Classify Positive Integers to Prove Collatz Conjecture by the Mathematical Induction

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Abstract

First, it is necessary to expound certain of basic concepts relating to prove this conjecture. After that, applies the mathematical induction, and classify positive integers to complete each classificatory proof orderly.

In addition, prepares several judging criteria before the proof starts, in order to use them in each classified proof.

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1. Introduction

The Collatz conjecture also called the *3x+1* mapping, *3n+1* problem, Hasse's algorithm, Kakutani's problem, Syracuse algorithm, Syracuse problem, Thwaites conjecture and Ulam's problem, etc.

Yet, it is still both unproved and un-negated a conjecture ever since named after Lothar Collatz in 1937; [1].

2. Certain of Basic Concepts

The Collatz conjecture states that take any positive integer n, if n is even, divide it by *2*; if n is odd, multiply it by *3* and add *1*. Repeat the process indefinitely, then, no matter what positive integer you start with, you will always eventually reach a result of *1*; [2].

Let us regard aforesaid operational stipulations as the operational rule.

Begin with any positive integer⁄integral expression to operate by the operational rule continuously, then, form successive integers⁄integral expressions. We regard such consecutive integers⁄integral expressions and synclastic arrowheads among them as an operational route.

Furthermore, let us use a capital letter plus the subscript "*ie*" to express a some positive integral expression such as P_{ie}, C_{ie} etc.

Where P_{ie} exists at an operational route, may term the operational route "an operational route via P_{ie}".

Generally speaking, integral expressions at an operational route contain a common variable or some variables which can be changed into a variable.

3. Mathematical Induction that Proves the Conjecture

Prove the Collatz conjecture by the mathematical induction as follows; [3].

(1) From 1→4→2→1; 2→1; 3→10→5→16→8→4→2→1; 4→2→1; 5→16→8→4→2→1; 6→3→10→5→16→8→4→2→1; 7→22→11→34→17→52→26→13→40→20→10→5→16→8→4→2→1; 8→4→2→1; 9→28→14→7→22→11→34→17→52→26→13→40→20→10→5→16→8→4→2→1; 10→5→16→8→4→2→1; 11→34→17→52→26→13→40→20→10→5→16→8→4→2→1; 12→6→3→10→5→16→8→4→2→1; 13→40→20→10→5→16→8→4→2→1; 14→7→22→11→34→17→52→26→13→40→20→10→5→16→8→4→2→1; 15→46→23→70→35→106→53→160→80→40→20→10→5→16→8→4→2→1; 16→8→4→2→1; 17→52→26→13→40→20→10→5→16→8→4→2→1; 18→9→28→14→7→22→11→34→17→52→26→13→40→20→10→5→16→8→4→2→1 and 19→58→29→88→44→22→11→34→17→52→26→13→40→20→10→5→16→8→4→2→1, it is observed that every positive integer≤ 19 suits the conjecture.

(2) Suppose that n suits the conjecture, where n is an integer≥ 19.

(3) Prove which $n+1$ suits the conjecture likewise.

4. Several Judging Criteria

Before the proof starts, must prepare judging criteria concerned, ut infra:

Theorem 1. If an integral expression at an operational route via P_{ie} is smaller than P_{ie}, and P_{ie} contains $n+1$, then P_{ie} and $n+1$ suit the conjecture.

For example, if there is $P_{ie}=31+3^2\eta$, and P_{ie} contains $n+1$, where $\eta\ge 0$, then from $27+2^3\eta\to 82+3\times 2^3\eta\to 41+3\times 2^2\eta\to 124+3^2\times 2^2\eta\to 62+3^2\times 2\eta\to 31+3^2\eta>27+2^3\eta$, infer that P_{ie} and $n+1$ suit the conjecture.

Also, if there is $P_{ie}=5+2^2\mu$, and P_{ie} contains $n+1$, where $\mu\geq 0$, then from $5+2^2\mu\to 16+3\times 2^2\mu\to 8+3\times 2\mu\to 4+3\mu<5+2^2\mu$, infer that P_{ie} and $n+1$ suit the conjecture.

Proof. Suppose that there is C_{ie} at an operational route via P_{ie} where $C_{ie}<P_{ie}$, then when their common variable equals some fixed value such that $P_{ie}=n+1$, let $C_{ie}=m$, so it has $m<n+1$. Thus, operations of $n+1$ can pass operations of m at the operational route via $n+1$ to reach 1 since every positive integer$<n+1$ has been supposed to suit the conjecture.

When their common variable is equal to each value, a value of P_{ie} can too be operated to 1 via a matching value of C_{ie}, so P_{ie} suits the conjecture.

Theorem 2. If an operational route via Q_{ie} and an operational route via P_{ie} intersect directly, and P_{ie} contains $n+1$, and that an integral expression at the operational route via Q_{ie} is smaller than P_{ie}, then P_{ie} and $n+1$ suit the conjecture, where P_{ie} and Q_{ie} exist not at an operational route.

For example, $P_{ie}=63+3\times 2^8\varphi$, and P_{ie} contains $n+1$, where $\varphi\geq 0$, then from $63+3\times 2^8\varphi\to 190+3^2\times 2^8\varphi\to 95+3^2\times 2^7\varphi\to 286+3^3\times 2^7\varphi\to 143+3^3\times 2^6\varphi\to 430+3^4\times 2^6\varphi\to 215+3^4\times 2^5\varphi\to 646+3^5\times 2^5\varphi\to 323+3^5\times 2^4\varphi\to 970+3^6\times 2^4\varphi\to 485+3^6\times 2^3\varphi\to 1456+3^7\times 2^3\varphi\to 728+3^7\times 2^2\varphi\to 364+3^7\times 2\varphi\to 182+3^7\varphi\to \ldots$

$\uparrow 121+3^6\times 2\varphi\leftarrow 242+3^6\times 2^2\varphi\leftarrow 484+3^6\times 2^3\varphi\leftarrow 161+3^5\times 2^3\varphi\leftarrow 322+3^5\times 2^4\varphi\leftarrow 107+3^4\times 2^4\varphi\leftarrow 214+3^4\times 2^5\varphi\leftarrow 71+3^3\times 2^5\varphi\leftarrow 142+3^3\times 2^6\varphi\leftarrow 47+3^2\times 2^6\varphi<63+3\times 2^8\varphi$,

infer that P_{ie} and $n+1$ suit the conjecture.

Proof. Suppose that D_{ie} at an operational route via Q_{ie} is smaller than P_{ie}, and the operational route via Q_{ie} and an operational route via P_{ie} intersect at A_{ie}, then when their common variable is endowed with some fixed value such that $P_{ie}= n+1$, let $D_{ie}=\mu$ and $A_{ie}=\xi$, so it has $\mu<n+1$.

Since ξ and μ exist at an operational route, so operations of ξ can pass operations of μ to reach 1. Since $n+1$ and ξ exist at an operational route, then operations of $n+1$ can pass continuous operations of ξ to reach 1.

When their common variable equals each value, a value of P_{ie} can too be operated to 1 via matching values of A_{ie} and D_{ie}, so P_{ie} suits the conjecture.

Lemma. If an operational route via Q_{ie} and an operational route via P_{ie} are at indirect connection, and P_{ie} contains $n+1$, and that an integral expression at the operational route via Q_{ie} is smaller than P_{ie}, then P_{ie} and $n+1$ suit the conjecture.

The indirect connection is directed to two operational routes without the intersection in which case many operational routes intersect orderly.

In addition to judging criteria, substitute d, e, f, g, h etc. for c to appear within integral expressions at operational routes of $15+12c$ and $19+12c$, actually, that is in order to avoid confusion and for convenience.

5. Initial Classified Proofs

By now, set about proving the Collatz conjecture gradually, as follows:

Proof. On balance, classify positive integers gradually, and synchronously find out a relation between each class which possibly contains $n+1$ and another class which is smaller than the former to prove that $n+1$ suits the conjecture, according to proven a theorem or the lemma.

First, divide integers≥ 1 into positive even numbers and odd numbers.

For even number $2k$ with $k\geq 1$, from $2k\rightarrow k<2k$, infer that if $n+1\in 2k$, then $2k$ and $n+1$ suit the conjecture according to Theorem *1*.

For positive odd numbers out of first step of the mathematical induction, divide them into *2* genera, i.e. $5+4k$ and $7+4k$, where $k\geq 4$.

For $5+4k$, from $5+4k\rightarrow 16+12k\rightarrow 8+6k\rightarrow 4+3k<5+4k$, infer that if $n+1\in 5+4k$, then $5+4k$ and $n+1$ suit the conjecture according to Theorem *1*.

Further divide $7+4k$ into *3* sorts: $15+12c$, $19+12c$ and $23+12c$, where $c\geq 0$.

For $23+12c$, from $15+8c\rightarrow 46+24c\rightarrow 23+12c>15+8c$, infer that if $n+1\in 23+12c$, then $23+12c$ and $n+1$ suits the conjecture according to Theorem *1*.

For $15+12c$ and $19+12c$ when $c=0$, they have proved to suit the conjecture in third section. So only need us to prove $15+12c$ and $19+12c$ where $c\geq 1$.

6. Proving which 15+12c and 19+12c Suit the Conjecture

For $15+12c/19+12c$ where $c\geq 1$, continue to operate them, so as to find a relation between each operational result and a judging criterion.

First, operate $15+12c$ by the operational rule successively as listed below.

15+12c→46+36c→23+18c→70+54c→35+27c ♣

d=2e+1: 29+27e (**1**)   e=2f: 142+486f→71+243f ♥
♣35+27c↓→c=2d+1: 31+27d↑→d=2e: 94+162e→47+81e↑→e=2f+1:64+81f (**2**)
c=2d: 106+162d→53+81d↓→d=2e+1:67+81e↓→e=2f+1:74+81f (**3**)
d=2e:160+486e♦   e=2f: 202+486f→101+243f ♠

g=2h+1: 200+243h (**4**)   **…**
♥ 71+243f↓→f=2g+1:157+243g↑→g=2h: 472+1458h→236+729h↑→ **…**
f=2g: 214+1458g→107+729g↓→g=2h+1: 418+729h↓→**…**
g=2h: 322+4374h→**… …**

g=2h: 86+243h (**5**)

♠101+243f↓→f=2g+1:172+243g↑→g=2h+1:1246+1458h→**…**

f=2g: 304+1458g→152+729g ↓→**…**

…

…

♦160+486e→80+243e↓→e=2f+1: 970+1458f→485+729f↑→ **…** **…**

e=2f:40+243f↓→f=2g+1:850+1458g→ 425+729g↑→ **…**

f=2g: 20+243g↓→g=2h: 10+243h (**6**) **…**

g=2h+1:790+1458h→395+729h↑→ **…**

Annotation:

(1) Each of letters c, d, e, f, g, h …etc at listed above operational routes expresses each of natural numbers plus 0.

(2) Also, there are ♣↔♣, ♥↔♥, ♠↔♠, and ♦↔♦.

(3) Aforesaid two points are suitable to latter operational routes of 19+12c similarly.

First the author defines a terminology. Namely in the course of operation of *15+12c/19+12c* by the operational rule, if an operational result is smaller than a kind of *15+12c/19+12c*, and that it first appears at an operational route of *15+12c/19+12c*, then term the operational result "№1 satisfactory operational result".

Hereby infer 3 kinds of *15+12c* derived from №1 satisfactory operational results at the bunch of operational routes of *15+12c* to suit the conjecture.

1. From c=2d+1 and d=2e+1, get c=2d+1=2(2e+1)+1=4e+3, so it has 15+12c= $51+48e=51+3\times2^4e\to154+3^2\times2^4e\to77+3^2\times2^3e\to232+3^3\times2^3e\to116+3^3\times2^2e\to58+3^3\times2e$ $\to29+27e$ where mark (**1**), and 29+27e<51+48e, thus, if it is $n+1\in 51+48e$, then 51+48e and *n+1* suit the conjecture according to Theorem *1*.

2. From c=2d+1, d=2e and e=2f+1, get c=2d+1=4e+1=4(2f+1)+1=8f+5, so it has $15+12c=75+96f=75+3\times2^5f\to226+3^2\times2^5f\to113+3^2\times2^4f\to340+3^3\times2^4f\to170+3^3\times2^3f\to$ $85+3^3\times2^2f\to256+3^4\times2^2f\to128+3^4\times2^1f\to64+81f$ where mark (**2**), and 64+81f<75+96f, thus, if there is $n+1\in 75+96f$, then *75+96f* and *n+1* suit the conjecture according to Theorem *1*.

3. From c=2d, d=2e+1 and e=2f+1, get c=2d=4e+2=4(2f+1)+2=8f+6, so it has $15+12c=87+96f=87+3\times2^5f\to262+3^2\times2^5f\to131+3^2\times2^4f\to394+3^3\times2^4f\to197+3^3\times2^3f$ $\to592+3^4\times2^3f\to296+3^4\times2^2f\to148+3^4\times2^1f\to74+81f$ where mark (**3**), and $74+81f<87+96f$, thus, if it is $n+1\in 87+96f$, then $87+96f$ and $n+1$ suit the conjecture according to Theorem *1*.

Like that, each reader oneself can likewise figure out following these:

For $15+12c=315+384h$, operate it to $200+243h$ where mark (**4**) at the bunch of operational routes of $15+12c$;

For $15+12c=135+384h$, operate it to $86+243h$ where mark (**5**) at the bunch of operational routes of $15+12c$;

For $15+12c=15+384h$, operate it to $10+243h$ where mark (**6**) at the bunch of operational routes of $15+12c$.

Thus it can be seen, if $n+1\in$ a kind of $15+12c$ derived from a №1 satisfactory operational result, then the kind and $n+1$ suit the conjecture.

Secondly, operate $19+12c$ by the operational rule successively, as follows.

19+12c→58+36c→ 29+18c→ 88+54c→ 44+27c ♣

d=2e: 11+27e (**α**) e=2f:37+81f (**β**)
♣ 44+27c↓→c=2d: 22+27d↑→d=2e+1:148+162e→74+81e↑→e=2f+1:466+486f ♥
c=2d+1: 214+162d→107+81d↓→d=2e:322+486e ♠
d=2e+1:94+81e↓→e=2f:47+81f (γ)
e=2f+1:526+486f ♦

g=2h: 119+243h (**δ**) …
f=2g+1:238+243g↑→g=2h+1:1444+1458h→722+729h↑→…
♥466+486f→233+243f↑→f=2g: 700+1458g→350+729g↓→g=2h+1:3238+4374h↓
g=2h: 175+729h↓→… …
…

$g=2h+1$:$172+243h$ (**ε**)
$f=2g$: $101+243g\uparrow\rightarrow g=2h$: $304+1458h\rightarrow$...
$e=2f+1$:$202+243f\uparrow\rightarrow f=2g+1$:$1336+1458g\rightarrow$...
♠$322+486e\rightarrow 161+243e\uparrow\rightarrow e=2f$:$484+1458f\rightarrow$...

♦$526+486f\rightarrow 263+243f\downarrow\rightarrow f=2g$: $790+1458g\rightarrow$...
$f=2g+1$: $253+243g\downarrow\rightarrow g=2h+1$: $248+243h$ (ζ)
$g=2h$: $760+1458h\rightarrow$...

As listed above, the author likewise first infers 3 kinds of $19+12c$ derived from №1 satisfactory operational results at the bunch of operational routes of $19+12c$ to suit the conjecture.

1. From $c=2d$ and $d=2e$, get $c=2d=4e$, so it has $19+12c=19+48e=19+3\times 2^4e\rightarrow 58+3^2\times 2^4e\rightarrow 29+3^2\times 2^3e\rightarrow 88+3^3\times 2^3e\rightarrow 44+3^3\times 2^2e\rightarrow 22+3^3\times 2e\rightarrow 11+27e$ where mark (**α**), and $11+27e<19+48e$, thus, if it is $n+1\in 19+48e$, then $19+48e$ and $n+1$ suit the conjecture according to Theorem *1*.

2. From $c=2d$, $d=2e+1$ and $e=2f$, get $c=2d=2(2e+1)=4e+2=8f+2$, so it has $19+12c=43+96f=43+3\times 2^5f\rightarrow 130+3^2\times 2^5f\rightarrow 65+3^2\times 2^4f\rightarrow 196+3^3\times 2^4f\rightarrow 98+3^3\times 2^3f\rightarrow 49+3^3\times 2^2f\rightarrow 148+3^4\times 2^2f\rightarrow 74+3^4\times 2^1f\rightarrow 37+81f$ where mark (**β**), and $37+81f<43+96f$, thus, if it is $n+1\in 43+96f$, then $43+96f$ and $n+1$ suit the conjecture according to Theorem *1*.

3. From $c=2d+1$, $d=2e+1$ and $e=2f$, get $c=2d+1=4e+3=8f+3$, so it has $19+12c=55+96f=55+3\times 2^5f\rightarrow 166+3^2\times 2^5f\rightarrow 83+3^2\times 2^4f\rightarrow 250+3^3\times 2^4f\rightarrow 125+3^3\times 2^3f\rightarrow 376+3^4\times 2^3f\rightarrow 188+3^4\times 2^2f\rightarrow 94+3^4\times 2^1f\rightarrow 47+81f$ where mark (γ), and $47+81f<55+96f$, thus, if it is $n+1\in 55+96f$, then $55+96f$ and $n+1$ suit the conjecture according to Theorem *1*.

Like that, each reader oneself can likewise figure out following these:

For $19+12c=187+384h$, operate it to $119+243h$ where mark (δ) at the bunch of operational routes of $19+12c$;

For $19+12c=271+384h$, operate it to $172+243h$ where mark (ε) at the bunch of operational routes of $19+12c$;

For $19+12c=391+384h$, operate it to $248+243h$ where mark (ζ) at the bunch of operational routes of $19+12c$.

Thus it can be seen, if $n+1\in$ a kind of $19+12c$ derived from a №1 satisfactory operational result, then the kind and $n+1$ suit the conjecture.

Overall, if $n+1\in$any kind of $15+12c/19+12c$ derived from a №1 satisfactory operational result, then the kind of $15+12c/19+12c$ and $n+1$suit the conjecture.

Undoubtedly, there is always an operational route from each kind of $15+12c/19+12c$ operating to a №1 satisfactory operational result, whether it is a proved kind of $15+12c/19+12c$, or an unproved kind of $15+12c/19+12c$.

Each kind of $15+12c/19+12c$ corresponding to a №1 satisfactory operational result, yet a №1 satisfactory operational result can derive many kinds of $15+12c/19+12c$ at operational routes of $15+12c/19+12c$.

Begin with $15+12c/19+12c$ to operate by the operational rule on and on, along continuation of operations, it will continuously educe more and more №1 satisfactory operational results, hereby derive more and more kinds of $15+12c/19+12c$ be proved, even there are infinitely many satisfactory operational results and infinitely many kinds of $15+12c/19+12c$ because there are $c\geq 1$.

In any case, all №1 satisfactory operational results exist at the bunch of operational routes of $15+12c/19+12c$ inevitably.

In addition, 3 operational routes of proved 3 kinds of $15+12c/19+12c$ exist also at the bunch of operational routes of $15+12c/ 19+12c$.

Thus, for each operational route from an unproved kind of $15+12c/19+12c$ to a №1 satisfactory operational result and an operational route from any proved kind of $15+12c/19+12c$ to a №1 satisfactory operational result, they either intersect directly or connect indirectly, therefore each unproved kind of $15+12c/ 19+12c$ is proved to suit the conjecture according to Theorem *2* or the Lemma.

In consequence, if $n+1\in$ any kind of $15+12c/19+12c$, then the kind of $15+12c/19+12c$ and $n+1$ suit the conjecture.

7. Make a Summary and Reach the Conclusion

To sum up, $n+1$ has been proved to suit the conjecture, whether $n+1$ belongs within which genus, which sort or which kind of odd numbers, or it is exactly an even number.

We can also prove positive integers $n+2$, $n+3$ etc. up to every positive integer to suit the conjecture in the light of the old way of doing the thing.

The proof was thus brought to a close. As a consequence, the Collatz conjecture is tenable.

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