A system theoretic point of view on the nonlinearity of the liquid flow

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Abstract: We acquaint the reader with the concept of *structural nonlinearity*. As the given example of the liquid flow shows, this concept is heuristically useful.

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1. Introduction

Modern science more and more deals with complicated structures, and physicists should be introduced to some system theoretic basics. These basics help one, in the present example, to think about the "structure" of an observed liquid flow, and to explain its mathematical nonlinearity. However, we start from something equationally more standard.

2. The nonlinearity of the liquid flow

As is well known, the nonlinearity of the main hydrodynamic Navier--Stocks equation follows from the very dynamics. This equation is Newton's second law equation applied to a *moving element of the flow*. The acceleration is

$$\frac{d\vec{v}}{dt} \equiv \frac{\partial \vec{v}}{\partial t} + (\vec{v}\nabla)\vec{v} \quad (1)$$

where the differential operator

$$\vec{v}\nabla \equiv \frac{d\vec{r}}{dt} \cdot \frac{\partial}{\partial \vec{r}} \quad (2)$$

appears because $\vec{v} = \vec{v}(\vec{r}, t)$ and $\vec{r} = \vec{r}(t)$.

The "quadratic" by the velocity expression $(\vec{v}\nabla)\vec{v}$ introduces the nonlinearity in the Navier--Stocks equation.

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v}\nabla)\vec{v} = -\nabla p + \eta \Delta \vec{v} \quad (3)$$

written here for an incompressible flow. For a compressible flow (more relevant to aerodynamics) this equation is changed [1], but the nonlinearity remains by the same reasons.

3. The "structural nonlinearity"

For general nonlinear systems, it is conventional, after Poincare, Lyapunov, then Andronov at al, to write

$$\frac{d\vec{x}}{dt} = \vec{F}(\vec{x}, \vec{u}(t)), \quad \vec{x}(0) = \vec{x}_0 \ . \qquad (4)$$

where $\vec{x}(t)$, or $\{x_k(t)\}$, is the vector of the state variables, $\vec{x}_0$ is the given initial value of $\vec{x}(t)$, i.e. all of the components $\{x_k(0)\}$ are given, and $\vec{u}(t)$ is the vector of the inputs.

It is generally not clear in (4) how $\vec{x}(t)$ is changed if we replace $\vec{u}(t)$ by $k\vec{u}(t)$ with a constant *k* (a typical test of linearity), and we suggest another presentation of nonlinear systems, closer (in the meaning of a structure) to the presentation of the *linear* systems which is [2]

$$\frac{d\vec{x}}{dt} = [A]\vec{x} + [B]\vec{u}(t), \quad \vec{x}(0) = \vec{x}_0, \qquad (5)$$

where the matrices [A] and [B] represent the *structure* of the system.

Remark:

Dimensions of $\vec{x}(t)$ and $\vec{u}(t)$ in [5] need not be the same, and then the respective dimensions of the matrices also will be different. However, one should see that since the human operator *both* applies to the system $\vec{u}(t)$ and defines $\vec{x}_0$ (indeed, neither $\vec{u}(t)$, nor $\vec{x}_0$ is defined by the producer of the real system), there is a *generalized input* $\{\vec{x}_0, \vec{u}(t)\}$, and the checking of the linearity has to be not $\vec{u}(t) \to k\,\vec{u}(t)$, but

$$\{\vec{x}_0, \vec{u}(t)\} \to k\{\vec{x}_0, \vec{u}(t)\} = \{k\vec{x}_0, k\vec{u}(t)\} \ . \qquad (6)$$

If the processes in the system, initiated in any way by the (generalized) input, do not change the system, i.e. the parameters of the matrices in (5) are fixed, then the very physical system remains just the same as it was in rest, *obviously linear*.

Thus, we write, after [3], for a *nonlinear* system, the state equations not as (4), but as

$$\frac{d\vec{x}}{dt} = [A(\vec{x})]\vec{x} + [B(\vec{x})]\vec{u}(t), \quad \vec{x}(0) = \vec{x}_0 \, . \qquad (7)$$

That is, we suggest to basically define the nonlinearity as an influence of the processes in the system on the system's structure (parameters).

To our point, it is sufficient to be focused only on [A], that is, to deal only with the homogeneous equation obtained for $\vec{u}(t) = \vec{0}$:

$$\frac{d\vec{x}}{dt} = [A(\vec{x})]\vec{x}, \quad \vec{x}(0) = \vec{x}_0 \qquad (8)$$

in which the nonlinearity is quite obvious. The extended version (7) does not add a lot to the point.

Let us show how observing the "structural nonlinearity", or the "A(x)-nonlinearity" can help one to understand something in a difficult physics situation. This appears to be a matter of a very simple logic.

4. Why the hydrodynamic (aerodynamic) equations must be nonlinear from the "structural" system-theoretic point of view?

Consider liquid (air) flow, namely, its velocity field $\vec{v}(\vec{r},t)$ from the system theoretic point of view. That is, let us see $\vec{v}(\vec{r},t)$ as some vector $\vec{x}(t)$ of the state variables of the "system", which has to be found. However, the structure of the "system", i.e. the analogy to the above matrix [A] is just the same vector field $\vec{v}(\vec{r},t)$ that we can observe. Indeed, in no other form the structure is given to the observer. If we add the pressure *p*, also included in the given below Navier-Stocks equation, to the components of $\vec{v}$, as an unknown, i.e. include *p* into vector $\vec{x}$, the fact that the "structure" of the system is organically associated with the state variables remains. For the flow, the connection of the structure [A] and the unknown $\vec{x}(t)$ is obvious simply because they are the same. That is, the flow obviously has the structural [A(x)]-nonlinearity.

Of course, this simple observation and argument cannot constructively explain the processes that take place in the flow, however, it makes, in particular, the appearance of the turbulence not surprising, because it is well known that a chaotic state (here the turbulence) can be obtained only in a nonlinear (and never in any finite linear) system, which is a physical conclusion. This shows the heuristic validity of the concept of "structural nonlinearity".

However, let us also directly observe the matrix [A(x)] in the Navier-Stocks equation. To rewrite this equation closer to (7) or (8) we have to agree about the following. Since the system state equation theory does NOT deal with moving systems, we should use not Lagrange's, but Euler's hydrodynamics presentation [1], that is, to be focused not on a moving element of the flow, but on a certain point $\vec{r}$. Then, $\vec{x}$ is just $\vec{v}$ at this point, and, *for the localized unmoving system*, we should interpret the partial derivative by time, $\frac{\partial \vec{v}}{\partial t}$*, as the full derivative* $\frac{d\vec{x}}{dt}$. With these actions, we rewrite Navier-Stocks equation:

$$\frac{\partial \vec{v}}{\partial t} = [-(\vec{v}\nabla) + \frac{\eta\Delta}{\rho}]\vec{v} - \frac{\nabla p}{\rho} \qquad (9)$$

thus:

$$\frac{d\vec{x}}{dt} = [-(\vec{x}\nabla) + \frac{\eta\Delta}{\rho}]\vec{x} - \frac{\nabla p}{\rho} \ , \qquad (10)$$

observing matrix $[A(\vec{x})]$ as

$$[A(\vec{x})] = [-(\vec{x}\nabla) + \frac{\eta\Delta}{\rho}] \ , \qquad (11)$$

or, returning to $\vec{v}$,

$$[A(\vec{v})] = [-(\vec{v}\nabla) + \frac{\eta\Delta}{\rho}] \qquad (12)$$

The structural nonlinearity of the flow is obvious.

5. Conclusions, final remarks, and some pedagogical philosophy

Motivated by the analysis of [2], we suggested to use the "structural" presentation of a nonlinearity, namely the [A(x)]-nonlinearity, for a qualitative analysis of complicated physical systems.

The two seemingly very different explanations of a nonlinearity of a liquid flow -- the spatial-dynamic and the structural-view ones well agree with each other. However, the main conclusion is not the well-known fact of nonlinearity of Navier-Stocks equation. The *heuristic point* is that -- when looking at liquid flow, and understanding that *the "system" is the same initially unknown vector field, i.e. is the same as the very process,* -- one who does not know anything about hydrodynamic theory, sees that there is a "structural nonlinearity", i.e. the hydrodynamic equations must be nonlinear. One thus sees that the turbulence is a chaos that should be (and often is) observed through these nonlinear equations.

The concept of "structural nonlinearity" originates from the conventional linear system representation (5), which is on the border between the typical linear and nonlinear presentations. That we start directly from the form (7), and not from constructions of [A(x)] and [B(x)] for some concrete cases, reflects our principal position that **nonlinearity as an influence of the processes in the system on the system's structure, which is clear in (7) or (8).**

One notices that we do *not* accept the *purely philological, i.e. in principle nonconstructive, though popular*, definition of "nonlinear" as "not a linear one". With such a "not"-definition, one could not *surely* see the liquid flow as a nonlinear system. As well, by saying that "nonlinear" is "not a linear one", one assumes that in order to speak about nonlinearity, one has to already know what linearity is. However, in variational calculus, straight lines appear as particular cases of the curves -- for instance, as realizing the minimal distance between two points. That is, nonlinearity *can be* seen as more initial concept than linearity.

It would be very interesting if one could, after revealing the nonlinearity of the flow as suggested here, to find a phenomenological convincing way to show that $(\vec{v}\nabla)$ must appear in $[A(\vec{v})]$, and thus to suggest a new derivation of the Navier-Stocks equation. That is, to try to inverse the way via (1), (3), and (9)-(12). On the heuristic-pedagogical regards, finding a new way to a known equation can be useful. For instance, the derivation by Einstein of the known Plank's formula of the black body radiation [4], led in [5] to foundation of quantum electronics.

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