i\frac{\widehat{H}}{\hbar}t}$, is conserved[1]; this is true for each analytic function $f(t) = \operatorname{Tr}\left[f(\widehat{\rho}(t))\right]$, so that there is an infinity of invariants of motion[2].

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[1] This is a simple generalization of the work of Ansari, Steensel and Nazarov

[2] particle dynamics can only occur due to variations in the force field (variation of the wave function of the boson, in phase, amplitude or spatial extension): the motion of the particles produces observables in the fields, the variation of the fields induces observables in the motion of particles

If there is a variation of the measured quantum entropy, then there must be unobserved particles that hold the total value: the gauge bosons.

For example the black hole emits gauge bosons (Hawking radiations, gravitons) that contain information on the evolution of the black hole.

This is true for the Universe, that contain itself, so that the entropy of the Universe, with the gauge bosons, could be an invariant.

The simplest gauge boson is the photon, but it is necessary a quantum equation for the photon to evaluate the entropy from a wave function.

I write the photon quantum equation deriving it from the Lagrangian[3] of the electromagnetic field

$$
\begin{array}{l}
F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_z \\ E_z & -B_y & B_z & 0 \end{pmatrix} \\
\mathcal{L} = -\frac{1}{16\pi} F^{\alpha\beta} F_{\alpha\beta} = -\frac{1}{16\pi} g_{\gamma\alpha} g_{\delta\beta} \left(\partial^\gamma A^\delta - \partial^\delta A^\gamma \right) \left(\partial^\alpha A^\beta - \partial^\beta A^\alpha \right) \\
p_\mu = \frac{\partial \mathcal{L}}{\partial \partial^0 A^\mu} = -\frac{F_{0\mu}}{4\pi} \\
\boldsymbol{E} = -4\pi \boldsymbol{p} = 4\pi i\hbar \boldsymbol{\nabla}_{\boldsymbol{A}} \\
\mathcal{H} = \partial_0 A_l \frac{\partial \mathcal{L}}{\partial \partial_0 A_l} - \mathcal{L} = -\frac{1}{4\pi} F^{0\mu} \partial_0 A_\mu + \frac{1}{16\pi} F^{\alpha\beta} F_{\alpha\beta} = \frac{\boldsymbol{E}^2}{8\pi} + \frac{\boldsymbol{B}^2}{8\pi}
\end{array}
$$

this result[4] permit to write the photon quantum equation:

$$
\boxed{i\hbar \partial_t \Psi(A^\mu) = -2\pi\hbar^2 \Delta_{\boldsymbol{A}} \Psi(A^\mu) + \frac{\boldsymbol{\nabla} \times \boldsymbol{A}}{8\pi} \Psi(A^\mu) = -2\pi\hbar^2 \Delta_{\boldsymbol{A}} \Psi(A^\mu) + \frac{\boldsymbol{B}^2}{8\pi} \Psi(A^\mu)}
$$

I write three solution of this equation, to verify the exactness of the equation.

[3] I use the book Classical Electodynamics of Jackson

[4] The step-by-step calculus in The classical theory of field of Landau and Lifshitz

The first wave function is for a constant electric field[5]:

$$
\begin{aligned}
&A^\mu = (-Ex, 0, 0, 0)\\
&\boldsymbol{B} = 0\\
&\boldsymbol{E} = \hat{\boldsymbol{x}}E\\
&i\hbar\partial_t\Psi = -2\pi\hbar^2\Delta_{\boldsymbol{A}}\Psi\\
&\Psi = e^{-i\frac{W}{\hbar}t}\psi\\
&W\psi = -2\pi\hbar^2\Delta_{\boldsymbol{A}}\psi\\
&\Psi = \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{i}{\hbar}\left(\boldsymbol{p}\cdot\boldsymbol{A}-2\pi\boldsymbol{p}^2t\right)}\\
&\Delta_{\boldsymbol{A}}\Psi - \frac{B^2}{16\pi^2\hbar^2}\Psi + \frac{i}{2\pi\hbar}\partial_t\Psi = 0\\
&\Delta_{\boldsymbol{A}}\Psi^* - \frac{B^2}{16\pi^2\hbar^2}\Psi^* - \frac{i}{h}\partial_t\Psi^* = 0\\
&\Psi^*\Delta_A\Psi - \Psi\Delta_A\Psi^* + \frac{i}{h}\Psi^*\partial_t\Psi + \frac{i}{h}\Psi\partial_t\Psi^* = 0\\
&\boldsymbol{\nabla}_{\boldsymbol{A}}\cdot\boldsymbol{J} + \partial_t(\Psi^*\Psi) = 0\\
&\boldsymbol{J} = -ih\left(\Psi^*\boldsymbol{\nabla}_{\boldsymbol{A}}\Psi - \Psi\boldsymbol{\nabla}_{\boldsymbol{A}}\Psi^*\right)\\
&\boldsymbol{E} = \Psi^*(-i\hbar\boldsymbol{\nabla}_{\boldsymbol{A}})\Phi = \frac{\boldsymbol{p}}{h}\\
&\Psi = \frac{1}{\sqrt{2\pi\hbar}}e^{2\pi i\left(\boldsymbol{E}\cdot\boldsymbol{A}-2\pi h\boldsymbol{E}^2t\right)}\\
&\boldsymbol{J} = 4\pi\boldsymbol{E}
\end{aligned}
$$

there is a constant probability current in the direction of the electric field.

The second wave function is for a constant magnetic field:

$$
\begin{aligned}
&A^\mu = (0, 0, By, 0)\\
&\boldsymbol{B} = (B, 0, 0)\\
&\boldsymbol{E} = 0\\
&i\hbar\partial_t\Psi = -2\pi\hbar^2\Delta_{\boldsymbol{A}}\Psi + \frac{B^2}{8\pi}\Psi\\
&\Psi = e^{-i\frac{W}{\hbar}t}\psi\\
&-2\pi\hbar^2\Delta_A\psi + \left(\frac{B^2}{8\pi} - W\right)\psi = 0\\
&\Psi = \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{i}{\hbar}\left(\boldsymbol{p}\cdot\boldsymbol{A}-2\pi\boldsymbol{p}^2t+\frac{\boldsymbol{B}^2}{8\pi}t\right)}\\
&\boldsymbol{E} = -i\hbar\Psi^*\boldsymbol{\nabla}_{\boldsymbol{A}}\Psi = \frac{\boldsymbol{p}}{h}\\
&\Psi = \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{i}{\hbar}\left(h\boldsymbol{E}\cdot\boldsymbol{A}-2\pi h^2\boldsymbol{E}^2t+\frac{\boldsymbol{B}^2}{8\pi}t\right)} = \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{i}{\hbar}\frac{\boldsymbol{B}^2}{8\pi}t}\\
&\boldsymbol{J} = 0
\end{aligned}
$$

there is not a flow of probability in a constant magnetic field.

[5]the phase of the wave function could be measured to verify the theory

The third wave function is for a circular polarized photon:

$$
\begin{aligned}
&A^{\mu} = (0, 0, A\sin(kx-\omega t), A\cos(kx-\omega t))\\
&\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A} = \tfrac{\omega}{c}A\ (0, \sin(kx-\omega t), \cos(kx-\omega t))\\
&\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \tfrac{1}{c}\tfrac{\partial \boldsymbol{A}}{\partial t} = \tfrac{\omega}{c}A\ (0, \cos(kx-\omega t), -\sin(kx-\omega t))\\
&\boldsymbol{E}^2 = \boldsymbol{B}^2 = \tfrac{\omega^2}{c^2}A^2\\
&i\hbar\partial_t\Psi = -2\pi\hbar^2\Delta_{\boldsymbol{A}}\Psi + \tfrac{\omega^2}{8\pi c^2}A^2\Psi\\
&\Psi = e^{-i\frac{W}{c\hbar}t}\psi\\
&\tfrac{W}{c}\psi = -2\pi\hbar^2\Delta_{\boldsymbol{A}}\psi + \tfrac{\omega^2}{8\pi c^2}A^2\psi\\
&\Delta_{\boldsymbol{A}}\psi + \tfrac{W}{2\pi\hbar^2 c}\psi - \tfrac{\omega^2}{16\pi^2\hbar^2c^2}A^2\psi = 0\\
&\boldsymbol{\xi} = \alpha\boldsymbol{A}\\
&\alpha^2\Delta_{\boldsymbol{\xi}}\psi + \left(\tfrac{W}{2\pi\hbar^2 c} - \tfrac{\omega^2}{16\pi^2\hbar^2c^2\alpha^2}\xi^2\right)\psi = 0\\
&\alpha = \sqrt[4]{\tfrac{\omega^2}{16\pi^2\hbar^2c^2}} = \sqrt{\tfrac{\nu}{2\hbar c}}\\
&\Delta_{\boldsymbol{\xi}}\psi + \left(\tfrac{2W}{h\nu} - \xi^2\right)\psi = 0\\
&W_{n_x,n_y} = h\nu\left(n_x + n_y + \tfrac{1}{2}\right)\\
&\Psi\left(W_{n_x,n_y}\right) = e^{-\frac{\alpha^2A^2}{2} - i\frac{W}{\hbar c}t}\prod_j\left(\tfrac{\alpha}{\pi^{1/2}2^{n_j}n_j!}\right)^{1/2}H_{n_j}(\alpha A^{\mu}) =\\
&= e^{-\frac{\alpha^2A^2}{2} - i\frac{W}{\hbar c}t}\tfrac{\alpha^{3/2}}{\pi^{3/2}2^{n_x+n_y+n_z}n_x!n_y!n_z!}H_{n_x}(\alpha A_x)H_{n_y}(\alpha A_y)H_{n_z}(\alpha A_z) =\\
&= (-1)^{n_y}e^{-\frac{\alpha^2A^2}{2} - i\frac{W}{\hbar c}t}\tfrac{\alpha^{3/2}}{\pi^{3/2}2^{n_x+n_y+n_z}n_x!n_y!n_z!}H_{n_y}(\alpha A\sin(kx-\omega t))H_{n_z}(\alpha A\cos(kx-\omega t))
\end{aligned}
$$

this is a three-dimensional quantum harmonic oscillator for a photon[6,7].

The photon wave function is a A^{μ} function, then:

$$\Psi(A^{\mu}) = \sum_{n_0,n_1,n_2,n_3} a_{n_0,n_1,n_2,n_3}A_0^{n_0}A_1^{n_1}A_2^{n_2}A_3^{n_3} = b_0 + c_{\mu}A^{\mu} + \cdots$$

so that the interaction in the quantum equations must contain[8] the product of vector potential and particle wave function[9,10].

[6] $n_x = n_y$ for the quantum number, and the vector potential

[7] the exponent $-i\frac{W}{c\hbar}t$ is used to obtain the harmonic oscillator, it is a temporary solution, until the problem can be solved more correctly

[8] to the first order of approximation

[9] this approximation is also true for probability

[10] the photon wave function seem to be present in the gauge covariant derivative, if $e\Psi_{\gamma}^{*}\psi_e = ec_{\mu}^{*}A^{\mu}\psi_e = -ieA^{\mu}\psi_e$ with $b_0 = 0$ and $c_{\mu} = -i$