A system-theoretic point of view on the nonlinearity of some dynamic systems

Emanuel Gluskin

Engineering Department, Ruppin Academic Center, Emek Hefer 40250, Israel

emanuel15@bezeqint.net

Abstract: We consider the concept of *structural nonlinearity*, that is, the nonlinearity expressed as an influence of the processes in the system on its structure. As the given examples show, this concept is very useful.

Keywords: State equations, Liquid flow, Order in Medical Treatment, Electrical Circuits.

1. Introduction

Modern science more and more deals with complicated structures, and physicists should be introduced to some system theoretic basics. These basics help one, in our first example, to think about the "structure" of an observed liquid flow, and to explain why the mathematical nonlinearity must appear in a description of the flow.

The second example is sociological, showing how the concept of 'structural nonlinearity' can be applied in a qualitative consideration of a medical treatment process.

Finally, we show some electrical circuits relevant to the 'structural nonlinearity'.

2. The nonlinearity of the liquid flow

The nonlinearity of the main hydrodynamic Navier--Stocks equation follows from the very dynamics of the flow. This equation is Newton's second law equation applied to a *moving element of the flow*. The acceleration is

$$\frac{d\vec{v}}{dt} \equiv \frac{\partial \vec{v}}{\partial t} + (\vec{v}\nabla)\vec{v} \qquad (1)$$

where the differential operator

$$\vec{v}\nabla \equiv \frac{d\vec{r}}{dt} \cdot \frac{\partial}{\partial \vec{r}} \qquad (2)$$

appears because $\vec{v} = \vec{v}(\vec{r}, t)$ and $\vec{r} = \vec{r}(t)$.

The "quadratic" by the velocity expression $(\vec{v}\nabla)\vec{v}$ introduces the nonlinearity in the Navier--Stocks equation.

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v}\nabla)\vec{v} = -\nabla p + \eta \Delta \vec{v} \qquad (3)$$

written here for an incompressible flow. For a compressible flow (more relevant to aerodynamics) this equation is changed [1], but the nonlinearity remains by the same reasons.

We shall obtain the nonlinearity of the flow also in another way. This less formal, but heuristically useful way explains why the equation *must be* nonlinear.

3. The "structural nonlinearity"

In the typical (e.g. [2]) *linear* systems presentation

$$\frac{d\vec{x}}{dt} = [A]\vec{x} + [B]\vec{u}(t), \quad \vec{x}(0) = \vec{x}_0, \qquad (4)$$

the matrices [A] and [B] represent the *structure* of the system; $\vec{x}(t)$, or $\{x_k(t)\}$, is the vector of the state variables, $\vec{x}_0$ is the given initial value of $\vec{x}(t)$, i.e. all of the components $\{x_k(0)\}$ are given, and $\vec{u}(t)$ is the vector of the inputs.

Dimensions of $\vec{x}(t)$ and $\vec{u}(t)$ in (4) need not be the same, and then the respective dimensions of the matrices also will be different. However, one should see that since the human operator *both* applies to the system $\vec{u}(t)$ and defines $\vec{x}_0$ (indeed, neither $\vec{u}(t)$, nor $\vec{x}_0$ is defined by the producer of the real system), there is a *generalized input* $\{\vec{x}_0, \vec{u}(t)\}$, and the checking of the linearity has to be not $\vec{u}(t) \to k\,\vec{u}(t)$, but

$$\{\vec{x}_0, \vec{u}(t)\} \to k\{\vec{x}_0, \vec{u}(t)\} = \{k\vec{x}_0, k\vec{u}(t)\}\ . \qquad (5)$$

That is, the initial conditions also have to be changed in the test.

For a general *nonlinear* system, it is conventional, to write after Poincare, Lyapunov, and then Andronov at al.,

$$\frac{d\vec{x}}{dt} = \vec{F}(\vec{x}, \vec{u}(t)), \quad \vec{x}(0) = \vec{x}_0\ , \qquad (6)$$

If the processes in the system, initiated in any way by the generalized input, do not change the system, i.e. the parameters of the matrices in (4) are fixed, then the very physical system remains just the same as it was in rest and *obviously is linear*.

Thus, we write for a *nonlinear* system, the state equations not as (6), but as

$$\frac{d\vec{x}}{dt} = [A(\vec{x})]\vec{x} + [B(\vec{x})]\vec{u}(t), \quad \vec{x}(0) = \vec{x}_0\,. \qquad (7)$$

That is, keeping the concept of structure also for a nonlinear system, we suggest to basically define the nonlinearity as an influence of the processes in the system on its structure (parameters).

It is sufficient to be focused only on [A], that is, to deal only with the homogeneous equation obtained for $\vec{u}(t) = \vec{0}$:

$$\frac{d\vec{x}}{dt} = [A(\vec{x})]\vec{x}, \quad \vec{x}(0) = \vec{x}_0 \qquad (8)$$

in which the nonlinearity is quite obvious. The extended version (7) does not add a lot to the point.

Let us show how observing the "structural nonlinearity", or the "[A(x)]-nonlinearity" can help one to understand something in a difficult physics situation. This is a matter of a very simple logic.

4. Why the hydrodynamic (aerodynamic) equations *must be* nonlinear from the "structural" system-theoretic point of view?

Consider liquid (air) flow, namely, its velocity field $\vec{v}(\vec{r},t)$ from the system theoretic point of view. That is, let us see $\vec{v}(\vec{r},t)$ as some vector $\vec{x}(t)$ of the state variables of the "system", which has to be found. However, the structure of the "system", i.e. the analogy to the above matrix [A] is just the same vector field $\vec{v}(\vec{r},t)$ that we can observe -- indeed, in no other form the structure is given to the observer. If we add the pressure *p*, also included in the given below Navier-Stocks equation, to the components of $\vec{v}$, as an unknown, i.e. include *p* into vector $\vec{x}$, the fact that the "structure" of the system is organically associated with the state variables remains. For the flow, the connection of the structure [A] and the unknown $\vec{x}(t)$ is obvious, simply because they are the same. That is, the flow obviously has the structural [A(x)]-nonlinearity, i.e. the hydrodynamic equations *must be* nonlinear.

This simple observation makes, in particular, the appearance of the turbulence not surprising, because a chaotic state (the turbulence) can be obtained only in a nonlinear system. This physical conclusion shows the heuristic validity of the concept of "structural nonlinearity".

However, let us also directly observe the matrix [A(x)] in the Navier-Stocks equation. To rewrite this equation closer to (7) or (8) we have to agree about the following. Since the system state equation theory does NOT deal with moving systems, we should use not Lagrange's, but Euler's hydrodynamics presentation [1], that is, to be focused not on a moving element of the flow, but on a certain point $\vec{r}$. Then, $\vec{x}$ is just $\vec{v}$ at this point, and, *for the localized unmoving system*, we should interpret the partial derivative by time, $\frac{\partial \vec{v}}{\partial t}$*, as the full derivative* $\frac{d\vec{x}}{dt}$. With these actions, we rewrite Navier-Stocks equation:

$$\frac{\partial \vec{v}}{\partial t} = [-(\vec{v}\nabla) + \frac{\eta\Delta}{\rho}]\vec{v} - \frac{\nabla p}{\rho} \qquad (9)$$

thus:

$$\frac{d\vec{x}}{dt} = [-(\vec{x}\nabla) + \frac{\eta\Delta}{\rho}]\vec{x} - \frac{\nabla p}{\rho} \quad , \qquad (10)$$

observing matrix $[A(\vec{x})]$ as

$$[A(\vec{x})] = [-(\vec{x}\nabla) + \frac{\eta\Delta}{\rho}] \; , \qquad (11)$$

or, returning to $\vec{v}$,

$$[A(\vec{v})] = [-(\vec{v}\nabla) + \frac{\eta\Delta}{\rho}] \qquad (12)$$

The structural nonlinearity of the flow is obvious.

5. A medical-sociological application of the structural outlook

For this application, we have to give a specific form to the [A(x)]-nonlinearity. This form is so important, that deserves some preliminaries, as follows.

5.1. On the role of switching systems and *operations* – a version of the structural nonlinearity

Modern technological world becomes more and more that of switched systems, as is most easily seen in electronics development. If an electronic modeling is needed for a physical situation, – it is most probably performed, today, using a switched, and not an analogous technology. This is most strongly expressed when we speak about nonlinear circuits. For instance, a *ferroelectric capacitor* [4] can actually demonstrate its nonlinearity only in a limited voltage range, -- and thus as a weak nonlinearity. For revealing its theoretically very strong nonlinearity, the capacitor has to be under so high a voltage stress that electrical breakdown of its layers can occur. That is, the associated analogous strongly nonlinear circuit is insufficiently reliable. However, nonlinear and even strongly nonlinear absolutely reliable capacitive unit can be obtained by proper switching of some linear capacitors. The point is that such a unit can be linear (time variant) and nonlinear, depending of what defines the instances of the switching – if the switching is defined by a state variable – that is, by a circuit function that has to be found – than it is a nonlinear circuit (capacitive unit). See Section 6.

While with analog circuits we have the reliability problem, there is also a fundamental problem with switched circuits [1]. Namely, switching is always associated with some jump (inflection, etc.,) of a time function, and such jump or inflection is always associated with an infinite frequency spectrum of the function. The latter is problematic if one speaks about a lumped circuit to which Kirchhoff's equations are wanted to be applied – application of these equations is associated with a limitation of the frequencies of the circuit operation. For the process with the too high frequencies, the circuit has to be treated, as a distributed one, using Maxwell's equations, which is a very different mathematical frame. The frequency problem means that we cannot speak about really isolated switching points, and have to smooth the singularity by defining the intervals in which the switching occurs as having some finite duration. This makes the switching not so sharp, and limits the frequency range, so that Kirchhoff's equations can possibly be applied. However, it is widely accepted to see the switching instances (points) as really isolated, and in what follows we use this common idealization.

For the switching processes, our [A(x)]-nonlinearity is reduced to the condition that in any function of the type

$$f(t-t_1) \qquad (13)$$

the switching instant t_1 is a zero- or level-crossing of a state variable $x(t)$, i.e. of an unknown function that has to be found. (See also Fig. 1 below where *f*(.) is written somewhat more generally, without the explicit shift.) The opposite case when t_1 is prescribed (either is a known number, or by a certain level-crossing of a *given* time function), expression (13) is linear, and its appearance in an equation cannot lead to a chaotic solution.

Considering the "level-crossing nonlinearity", $t_1(x)$, as a case of the [A(x)]-nonlinearity, let us turn to our next example. The topic here is a nurse-house, or a hospital, though one can find many other examples of such a nonlinearity. In such a place of the human treatment, the vector of the state variables $\vec{x}(t)$ is the physical states of the patients.

5.2. Are the actions of the staff of a nurse-house, or a hospital, linear or nonlinear processes?

In the here-relevant terms, the "switching instances" are the moments when the stuff of such an organization:

Distributes meals,

Gives medicine (drugs),

Performs medical checking (measurement of the temperatures, etc.),

Replaces the pumpers.

With the accepted here idealization, these actions are done at some certain time instants, that is, we speak about a switching process for which the form (13) is relevant, and the crucial question is whether these instants are defined *a-priori*, or the vector $\vec{x}(t)$ of the physical (and moral, i.e. with the natural human wishes and requests) states of the patients, *influence*s the instants of the actions. In the latter case, we have nonlinearity.

As the matter of fact, besides some exceptional cases, all of the actions are done at the prescribed moments t_1. If it is *not* so, then (13) becomes a nonlinear expression because of $t_1(x)$, and, being involved in some equations, could make these equations nonlinear. The latter could, in principle cause, a chaotic $x(t)$ – a situation to which any nurse-house is even not allowed to be close, since then a patient could find himself in unexpected troubles.

6. Electrical circuits with the switching structural nonlinearity; the role of the comparator

Electrical circuits present a good field for modeling our mathematical scheme. Figure 1 shows how the dependence $t_1(\vec{x})$, i.e. the switching nonlinearity, can be realized. It is assumed that at least one of the inputs of the comparator, i.e. $f_1(t)$, or $f_2(t)$, or both, belong(s) to the unknown $\vec{x}$. The comparator selects the point *t** depending on $\vec{x}$, where $f_1(t) = f_2(t)$, which finally leads to some *nonlinear* expression $f(t, t^*(x))$. In the notations of (13), it may be some $f(t - t_1)$, with the simple shift

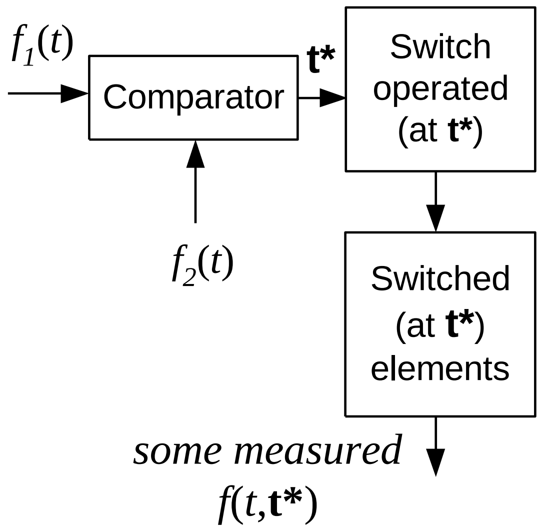

Fig. 1: Thus, using a comparator, on whose inputs at least one of the functions $f_1(t)$, or $f_2(t)$ belongs to $\vec{x}$, the nonlinear (by the level-crossing) expression can be obtained.

Figure 2 shows, furthermore, the *circuit* distinction caused by either linear, or nonlinear switching.

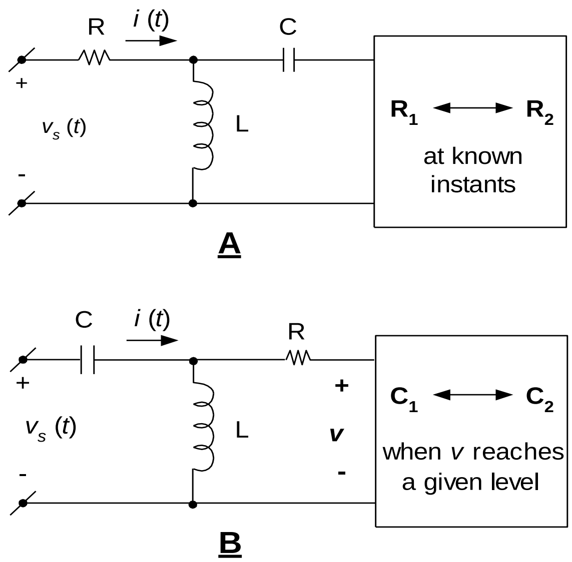

Fig. 2: Circuit **A** is linear, and **B** nonlinear. One can check this, e.g., by study the map $v_s(t)$ → $i(t)$; is it linear convolution, or not? In case **B** , the switched unit has a nonlinear "characteristic", and in case **A** it has a prescribed one. Case **B** is of the type [A(**x**)] (*v* is our *x*), and case **A** is of the type [A(*t*)]. That is, in case **A** we have **R**(*t*), and in case **B** , **C**(*t*,*v*) is obtained as **C**(*t*,**t***(*v*)) – the capacitive unit depends on the voltage that has to be found.

In [5], a transferring from one linear circuit to another linear one, leads, as [5] shows, to a chaotic state. However, as [6] explains -- there is, in fact, nothing surprising in [5], because though the (sub)circuits between which the switching is done are linear, — *the switching is nonlinear*, as it is performed when a state variable, a non-prescribed voltage, reaches a certain level. That is, it is some $t^*(x)$-switching, i.e. some $t^*(x)$-nonlinearity. Thus, the whole system with the two linear subsystems, in which the chaos is observed *is nonlinear*. Since the mistake in [5] is not a trivial one, the reader is encouraged to study [5] and [6] well. The importance of [5] is in particular associated with the interesting problem of overviewing all of the possibilities to obtain chaos in electronics.

6. Conclusions and final remarks

Motivated by the analysis of [3], we suggested to use the "structural" presentation of a nonlinearity, namely the [A(x)]-nonlinearity, for a qualitative analysis of complicated physical systems. For some switched systems, the [A(x)]-nonlinearity becomes $t^*(x)$-nonlinearity.

In the first example, the two seemingly very different explanations of a nonlinearity of a liquid flow -- the spatial-dynamic and the structural-view one, well agree with each other. The main conclusion is the *heuristic point* that -- when looking at liquid flow, and understanding that *the "system" is the same initially unknown vector field, i.e. is the same as the very process* that has to be described, -- one who does not know anything about hydrodynamic theory, sees that there is a "structural nonlinearity", i.e. the hydrodynamic equations *must* be nonlinear. The turbulence is a chaos that should be observed through these nonlinear equations.

It would be very interesting if one could try to inverse the way via (1), (3), and (9)-(12), finding some phenomenological convincing way to show that $(\vec{v}\nabla)$ must appear in $[A(\vec{v})]$, and thus to suggest a new derivation of the Navier-Stocks equation.

The concept of "structural nonlinearity" originates from the conventional linear system representation (4), which is on the border between the typical linear and nonlinear presentations. That we start directly from the form (7), and not from constructions of [A(x)] and [B(x)] for some concrete cases, reflects our principal position that **nonlinearity is an influence of the processes in the system on the system's structure, which is clearly seen in (7) or (8).**

The heuristic value of this position is also demonstrated by a medical example, and some electrical circuit examples. For the latter, see also [7,8].

Acknowledgement

I am grateful to Vladimir E. Zakharov (Univ. Arizona) for a discussion of the example with fluid.

References

[1] L.D. Landau and E.M. Lifshitz, "*Fluid Mechanics*", Butterworth Heinemann, Oxford, 1987. (See chapter II).

[2] Lotfi A. Zadeh and Ch. A Desoer, "*Linear system theory, the state space approach*", McGraw-Hill, New York, 1963.

[3] E. Gluskin, "Structure, Nonlinearity, and System Theory" (2015) International Journal of Circuit Theory and Applications vol. 43, Issue 4, pp. 534-543, DOI: 10.1002/cta.1960.

[4] E. Gluskin, "The use of nonlinear capacitors", Intern. J. Electronics, 1985, vol. 58, no/1, 63-81,

[5] Liu X, Teo K-L, Zhang H, Chen G. Switching control of linear systems for generating chaos. Chaos, Solitons & Fractals 2006;30:725–33.

[6] E. Gluskin, "On one interesting case of switching nonlinearity," Chaos, Solitons and Fractals 41 (2009) 1870–1872.

[7] E. Gluskin, "A small theorem on the nonlinearity of a switched system", Intern. J. Commun. (AEU). 62 (2008), 260-265.

[8] E. Gluskin, "The zerocrossing nonlinearity", Physics Letters A, 250 (1998) 93-98.