The Geometrization of Quantum Mechanics, the Nonlinear Klein-Gordon Equation, Finsler Gravity and Phase Spaces

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Abstract

The Geometrization of Quantum Mechanics proposed in this work is based on the postulate that the quantum probability density can *curve* the classical spacetime. It is shown that the gravitational field produced by *smearing* a point-mass M_o at $r = 0$ throughout all of space (in an spherically symmetric fashion) can be interpreted as the gravitational field generated by a self-gravitating anisotropic fluid droplet of mass density $4\pi M_o r^2 \varphi^*(r)\varphi(r)$ and which is sourced by the *probability cloud* (associated with a spinless point-particle of mass M_o) *permeating* a 3-spatial domain region $\mathcal{D}_3 = \int 4\pi r^2 dr$ at any time t. Classically one may smear the point mass in any way we wish leading to arbitrary density configurations $\rho(r)$. However, Quantum Mechanically this is *not* the case because the radial mass configuration $M(r)$ must obey a key third order nonlinear differential equation (nonlinear extension of the Klein-Gordon equation) displayed in this work and which is the static spherically symmetric relativistic analog of the Newton-Schrödinger equation. We conclude by extending our proposal to the Lagrange-Finsler and Hamilton-Cartan geometry of (co) tangent spaces and involving the relativistic version of Bohm's Quantum Potential. By further postulating that the quasi-probability Wigner distribution $W(x,p)$ *curves* phase spaces, and by encompassing the Finsler-like geometry of the cotangent-bundle with phase space quantum mechanics, one can naturally incorporate the *noncommutative* and non-local Moyal star product (there are also non-associative star products as well). Phase space is the arena where to implement the space-time-matter unification program. It is our belief this is the right platform where the quantization *of* spacetime and the quantization *in* spacetime will coalesce.

1 Introduction : On Geometry, Quantum Mechanics and Bohm's Potential

The Newton-Schrödinger equation has had a long history since the 1950's [1], [3]. It is the name given to the system coupling the Schrödinger equation to the Poisson equation. In the case of a single particle, this coupling is effected as follows: for the potential energy term in the Schrödinger equation take the gravitational potential energy determined by the Poisson equation from a matter density proportional to the probability density obtained from the wave-function. For a single particle of mass m the coupled system of equations leads to the nonlinear and nonlocal Newton-Schrödinger integro-differential equation

$$i\hbar \frac{\partial \Psi(\vec{r},t)}{\partial t} \ = \ - \ \frac{\hbar^2}{2m} \nabla^2 \Psi(\vec{r},t) \ + \ V(\vec{r},t) \ \Psi(\vec{r},t) \ - \ \left(Gm^2 \ \int \ \frac{|\Psi(\vec{r}',t)|^2}{|\vec{r} - \vec{r}'|} \ d^3r' \right) \ \Psi(\vec{r},t) \tag{1.1}$$

In [4] we found exact solutions to the stationary spherically symmetric Newton-Schrödinger equation in terms of integrals involving *generalized* Gaussians. The energy eigenvalues were also obtained in terms of these integrals which agree with the numerical results in the literature.

The authors [2] have shown that the Schrödinger-Newton equation for spherically symmetric gravitational fields can be derived in a WKB-like expansion from the Einstein-Klein-Gordon, and Einstein-Dirac-Cartan system. As emphasized by these authors, the central question is whether this is the right way to represent the gravitational field of quantum systems. Reading the classical fields as one-particle probability amplitudes Ψ, it amounts to assuming the validity of the semi-classical Einstein equations

$$R_{\mu\nu} \ - \ \frac{1}{2} \ g_{\mu\nu} R \ = 8\pi G < \Psi | \hat{T}_{\mu\nu} | \Psi > \tag{1.2}$$

where the left hand side is treated classically, but in the right hand side one is taking the expectation value of the stress energy operator in the state $|\Psi >$. However it has been argued by several authors that this approach is incorrect because the collapse of the wave function in the measurement process leads to a discontinuity such that the local energy conservation law $\nabla^\mu < \Psi | \hat{T}_{\mu\nu} | \Psi > \ = \ 0$ is violated. For this reason we shall follow a different approach than the one indicated by eq-(1.2) and that is based on the geometrization process of Quantum Mechanics (which is *not* the same as geometric quantization).

An early geometric approach was based on the connection between Bohm's quantum potential and the scalar curvature in Weyl geometry. It was noticed long ago by [5] that the relativistic version of Bohm's potential Q is proportional to the Weyl scalar curvature R_W in $flat$ spacetime backgrounds when the Weyl's gauge field of dilatations is $A_\mu \sim \partial_\mu ln(\phi^* \phi)$, with $\phi(x)$ being a complex scalar field. In other words, when the scalar field is Weyl-covariantly constant $D_\mu^{Weyl} \phi \ = \ 0$. Because A_μ is pure gauge (total derivative) the Weyl's field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 0$, which implies that the rate of the ticking of clocks (in flat spacetime) will be independent of their paths taken from point A to B. Consequently, atomic clocks arriving on the earth via different trajectories will tick at the same rate (same spectral lines). In this fashion one can avoid Einstein's criticism of Weyl's geometry.

Based on the findings by [5], that the Weyl scalar curvature $R_W \sim Q$, when $A_\mu \sim \partial_\mu ln(\phi^* \phi)$, one can interpret $m(1+\gamma(\frac{\hbar}{m})^2 R_W) = m + Q(x^\mu)$ (with γ a numerical coefficient) as an $effective$ mass function $m_{eff}(x^\mu)$ so that the relativistic Hamilton Jacobi equation, with signature $(+,-,-,-)$ and $c=1$, becomes

$$(\partial_\mu S)^2 \ = \ (m+Q)^2 \ = \ m^2 \ + \ 2mQ \ + \ Q^2 \tag{1.3}$$

The relativistic version of the Bohm potential for a scalar field is [5], [6]

$$Q \ = \ \frac{\hbar^2}{2m} \ \frac{\Box(\sqrt{\phi^*(\vec{r},t)\phi(\vec{r},t)})}{\sqrt{\phi^*(\vec{r},t)\phi(\vec{r},t)}} \tag{1.4}$$

To *leading* order in Q the above Hamilton-Jacobi equation (1.3) gives

$$(\partial_\mu S)^2 \ = \ m^2 \ + \ \hbar^2 \ \frac{\Box(\sqrt{\phi^*(\vec{r},t)\phi(\vec{r},t)})}{\sqrt{\phi^*(\vec{r},t)\phi(\vec{r},t)}} \tag{1.5}$$

The four-current is

$$J_\mu \ = \ i \ (\ \phi^*(\vec{r},t) \ \partial_\mu \phi(\vec{r},t) \ - \ \phi(\vec{r},t) \ \partial_\mu \phi^*(\vec{r},t) \ ) \tag{1.6}$$

and obeys the conservation law (continuity equation)

$$\partial_\mu J^\mu = 0 \tag{1.7}$$

related to the conservation of a Noether charge $\int J_\mu d\Sigma^\mu$ that is given by the flux of the current J_μ through a spatial 3-surface Σ^μ. The charge counts the number of scalar particles minus the number of anti-particles flowing through the 3-spatial surface. In QFT (relativistic QM) the scalar field ϕ is *no* longer a wave function, hence it is not related to a one-particle probability amplitude as such but to many-particles (second-quantization).

Writing the complex scalar field in the polar form

$$\phi \ \equiv \ ||\phi(\vec{r},t)|| \ e^{iS(\vec{r},t)/\hbar} \ = \ \sqrt{\phi^*(\vec{r},t)\phi(\vec{r},t)} \ e^{iS(\vec{r},t)/\hbar} \tag{1.8}$$

allows to solve for $S = -\frac{i\hbar}{2} ln(\frac{\phi}{\phi^*})$. After a lengthy but straightforward algebra the eqs-(1.5, 1.6, 1.7) lead to the Klein-Gordon equation [1]

$$(\hbar^2 \ \Box \ + \ m^2) \ \phi(\vec{r},t) \ = \ 0, \ \ (\hbar^2 \ \Box \ + \ m^2) \ \phi^*(\vec{r},t) \ = \ 0 \tag{1.9}$$

If one includes the Q^2 terms above in eq-(1.3) one will end up with a more complicated equation than the Klein-Gordon equation involving *quartic* derivatives. A stringy deformation of the Weyl Heisenberg algebra $[x,p] = i(\hbar + \alpha' p^2)$, with inverse string tension α',

[1] Using a different signature $(-,+,+,+)$ requires changing the signs in the right hand side of (1.5) and it leads to the Klein-Goldon equation with a sign change in the m^2 term

and leading to a minimal-length uncertainty relation, can be represented via the operators $X = x$ and $P = p(1 + \frac{\alpha'}{3}p^2)$. Replacing these redefined operators X, P into a Hamiltonian $H = \frac{P^2}{2m} + V(X)$, and using the correspondence $p \rightarrow -i\hbar\frac{d}{dx}$, yields a *modified* Schrödinger equation with *higher* derivatives encoding the minimal length and with bounded states in the continuum. See [31] and references therein for details.

A conformally *covariant* equation[2] equation in curved backgrounds in $4D$ with a curvature scalar coupling, can also be obtained via Bohm's quantum potential [5], [6]

$$(\hbar^2 \ g^{\mu\nu} D_\mu D_\nu \ + \ m^2 \ + \ \frac{R_W}{6}) \ \phi(\vec{r}, t) \ = \ (\hbar^2 \ g^{\mu\nu} \nabla_\mu \nabla_\nu \ + \ m^2 \ + \ \frac{R}{6}) \ \phi(\vec{r}, t) \ = \ 0 \quad (1.10)$$

where $D_\mu = \nabla_\mu + A_\mu$ is the Weyl covariant derivative and R_W is the Weyl scalar curvature. The "conformal" mass m parameter is posited to scale under Weyl scalings with a Weyl weight of -1. The weight of $g^{\mu\nu}$ and R_W is -2, while the weight of ϕ is -1. Due to key factor of $\kappa = \frac{1}{6}$ (that varies with the spacetime dimension as $\kappa_d = \frac{d-2}{4(d-1)}$) in the Weyl scalar curvature R_W, the field A_μ *decouples* entirely from the left hand side of the equation leading to the right hand side expressed solely in terms of the Riemannian scalar curvature R and covariant derivatives ∇_μ based on the Christoffel connection.

A Weyl-gauge invariant proof of the spin-statistics theorem, and solving the Quantum nonlocality enigma by Weyl's Conformal Geometry can be found in more recent work by [**?**]. The coupling to the Electromagnetic field via the prescription $p_\mu \rightarrow p_\mu - eA_\mu$ leads to a modified Klein-Gordon equation by simply replacing $\Box$ with $(\partial_\mu - ieA_\mu)(\partial^\mu - ieA^\mu)$.

The deep question of whether or not Bohmian mechanics can be be made relativistic was studied in [7]. In relativistic space-time, Bohmian theories can be formulated by introducing a privileged foliation of space-time. The introduction of such a foliation - as extra absolute space-time structure - would seem to imply a clear violation of Lorentz invariance, and thus a conflict with fundamental relativity. The authors [7] considered the possibility that, instead of positing it as extra structure, the required foliation could be covariantly determined by the wave function. This allowed for the formulation of Bohmian theories that seem to qualify as fundamentally Lorentz invariant. They concluded with some discussion of whether or not they might also qualify as fundamentally relativistic.

In [9] we found that in certain physical scenarios Bohm's quantum potential coincided with the gravitational potential energy and that a notion of Classical/Quantum Duality existed in the Quantum Hamilton Jacobi equation casting further light into the deep interplay between gravity and quantum mechanics. Related to the connection between Bohm's quantum potential and the gravitational potential energy is the nonlinear and novel Bohm-Poisson-Schrödinger equation proposed by us in [**?**]

$$\nabla^2 Q \ = \ 4\pi G m \rho \ \Rightarrow \ - \ \frac{\hbar^2}{2m} \ \nabla^2 \ (\frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}}) \ = \ 4\pi G m \rho \quad (1.11)$$

The physical motivation behind (1.11) is that the laws of Physics should themselves determine the distribution density ρ of matter. It has solutions leading to *repulsive* gravitational behavior because eq-(1.11) is invariant under the transformations $G \rightarrow -G; \rho \rightarrow$

[2]The homogeneous differential equation is also conformally invariant

$-\rho$. The Bohm-Poisson equation was extended to the relativistic regime in [10]. Two specific solutions to the Relativistic Bohm-Poisson equation (associated to a real scalar field) were provided encoding the repulsive nature of dark energy. One solution leads to an exact cancellation of the cosmological constant, but an expanding decelerating cosmos; while the other solution leads to an exponential accelerated cosmos consistent with a de Sitter phase, and whose extremely small cosmological constant is $\Lambda = \frac{3}{R_H^2}$, consistent with current observations.

A different quantum potential than Bohm's was proposed by [11] based on the Quantum Equivalence postulate of Quantum Mechanics under D-dimensional Mobius transformations. In one-dimension, their quantum potential Q was given in terms of the Schwarzian derivative of the action with respect to x by $Q = \frac{\hbar^2}{4m}\{S, x\}$. The Schwarzian derivative is defined by $\{S, x\} = (S'''/S') - \frac{3}{2}(S''/S')^2$. The Schwarzian derivative is Mobius invariant $\{\gamma(S), x\} = \{S, x\}$, where the Mobius transformation is defined as $\gamma(S) = \frac{aS+b}{cS+d}, ad - bc = 1$. In one-dimension the continuity equation in the stationary case is $\frac{d}{dx}[(\rho(x)/m)(dS/dx)] = 0 \Rightarrow \rho(dS/dx) = constant$. Inserting $\sqrt{\rho} \sim (dS/dx)^{-\frac{1}{2}}$ into $Q = \frac{\hbar^2}{4m}\{S, x\}$ yields the expression for Bohm's quantum potential after some straightforward algebra [11].

Schwarzian Quantum Mechanics has recently been a very active topic of research in connection to the Sachdev-Ye-Kitaev (SYK) model [12]. Another very relevant topics of current research related to the emergence of gravity are holographic quantum complexity, entanglement entropy, information geometry, quantum computation and information theory, black holes, Cayley graphs, $\cdots$, see [13] and the references therein. The close relation between gravity and quantum mechanics has been analyzed by Susskind [14]. Our main goal, if possible, is to geometrize quantum mechanics. The emergence of quantum mechanics from the fractal geometry of spacetime has been advanced long ago by Nottale [15].

This completes a brief overview on the interplay of Weyl geometry, Quantum Mechanics and the Bohm potential. In the next sections we shall present other novel approaches to the Geometrization process of Quantum Mechanics and leave the perennial quest of the quantization *of* spacetime versus quantization *in* spacetime, and the superposition of spacetimes for another occassion.

2 The Geometrization of Quantum Mechanics

In this section we shall postulate that the quantum probability density can *curve* the classical spacetime. For simplicity, we focus on spherically symmetric static gravitational backgrounds. Let us start with the Schwarzschild-like static spherically symmetric metric

$$(ds)^2 = - (1 - \frac{2GM(r)}{r})(dt)^2 + (1 - \frac{2GM(r)}{r})^{-1}(dr)^2 + r^2(d\Omega_2)^2 \qquad (2.1)$$

based on a mass function $M(r)$. The metric (2.1) is *not* a solution of the vacuum field equations but instead is a solution to the Einstein field equations with sources $G^\mu_\nu =$

$8\pi G T^\mu_\nu$. The stress energy tensor is given by

$$T^\mu_\nu \equiv diag\ (-\rho(r),\ p_r(r),\ p_\theta(r),\ p_\varphi(r)) \tag{2.2}$$

and the Einstein field equations are

$$R_{tt} - \frac{1}{2} g_{tt} R = - 8\pi G\ g_{tt}\ \rho \tag{2.3a}$$

$$R_{rr} - \frac{1}{2} g_{rr} R = 8\pi G\ g_{rr}\ p_{(r)}, \quad \ldots \tag{2.3b}$$

where

$$\rho(r) = - p_r(r) = \frac{1}{4\pi r^2} \frac{dM(r)}{dr}, \quad p_\theta(r) = p_\varphi(r) = - \frac{1}{8\pi r} \frac{d^2M(r)}{dr^2} \tag{2.4}$$

The conservation law $\nabla_\mu T^\mu_\nu = 0$, after laborious algebra gives

$$p_\theta = p_\phi = - \rho - \frac{r}{2} \frac{d\rho}{dr} \tag{2.5}$$

which is consistent with (2.4).

In the case of a point mass located at $r = 0$, the mass function is $M(r) = M_o \Theta(r)$, where the antisymmetric Heaviside step function $\Theta(r)$ is 1 for $r > 0$; -1 for $r < 0$, and 0 at $r = 0$ since an antisymmetric function must vanish at the origin $r = 0$. The value of $\Theta(0) = 0$ is consistent with the arithmetic mean between $\{1, -1\}$. The derivative is $\frac{d\Theta(r)}{dr} = 2\delta(r)$ so that in the case when $M(r) = M_o\Theta(r)$, we have

$$\rho(r) = - p_r(r) = \frac{1}{4\pi r^2} \frac{dM(r)}{dr} = \frac{2M_o}{4\pi r^2}\ \delta(r)$$

$$p_\theta(r) = p_\varphi(r) = - \frac{1}{8\pi r} \frac{d^2M(r)}{dr^2} = \frac{2M_o}{8\pi r^2}\ \delta(r), \quad r\delta'(r) = -\delta(r) \tag{2.6}$$

Taking the trace of the Einstein field equations yields the scalar curvature R in terms of the trace of the stress energy tensor $T = -\rho + p_r + p_\theta + p_\phi$

$$-R = 8\pi G\ T = - 8\pi G \frac{2M_o}{4\pi r^2}\ \delta(r) \Rightarrow R = \frac{4GM_o}{r^2}\ \delta(r) \tag{2.7}$$

Because $\Theta(r)$ can be represented by $\frac{|r|}{r}$, when $M(r) = M_o \frac{|r|}{r}$ the metric (2.1) becomes in this special case

$$(ds)^2 = - (1 - \frac{2GM(r)}{r})(dt)^2 + (1 - \frac{2GM(r)}{r})^{-1}(dr)^2 + r^2(d\Omega_2)^2 =$$

$$- (1 - \frac{2GM_o|r|}{r^2})(dt)^2 + (1 - \frac{2GM_o|r|}{r^2})^{-1}(dr)^2 + r^2(d\Omega_2)^2 \tag{2.8}$$

Note the key presence of the absolute value $|r|$ in eq-(2.8) compared to the standard Hilbert-Schwarzschild metric. Because the metric (2.8) is explicitly *invariant* under $r \rightarrow$

$-r$ it is automatically extended to regions where $r < 0$. Rigorously speaking r is given by $\pm\sqrt{x^2+y^2+z^2}$ so one must also take into consideration the negative branch of r.

Because the derivative of the function $|r|$ is $\Theta(r)$, and the latter function has a *discontinuity* at $r = 0$, the second derivatives of the metric components g_{tt}, g_{rr} will generate a key delta function $\delta(r)$. Consequently the metric (2.8) is *no* longer Ricci flat, and the scalar curvature R is *no* longer vanishing as it occurs with the textbook Hilbert-Schwarzschild metric. The non-vanishing expression for R is given by eq-(2.7) in terms of the delta function. One should add that if one wishes to retain a full mathematical rigour, one would be required to recur to Colombeau's distributional calculus to deal with point-mass source distributions in General Relativity rather than using the Dirac delta function.

In [16] we proceeded to evaluate the Euclideanized Einstein-Hilbert action when the scalar curvature was $R = \frac{4GM_o}{r^2}\delta(r)$. The Euclideanized action is

$$S_E \;=\; \frac{1}{16\pi G}\int_0^\infty R\;4\pi r^2\;dr\;\int_0^\beta dt \tag{2.9}$$

where the Euclidean temporal interval is bounded by the inverse of the Hawking temperature $\beta = \frac{1}{k_B T_H} = 8\pi G M_o$. Inserting the expression for $R = \frac{4GM_o}{r^2}\delta(r)$, and taking into account $\int_0^\infty \delta(r)dr = \frac{1}{2}\int_{-\infty}^\infty \delta(r)dr = \frac{1}{2}$, due to the symmetry of the delta function $\delta(-r) = \delta(r)$, we showed [16] that the Euclideanized action (2.9)

$$S_E \;=\; 4\pi G M_o^2 \;=\; \frac{1}{4}\,\frac{4\pi(2GM_o)^2}{G} \;=\; \frac{Area}{4L_P^2} \tag{2.10}$$

is precisely the black-hole *entropy* associated with a horizon radius of $r_H = 2GM_o$. This construction where the Euclideanized action matches the black hole entropy can be generalized to other dimensions. These findings suggest that the true source of the black hole entropy is its *mass*, and that there is a correspondence between the "atoms" of matter and the "atoms" of spacetime consistent with the notion of space-time-matter unification. Furthermore, it was shown also in [16] that one can always perform an *active* diffeomorphism $r \to f(r)$ (not to be confused with a passive diffeomorphism which is tantamount of a coordinate transformation) of the Hilbert-Schwarzschild solution which physically *displaces* the location of the horizon towards the singularity and which was relevant to the *firewall* controversy.

In the case of a point mass, there is a spacelike singularity at $r = 0$. If the point mass is *smeared* throughout all of space, the singularity at the origin can be removed when the $lim_{r\to 0}(\frac{M(r)}{r})$ is finite. Namely if $M(r)$ scales as r^γ with $\gamma \geq 1$ as r goes to zero.

For example, if one chooses the density $\rho(r) = \frac{3}{4\pi l^3}e^{-r^3/l^3}$, the mass enclosed in a spherical region whose radius ranges from $0 \leq r' \leq r$ is

$$M(r) \;=\; M_o\int_0^r 4\pi r'^2\;\frac{3}{4\pi l^3}\;e^{-r'^3/l^3}\;dr' \;=\; M_o\;(\;1\;-\;e^{-r^3/l^3}\;) \tag{2.11}$$

and one recovers the Dymnikova metric [17] devoid of a singularity at the origin. In the vecinity of $r = 0$ one has $M(r)/r \sim r^2$ leading to a metric with a de Sitter-like core. If one chooses Gaussian density distributions the mass function $M(r)$ is given by an incomplete-gamma function leading also to a de Sitter-like core near the origin [18].

Einstein, Infeld and Hoffmann [19] showed in 1938 that if elementary particles (point-masses) are treated as *singularities* in spacetime, it is unnecessary to postulate the geodesic motion of point test masses as part of General Relativity. For instance, the electron may be treated as such a singularity; i.e as a microscopic black-hole. Next we shall show how the smearing process of a point-mass can be attained via a *probability cloud* corresponding to the quantum probability density $\Psi^*(x^\mu)\Psi(x^\mu)$, with $\Psi(x^\mu)$ obeying the Klein-Gordon equation, and associated to a spinless scalar particle of mass M_o. To smear the point mass of an spinning electron is more complicated due to its spin. For instance, one would have to modify the Kerr metric by smearing the singularity at $r = 0$, as well as the ring singularity at $r = a$. The smearing process will *no* longer be spherically symmetric but rotationally symmetric with respect to the z-axis. Instead of the Klein-Gordon equation one must use the Dirac equation for the Dirac spinor $\Psi_D(x^\mu)$ and for mass density $M_o\bar{\Psi}_D\Psi_D$.

To proceed we shall follow very closely the construction of relativistic wave-functions by [20] (not to be confused with second-quantized fields in Quantum Field Theory) . If a *one*-particle wave function can be denoted by $\Psi(x^\mu)$, it is natural to introduce the spacetime scalar product

$$< \Psi|\Psi > \;=\; \int d^4x\ \Psi^*(x^\mu)\ \Psi(x^\mu) \tag{2.12}$$

and to normalize Ψ such that

$$1 \;=\; \int d^4x\ \Psi^*(x^\mu)\ \Psi(x^\mu) \tag{2.13}$$

The quantity

$$dP_{(4)} \;=\; \Psi(x^\mu)^*\ \Psi(x^\mu)\ d^4x \tag{2.14}$$

is naturally interpreted as probability that the particle will be found in the (infinitesimal) spacetime 4-volume d^4x.

If eq-(2.14) is the fundamental 4-probability, then

$$\Psi^*_{(3)}(x^\mu)\ \Psi_{(3)}(x^\mu) \;=\; \frac{\Psi^*(x^\mu)\ \Psi(x^\mu)}{N_t},\quad N_t \;=\; \int d^3x\ \Psi^*(x^\mu)\ \Psi(x^\mu) \tag{2.15}$$

can be interpreted as the conditional 3-probability such that

$$dP_{(3)} \;=\; \Psi_{(3)}(x^\mu)^*\ \Psi_{(3)}(x^\mu)\ d^3x \tag{2.16}$$

is the probability that the particle will be found in the (infinitesimal) 3-volume d^3x, in the case one knows that the particle is detected at time t. Since $\Psi(x^\mu)$ is normalized to unity one can infer that N_t is also the marginal probability that the particle will be found at time t over the whole 3-dimensional region $\Sigma_t = \int d^3x$.

Having briefly introduced the relativistic wave function proposal by [20] let us focus now in the case where Ψ can be decomposed (factorized) as

$$\Psi(x^\mu) \ = \ \varphi(\vec{x}) \ \xi(t) \tag{2.17}$$

so that the 3-probability density

$$\Psi^*_{(3)}(\vec{x}) \ \Psi_{(3)}(\vec{x}) \ = \ \frac{\varphi^*(\vec{x}) \ \varphi(\vec{x})}{\int d^3x \ \varphi^*(\vec{x}) \ \varphi(\vec{x})} \tag{2.18}$$

is independent on t and is automatically normalized to unity

$$1 \ = \ \int d^3x \ \Psi^*_{(3)}(\vec{x}) \ \Psi_{(3)}(\vec{x}) \tag{2.18}$$

In the spherically symmetric case $\Psi(x^\mu) = \varphi(r)\xi(t)$, the overall normalization condition

$$1 \ = \ \int d^4x \ \Psi^*(r,t) \ \Psi(r,t) \ = \ \int_0^\infty \ \varphi^*(r) \ \varphi(r) \ 4\pi r^2 \ dr \ \int_0^\infty \xi^*(t) \ \xi(t) \ dt \tag{2.19}$$

leads to

$$N \ = \ \int_0^\infty \ \varphi^*(r) \ \varphi(r) \ 4\pi r^2 \ dr, \ \ \frac{1}{N} \ = \ \int_0^\infty \xi^*(t) \ \xi(t) \ dt \tag{2.20}$$

We have taken the temporal domain's range from $t = 0$ to $t = \infty$. One could have taken it instead to range from $t = -\infty$ (infinite past) to $t = \infty$ (infinite future) . But for now we concentrate in the *former* case. Given the mass $M(r)$ enclosed in the spherical region $0 \leq r' \leq r$

$$M[\varphi(r)] \ = \ M(r) \ = \ M_o \ \int_0^r \varphi^*(r') \ \varphi(r') \ 4\pi r'^2 \ dr' \tag{2.21}$$

the D' Alambertian is given by

$$\Box \equiv \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} \ g^{\mu\nu} \ \partial_\nu), \quad \hbar = c = 1 \tag{2.22}$$

and the analog of the Klein-Gordon-like equation is

$$(\ \Box \ - \ M_o{}^2 \ ) \ \Psi(x^\mu) \ = \ 0 \tag{2.23}$$

where, once again, $\Psi(x^\mu)$ must *not* be confused with the second-quantized scalar field $\Phi(x^\mu)$. Given the metric (2.1), the KG-like equation becomes

$$\frac{1}{r^2}\partial_r \left(\ r^2 \ (1 - \frac{2GM(r)}{r}) \ \xi(t) \ \partial_r \varphi(r) \ \right) \ -$$

$$\frac{1}{r^2}\partial_t \left(\ r^2 \ (1 - \frac{2GM(r)}{r})^{-1} \ \varphi(r) \ \partial_t \xi(t) \ \right) \ - \ M_o^2 \ \varphi(r) \ \xi(t) \ = \ 0 \tag{2.24}$$

Upon choosing $\xi(t) = e^{-\omega t/2}$, eq-(2.24) leads to the integro-differential equation

$$\frac{1}{r^2}\partial_r \left(r^2 \ (1 - \frac{2GM(r)}{r}) \ \partial_r \varphi(r) \right) \ -$$

$$\frac{\omega^2}{4r^2} \ \left(r^2 \ (1 - \frac{2GM(r)}{r})^{-1} \ \varphi(r) \right) \ - \ M_o^2 \ \varphi(r) \ = \ 0 \tag{2.25}$$

where the mass function $M[\varphi(r)] = M(r)$ is defined in terms of $\varphi(r)$ by the integral (2.21).

Some brief remarks are in order. Choosing $\xi(t) = e^{-i\omega t/2}$ does not provide a normalizable wave-function, for this reason we discard it. Because $\xi(t) = e^{-\omega t/2}$ vanishes in the $t \rightarrow \infty$ limit, the probability to detect the particle in an infinite 3-dim spatial slice of the 4-dim spacetime at time $t = \infty$ is zero. Compare this picture with the black hole evaporation time via Hawking radiation that is proportional to the $(mass)^3$, and that is astronomically large for solar mass size black holes. It is interesting that the double-scaling limit $\omega \rightarrow 0; t \rightarrow \infty; \omega t =$ finite does yield a nonzero probability at $t = \infty$. The zero energy $\omega = 0$ limit reminds us of the role that soft particles have in the proposals for the resolution to the black hole information paradox involving the Bondi-Matzner-van der Burg-Sachs algebra of super-translations and super-rotations at null infinity and Weinberg soft-hair theorems.

When $\varphi(r)$ is real-valued $\varphi^*(r) = \varphi(r)$, the above integro-differential equation (2.25) can be converted into a *nonlinear* differential equation involving the mass function $M(r)$, after expressing $\varphi(r)$ in terms of $M(r)$ via eq-(2.21), as follows

$$\frac{dM(r)}{dr} = M'(r) \ = \ 4\pi M_o r^2 \ \varphi^*(r) \ \varphi(r) \ = \ 4\pi M_o r^2 \ \varphi(r)^2 \ \Rightarrow \ \varphi(r) \ = \ (\frac{M'(r)}{4\pi M_o r^2})^{1/2} \tag{2.26}$$

In this fashion, after writing $\varphi(r)$ in terms of $M'(r)$, one ends up with a complicated *third* order nonlinear differential equation for the mass function $M(r)$

$$\frac{1}{r^2}\partial_r \left(r^2 \ (1 - \frac{2GM(r)}{r}) \ \partial_r \ (\frac{M'(r)}{4\pi M_o r^2})^{1/2} \right) \ -$$

$$\frac{\omega^2}{4r^2} \ \left(r^2 \ (1 - \frac{2GM(r)}{r})^{-1} \ (\frac{M'(r)}{4\pi M_o r^2})^{1/2} \right) \ - \ M_o^2 \ (\frac{M'(r)}{4\pi M_o r^2})^{1/2} \ = \ 0 \tag{2.27}$$

Thus eq-(2.27) is the static spherically symmetric relativistic analog of the Newton-Schrödinger equation. Suffice to say is that to find non-trivial solutions to eq-(2.27) is very difficult.

In essence what eqs-(2.1, 2.25, 2.27) encapsulate is that the gravitational field produced by *smearing* a point mass M_o at $r = 0$ throughout all of space (in an spherically symmetric fashion) can be interpreted as the gravitational field generated by a self-gravitating anisotropic fluid droplet of mass density $4\pi M_o r^2 \varphi^*(r)\varphi(r)$ and which is sourced by the *probability cloud* (associated with a spinless point-particle of mass M_o) *permeating* a 3-spatial domain region $\mathcal{D}_3 = \int_0^\infty 4\pi r^2 dr$ at any time t. Classically one may smear the point mass in any way we wish leading to arbitrary density configurations $\rho(r)$. However, Quantum Mechanically this is *not* the case because the radial mass configuration $M(r)$ must *obey* the third order nonlinear differential eq-(2.27).

If the solutions $\varphi(r)$ to eq-(2.25) are not normalized to unity, from the conditions (2.20) one can always perform the following scaling by an energy-dependent factor $N = N(\omega)$ as follows

$$\varphi \to \frac{\varphi}{\sqrt{N}}, \quad \xi(t) \to \sqrt{N}\xi(t), \quad M_o \to M_o, \quad M(r) \to \frac{M(r)}{N}, \quad G \to GN, \quad GM(r) \to GM(r) \tag{2.28}$$

and leading to properly normalized solutions of eq-(2.25). The scalings (2.28) will ensure that the metric (2.1) remains invariant as well as the D'Alambertian operator. The scaling of G is consistent with the scaling behavior of $G = G(k^2)$ in the Asymptotic Safety program of Quantum Gravity [21], [22]. If $N > 1$, then at lower energies (smaller mass $M(r) \to \frac{M(r)}{N}$) there is a higher value of $G \to GN$. And vice versa in case that $N < 1$.

To finalize this section we must remark that the Renormalization-Group (RG) improvement of the Schwarzschild black hole metric leading also to a resolution of the singularity at $r = 0$ is *not* the same as the smearing of a point mass M_o into a continuum mass distribution $M(r)$. The renormalization-group improved Schwarzschild black-hole metric [22] is given by

$$(ds)^2 \ = \ - \ (1 - \frac{2G(r)M_o}{r})(dt)^2 \ + \ (1 - \frac{2G(r)M_o}{r})^{-1}(dr)^2 \ + \ r^2(d\Omega_2)^2 \tag{2.29}$$

and is based on the Renormalization group flow of the Newtonian coupling $G[k^2(r)]$ after introducing a cutoff identification procedure $k \leftrightarrow r$ that relates energy scales to length scales, for example, like $k \sim \frac{1}{d(r)}$, where $d(r)$ is a proper radial distance [22].

The metric (2.29) is *not* a solution of the vacuum field equations but instead is a solution to the modified Einstein equations $G^\mu_\nu = 8\pi G(r) T^\mu_\nu$ where the *running* Newtonian coupling $G(r)$, and an effective stress energy tensor

$$T^\mu_\nu \ \equiv \ diag\ (-\rho(r), \ p_r(r), \ p_\theta(r), \ p_\varphi(r)) \tag{2.30}$$

appear in the right hand side. The components of T^μ_ν associated to the modified Einstein equations $G^\mu_\nu = 8\pi G(r) T^\mu_\nu$ are respectively given by

$$\rho \ = \ - \ p_r \ = \ \frac{M_o}{4\pi r^2 G(r)} \ \frac{dG(r)}{dr}, \quad p_\theta \ = \ p_\varphi \ = \ - \ \frac{M_o}{8\pi r G(r)} \ \frac{d^2G(r)}{dr^2} \tag{2.31}$$

The energy-momentum tensor is in this case an $effective$ stress energy tensor resulting from vacuum polarizations effects of the quantum gravitational field [24] (like a quantum-gravitational self-energy). As explained by [23], the quantum system is self-sustaining: a small variation of the Newton's constant triggers a ripple effect, consisting of successive back-reactions of the semi-classical background spacetime which, in turn, provokes further variations of the Newton's coupling and so forth. For recent findings pertaining Asymptotic Safety, Black-Hole Cosmology and the Universe as a Gravitating Vacuum State see [25]. In all of these works [23], [22], [25] there are no singularities at $r = 0$, nor at $t = 0$ as a result of the RG-improvement procedures based on Asymptotic Safety.

3 Geometrization of Quantum Mechanics via Finsler Geometry

3.1 Lagrange-Finsler Geometry and the Relativistic Quantum Potential

In the attempts to apply Finsler geometry to construct an extension of general relativity, the question about a suitable generalization of the Einstein equations is still under debate. Since Finsler geometry is based on a scalar function on the tangent bundle, the field equation which determines this function should also be a scalar equation. In the literature two such equations have been suggested: the one by Rutz and the one by Pfeifer and Wohlfarth (which was independently also found by Chen and Shen in the context of positive definite Finsler geometry).

Given a Finsler-Lagrange function

$$F^2 \;=\; L(x,\dot{x}) \;=\; g_{ij}(x,\dot{x})\dot{x}^i\dot{x}^j, \quad F \;=\; \sqrt{g_{ij}(x,\dot{x})\dot{x}^i\dot{x}^j}; \quad \dot{x}^i = \frac{dx^i}{d\tau} \tag{3.1}$$

the L-metric is defined as the vertical Hessian

$${}^L g_{ij} = \frac{1}{2}\partial_{\dot{x}^i}\partial_{\dot{x}^j} L(x,\dot{x}), \quad F^2(x,\dot{x}) \;=\; L(x,\dot{x}) \tag{3.2}$$

The geodesic spray coefficients $G^i : \ddot{x}^i + G^i = 0$, obtained from an Euler-Lagrange variational equation of the length measure of curves $l = \int_a^b \sqrt{|L(x,\dot{x}|}d\tau$, are given by

$$G^i \;=\; \frac{1}{4}\;{}^L g^{ij}\;(\;\dot{x}^k\;\partial_k\;\partial_{\dot{x}^j}\;L\;-\;\partial_{x^j}\;L\;) \tag{3.3}$$

which define the coefficients $N^i_j = \partial_{\dot{x}^j} G^i$ of the canonical Cartan non-linear connection

$$N^i_{jk} \;\equiv\; \partial_{\dot{x}^k}\;N^i_j \;=\; \partial_{\dot{x}^k}\;\partial_{\dot{x}^j}\;G^i \tag{3.4}$$

and permits the splitting of the tangent bundle into a vertical sub-bundle and a horizontal sub-bundle. The local adapted basis are denoted by the elongated horizontal derivatives $\delta_i = \partial_i - N^j_i\dot{\partial}_j$, and the fiber vertical derivatives are denoted by $\dot{\partial}_j \equiv \frac{\partial}{\partial\dot{x}^j} = \partial_{\dot{x}^j}$.

The so called Chern-Rund linear connection $\mathbf{D}$ on the tangent bundle TM is given by

$${}^{(\delta)}\Gamma^i_{jk} \;=\; \frac{1}{2}\;{}^L g^{ih}\;(\delta_k\;{}^L g_{hj}\;+\;\delta_j\;{}^L g_{hk}\;-\;\delta_h\;{}^L g_{jk}) \tag{3.5}$$

which allows to define the horizontal covariant derivative as follows

$$\nabla_j V_i \ = \ \delta_j V_i \ - \ {}^{(\delta)}\Gamma^k_{ij} \ V_k \ = \ \partial_j V_j \ - \ N^k_j \ \partial_{\dot{x}^k} \ V_i \ - \ {}^{(\delta)}\Gamma^k_{ij} \ V_k \tag{3.6}$$

The curvature of the non-linear connection is

$$[\delta_j, \delta_k] = R^i_{jk}\dot{\partial}_i = (\delta_j N^i_k - \delta_k N^i_j) \ \dot{\partial}_i \tag{3.7}$$

The Finsler Ricci tensor is $R^i_j = R^i_{jk}\dot{x}^k$, and the non-homogenized Finsler Ricci scalar R is given by its trace $R = R^i_i = R^i_{ik}\dot{x}^k$.

The *difference* between the derivative of the non-linear connection coefficients N^i_j and the Chern-Rund connection coefficients defines the Landsberg tensor

$$P^i_{jk}(x, \dot{x}) \ \equiv \ {}^{(\delta)}\Gamma^i_{jk} \ - \ N^i_{jk}, \ \ \dot{x}^j \ P^i_{jk} \ = \ 0 \tag{3.8}$$

The authors [26] proceed to construct an action for Finsler gravity on a compact domain D^+ of the positive projective tangent bundle PTM^+ with 0-homogeneous objects by introducing the 0-homogenized Ricci scalar $R_o = \frac{R}{L}$. The action is based on the Vainberg-Tonti Lagrangian and given by

$$S \ = \ \int_{D^+} R_o \ dV_o^+ \ = \ \int_{D^+} \frac{R}{L^3} \ |det \ {}^L g| \ \mathbf{i}_C (dx^0 \wedge dx^1 \cdots \wedge d\dot{x}^0 \wedge \cdots \wedge d\dot{x}^3) \tag{3.9}$$

$\mathbf{i}_C(\cdots)$ is the volume 7-form of $D^+ \subset PTM^+$ obtained by a contraction of a volume 8-form.

This construction allows a mathematically rigorous formulation of the Finsler gravity action as well as a technically precise derivation of the Euler-Lagrange equations from the action. The critical points of the Finsler gravity action formulated on subsets of the positive projective tangent bundle PTM^+ are obtained via variational methods and yield the vacuum field equation (fundamental result of [26])

$$\frac{1}{2} \ {}^L g^{ij} \ \partial_{\dot{x}^i}\partial_{\dot{x}^j} \ R \ - \ \frac{3}{L} \ R \ - \ {}^L g^{ij} \ (\nabla_j P_i - P_i P_j + \partial_{\dot{x}^i}(\nabla P_i)) \ = \ 0 \tag{3.10}$$

with $P_j = P^i_{ij}$, and $\nabla = \dot{x}^k D_{\delta_k} = \dot{x}^k \nabla_k$.

In the metric limit $L(x; \dot{x}) = g_{ij}(x)\dot{x}^i\dot{x}^j$ they reduce to known constructions from the ordinary metric (Riemannian) geometry: the curvature essentially becomes the Riemann curvature tensor, $R^i_{jk}(x; \dot{x}) = R^i_{jkl}(x)\dot{x}^l$; the generalized Christoffel symbols become the usual Christoffel symbols; and the non-linear connection now is a linear connection $N^i_j(x; \dot{x}) = \Gamma^i_{jk}(x)\dot{x}^k$. The Cartan linear covariant derivative in horizontal directions becomes the Levi-Civita covariant derivative while it becomes trivial in vertical directions.

After denoting $\dot{x}^\mu \equiv \frac{dx^\mu}{d\tau}$, we shall choose for the Finsler function the following one

$$F \ = \ (1 \ + \ \frac{Q(x)}{m}) \ \sqrt{g_{\mu\nu}(x, \dot{x}) \ \dot{x}^\mu \ \dot{x}^\nu}; \ \ L(x, \dot{x}) \ = \ F^2 \ = \ (1 \ + \ \frac{Q(x)}{m})^2 \ g_{\mu\nu}(x, \dot{x}) \ \dot{x}^\mu \ \dot{x}^\nu \tag{3.11}$$

where

$$Q \ = \ \frac{\hbar^2}{2m} \ \frac{\Box(\sqrt{\Psi^*(x^\mu)\Psi(x^\mu)})}{\sqrt{\Psi^*(x^\mu)\Psi(x^\mu)}} \tag{3.12}$$

is the relativistic analog of the Bohm's quantum potential with units of mass. Note that our choice for the above Lagrange-Finsler function (3.11) $differs$ from the one in [8].

In the simplest scenario when the background metric does *not* depend on the velocities $g_{\mu\nu}(x,\dot{x}) = g_{\mu\nu}(x)$, the Finsler metric

$$^L g_{\mu\nu}(x) \ = (\ 1 \ + \ \frac{2Q(x)}{m} \ + \ (\frac{Q(x)}{m})^2 \ ) \ g_{\mu\nu}(x) \tag{3.14}$$

is a just a $Q(x)$-dependent conformal scaling of the background metric $g_{\mu\nu}(x)$. Given a flat background metric $g_{\mu\nu}(x) = \eta_{\mu\nu}$, to leading order in Q one has

$$^L g_{\mu\nu} \ \sim \ \eta_{\mu\nu} \ + \frac{2Q}{m} \ \eta_{\mu\nu} \ = \ \eta_{\mu\nu} \ + \ h_{\mu\nu} \tag{3.15}$$

Hence, the linearized perturbation $h_{\mu\nu}$ of the flat metric $\eta_{\mu\nu}$ is provided by the relativistic version of Bohm's potential $h_{\mu\nu} = \frac{2Q(x)}{m}\eta_{\mu\nu}$. In the weak field approximation and slow moving bodies, the h_{00} component of the perturbation can be interpreted as twice an effective Newtonian potential $2V_{eff} \sim \frac{2Q}{m}$, which is consistent with the formulation of the Bohm-Poisson equation (1.11), and the results in [9] showing how the Newtonian gravitational potential energy coincides with the Bohm's potential in certain scenarios.

Given a standard matter action of the form (for a $(-,+,+,+)$ signature) with the inclusion of the quantum potential Q

$$S_m \ = \ - \ \int \ d\tau \ [\ m \ \sqrt{g_{\mu\nu}(x^\sigma) \ \dot{x}^\mu \ \dot{x}^\nu} \ - \ Q(x^\mu) \ ] \tag{3.16}$$

the Euler-Lagrange equations are given in terms of the Levi-Civita torsionless connection $\Gamma^\mu_{\alpha\beta}[g_{\mu\nu}]$ as follows

$$m \ \left(\ \ddot{x}^\mu \ + \ \Gamma^\mu_{\alpha\beta}[g_{\mu\nu}] \ \dot{x}^\alpha \ \dot{x}^\beta \ \right) \ = \ - \ \frac{\partial Q}{\partial x_\mu} \tag{3.17}$$

The term in the right hand side is the "quantum force" acting on the particle. The solutions $x^\mu = x^\mu(\tau)$ to eqs-(3.17) correspond to the Bohmian trajectories. If one had instead a Lagrangian of the form $\mathcal{L}_m = (m+Q)\sqrt{g_{\mu\nu}(x^\sigma)\dot{x}^\mu\dot{x}^\nu}$ based on an effective mass $m_{eff}(x) = m + Q(x)$, the Euler-Lagrange equations would have been more complicated. Consequently, after invoking the notion of wave-particle duality, the wave equation associated with the latter Euler-Lagrange equations will be much more complicated as indicated earlier in section **1** when we discussed the modifications of the Klein-Gordon equation.

In general when $g_{\mu\nu}(x,\dot{x})$ depends on the position and velocities and the expression (3.2) for the Finsler metric $^L g_{\mu\nu}$ is a much more complicated expression. The key result is that the *single* fundamental equation (3.10) for the Finsler-Lagrange function $L(x,\dot{x})$ furnishes the sought-after differential equation involving the coupling between the metric $g_{\mu\nu}(x,\dot{x})$ *and* the quantum probability density $\Psi^*(x)\Psi(x)$. This is a direct result of the introduction of an effective mass function $m_{eff} = m + Q$, where $Q(x^\mu)$ (3.12) is the relativistic version of Bohm's quantum potential. Whereas in the work of [5], the mass function was given in terms of the Weyl scalar curvature $m_{eff} = m(1 + \gamma(\frac{\hbar}{m})^2 R_{Weyl})$. To sum up, the findings in section **2** and **3.1** implement the gist of our geometrization program of Quantum Mechanics. We must note that now eq-(3.10) replaces the role of the Klein-Gordon equation of the previous section yielding another differential equation involving a coupling between $g_{\mu\nu}$ (geometry) and Ψ (matter).

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3.2 Finsler-like Gravity in the Cotangent Bundle and Phase Space Quantum Mechanics

In section **2** we postulated that the quantum probability density *curved* spacetime. In this section we postulate that the Weyl-Wigner-Groenewold-Moyal quasi-probability distribution $W(x,p)$ *curves* phase space. The literature on the geometry of Lagrange-Finsler, Hamilton-Cartan spaces and higher order (jet bundles) generalizations is ample, see [**?**], [28], and references therein. Let us begin with the Sasaki-Finsler metric of the cotangent space of a d-dim manifold T^*M_d, and which is given by the following metric in *block diagonal* form

$$(d\sigma)^2 \ = \ g_{ij}(x^k, p_a)\ dx^i d\ x^j \ + \ h^{ab}(x^k, p_c)\ \delta p_a\ \delta p_b \ = $$
$$g_{ij}(x^k, p_a)\ dx^i d\ x^j \ + \ h_{ab}(x^k, p_c)\ \delta p^a\ \delta p^b \qquad (3.18)$$

The range of the base manifold indices is $i,j,k = 0,1,2,3,.....d-1$; whereas the range of the fiber indices is $a,b,c = 0,1,2,3,.....d-1$. The standard coordinate basis frame has been replaced by the following anholonomic non-coordinate basis frame comprised of the following elongated and ordinary derivatives, respectively,

$$\delta_i \ = \ \delta/\delta x^i \ = \ \partial_{x^i} \ + \ N_{ia}\ \partial^a \ = \ \partial_{x^i} \ + \ N_{ia}\ \partial_{p_a};\ \ \partial^a \ \equiv \ \partial_{p_a} \ = \ \frac{\partial}{\partial p_a} \qquad (3.19)$$

The signature is chosen to be Lorentzian $(-,+,+,+,\cdots,+)$ for both g_{ij} and h_{ab}. It is important to emphasize that one does *not* have a theory with two times because the energy coordinate is not time. One should note the *key* position of the indices that allows us to distinguish between derivatives with respect to x^i and those with respect to p_a. The dual basis of $(\delta_i = \delta/\delta x^i; \partial^a = \partial/\partial p_a)$ is

$$dx^i,\ \ \delta p_a \ = \ dp_a \ - \ N_{ja}\ dx^j,\ \ \delta p^a \ = \ dp^a \ - \ N^a_j\ dx^j \qquad (3.20)$$

where the N-coefficients define a nonlinear connection, N–connection structure.

A gravity-matter action is $\mathcal{S} = \mathcal{S}_{grav} + \mathcal{S}_{matter}$, with

$$\mathcal{S}_{grav} \ = \ \frac{1}{2\kappa}\int d^4x\ d^4p\ \sqrt{|\det g_{AB}|}\ \left(\ g^{ij}\ R_{(ij)} \ + \ h_{ab}\ S^{(ab)}\ \right)$$

$$\mathcal{S}_{matter} \;=\; \int d^4x\ d^4p\ \sqrt{|\det g_{AB}|}\ \mathcal{L}_m[g_{ij}, h_{ab}, N_{ia}, \Phi, \Psi] \tag{3.21}$$

The determinant factorizes $det(g_{AB}) = det(g_{ij})det(h_{ab})$ in an anhololomic basis adapted to the nonlinear connection (the metric assumes the block diagonal form (1)). κ is the gravitational coupling constant. If the phase space action action (21) is dimensionless, after reintroducing the physical constants that were set to unity, gives $\kappa = 8\pi \to (8\pi G/c^4)(M_p c)^4$.

After a very laborious procedure the authors [29] have shown that variation of the action (3.21)

$$\frac{\delta \mathcal{S}}{\delta g_{ij}} \;=\; 0, \quad \frac{\delta \mathcal{S}}{\delta h_{ab}} \;=\; 0, \quad \frac{\delta \mathcal{S}}{\delta N_{ia}} \;=\; 0, \tag{3.21}$$

leads to the following field equations

$$R_{(ij)}(x,p) - \frac{1}{2}\, g_{ij}(x,p)\ (R + S) + R_{k(ia}\, C^{ka}_{j)} \;=\; \kappa\, T_{ij} \;=\; -\kappa\ \frac{2}{\sqrt{|\det g_{AB}|}}\ \frac{\delta(\sqrt{|\det g_{AB}|}\mathcal{L}_m)}{g^{ij}} \tag{3.22}$$

$$S_{(ab)}(x,p)\ - \frac{1}{2}\ h_{ab}(x,p)\ (R+S) \;=\; \kappa\ T_{ab} \;=\; -\,\kappa\ \frac{2}{\sqrt{|\det g_{AB}|}}\ \frac{\delta(\sqrt{|\det g_{AB}|}\mathcal{L}_m)}{h^{ab}} \tag{3.23}$$

$$g^{ik}\ \partial^a H^j_{kj}\ -\ g^{kl}\ \partial^a H^i_{kl} \;=\; \kappa\ T^{ia} \;=\; -\,\kappa\ \frac{2}{\sqrt{|\det g_{AB}|}}\ \frac{\delta(\sqrt{|\det g_{AB}|}\mathcal{L}_m)}{N_{ia}} \tag{3.24}$$

where

$$H^i{}_{jk} \;=\; \frac{1}{2} g^{in}\ (\delta_k g_{nj} + \delta_j g_{nk} - \delta_n g_{jk}) \tag{3.25}$$

$$C^{ab}_c \;=\; -\ \frac{1}{2}\ h_{cd}\ \left(\ \partial^b h^{ad}\ +\ \partial^a h^{bd}\ -\ \partial^d h^{ab}\ \right) \tag{3.26}$$

$$R^i_{kjh} \;=\; \delta_h H^i_{kj}\ -\ \delta_j H^i_{kh}\ +\ H^l_{kj}\ H^i_{lh}\ -\ H^l_{kh}\ H^i_{lj}\ -\ C^{ia}_k\ R_{jha} \tag{3.27}$$

$$S^{abc}_d \;=\; \partial^c\ C^{ab}_d\ -\ \partial^b\ C^{ac}_d\ +\ C^{eb}_d\ C^{ac}_e\ -\ C^{ec}_d\ C^{ab}_e; \tag{3.28}$$

$$R_{ija} \;=\; \frac{\delta N_{ja}}{\delta x^i}\ -\ \frac{\delta N_{ia}}{\delta x^j} \tag{3.29}$$

The above field equations are supplemented by the additional equations $\frac{\delta \mathcal{S}}{\delta \Psi} = 0, \frac{\delta \mathcal{S}}{\delta \Phi} = 0$ associated to the fermionic $\Psi(x,p)$ (Dirac spinors) and scalar fields $\Phi(x,p)$ living in the cotangent bundle. For a recent study of these equations within the context of Born's reciprocal relativity theory, curved phase space, Finsler geometry and the cosmological constant see [30].

Having outlined the essential elements of gravity in the cotangent bundle (phase space) we turn attention to phase space Quantum Mechanics. The key concept in phase space

Quantum Mechanics is the Wigner quasi-probability distribution [32] that was introduced by Wigner in 1932 to study quantum corrections to classical statistical mechanics. The goal was to link the wave-function that appears in Schrödinger's equation to a probability distribution in phase space. Moyal [32] had derived it independently, and allows one to study the classical limit, offering a comparison of the classical and quantum dynamics in phase space. It has been shown that the Wigner quasi-probability distribution function can be regarded as an $\hbar$-deformation of another phase space distribution function that describes an ensemble of de Broglie-Bohm causal trajectories [34].

The Wigner distribution $W(x,p)$ of a pure state is defined as

$$W(q,p) \ = \ \frac{1}{\pi\hbar} \int_{-\infty}^{+\infty} \ dy \ e^{\frac{2ipy}{\hbar}} \ \Psi^*(x+y) \ \Psi(x-y) \tag{3.30}$$

In the general case, which includes *mixed* states, it is the Wigner transform of the density matrix $\hat{\rho}$

$$W(x,p) = \frac{1}{\pi\hbar} \int_{-\infty}^{\infty} \langle x+y|\hat{\rho}|x-y\rangle e^{-2ipy/\hbar} \, dy, \tag{3.31}$$

This construction can be generalized to other dimensions. Despite that the Wigner function of quantum state typically takes some negative values (it is a quasi-probability distribution), however, the integral $\int dp \ W(q,p)$ is positive definite.

The Moyal evolution equation for the Wigner function is

$$\frac{\partial W(x,p,t)}{\partial t} = -\{\{W(x,p,t) \ , \ H(x,p)\}\} \ , \tag{3.32}$$

$H(x,p)$ is the Hamiltonian and $\{\{,\}\}$ is the Moyal bracket given by $W * H - H * W$, and where $*$ is the *noncommutative* Moyal star product of functions in phase space. In the classical limit $\hbar \rightarrow 0$, the Moyal bracket reduces to the Poisson bracket, while this evolution equation reduces to the Liouville equation of classical statistical mechanics.

A relativistic generalization of the Wigner distribution can be found in [33] and references therein. The idea now is to write an ansatz for $g_{ij}(x,p); h_{ab}(x,p), N_{ia}(x,p)$ in terms of the relativistic Wigner distribution $W(x,p)$, in the same vein that in section **2** we wrote down the metric (2.1) in terms of a mass function $M(r)$, and which in turn, was given in terms of an integral of the probability density $M(r) = M_o \int_0^r \varphi^* \varphi 4\pi r'^2 dr'$. Having done so we can proceed to evaluate the left hand side of eqs-(3.22-3.24) and then read-off the components of T_{AB} in the right-hand-side in terms of $W(q,p)$ and its derivatives. Finally, the Moyal evolution equation (3.32) associated with the relativistic Hamiltonian $H(x,p) = g_{\mu\nu}(x,p)p^\mu p^\nu$ yields the differential equation involving the coupling between the metric (geometry of phase space) $g_{\mu\nu}(x,p)$ and $W(x,p)$ (matter). Because the Moyal star product has infinite derivatives, the Moyal evolution equation will be *nonlocal*.

The relevance of this road map based on the postulate that quasi-probability distribution $W(x,p)$ *curves* phase spaces, and encompassing the Finsler-like geometry of the cotangent-bundle with phase space quantum mechanics, is that it naturally incorporates the *noncommutative* and non-local Moyal star product (there are also non-associative star products as well). Because different foliations of phase space leads to different spacetimes, the superposition of different spacetimes could be realized by assembling those different foliations into an ensemble. Each observer will see a different spacetime. For example, the tangent at a point in momentum space provides a spacetime. The tangent at another point in momentum space provides another spacetime. Phase space is the arena where to implement space-time-matter unification as advocated in [30]. It is our belief this is the right platform where the quantization *of* spacetime and the quantization *in* spacetime will coalesce.

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