A pair of congruent circles in Tenzan Tebikigusa Furoku

Hiroshi Okumura
Maebashi Gunma 371-0123, Japan
e-mail: hokmr@yandex.com

Abstract. We generalize non-Archimedean congruent circles appeared in Sampō Tenzan Tebikigusa Furoku to the collinear arbelos.

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1. Introduction

For two points P and Q on a line AB in the plane, we denote the semicircle of diameter PQ by (PQ), where all the semicircles with diameters on AB are constructed on the same side. We consider an arbelos formed by the three semicircles (AO), (BO) and (AB) for a point O on the segment AB, where $|AO| = 2a$ and $|BO| = 2b$ (see Figure 1). The perpendicular to AB at the point O is called the axis. Inradius of the curvilinear triangle made by (AB) and the axis and one of (AO) and (BO) equals $ab/(a + b)$, and circles of the same radius are called Archimedean circles of the arbelos.

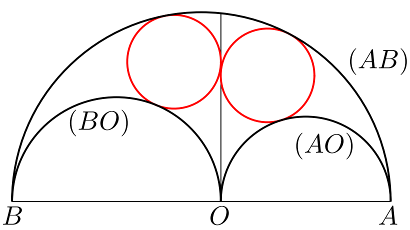

Figure 1.

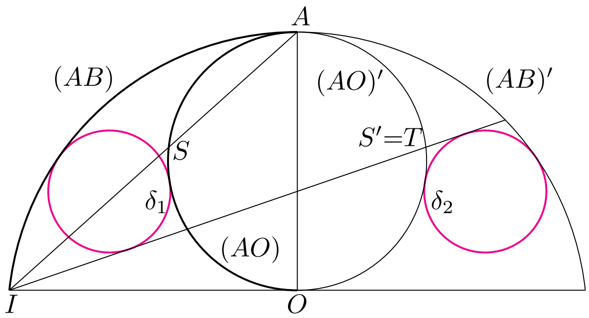

Figure 2.

We denote the reflection in the line AB by $'$. Let I be the point of intersection of (AB) and the axis. In [3], we have shown the following theorem (see Figure 2):

Theorem 1. *Let S be the point of intersection of AI and (AO). For a point T on $(AO)'$, let δ_1 be the incircle of the curvilinear triangle made by (AO), (AB) and the line IT, and let δ_2 be the circle touching $(AO)'$, $(AB)'$ and IT from the side opposite to δ_1. Then the two circles δ_1 and δ_2 are congruent if and only if $T = S'$. In this event their common radius equals*

$$\frac{4a^2b}{(2a+b)^2}.$$

The congruent circles are stated in [1], and Theorem 1 gives a necessary and sufficient condition giving the two congruent circles. In this article we generalize the theorem to a generalized arbelos called the collinear arbelos.

2. Collinear arbelos

For a point P on the half line with initial point A passing through B, let Q be the point on the line AB such that $\overrightarrow{OA} \cdot \overrightarrow{OP} = \overrightarrow{OB} \cdot \overrightarrow{OQ}$, where $\cdot$ is the inner product of the vectors. Let $\alpha = (AP)$, $\beta = (BQ)$ and $\gamma = (AB)$. The configuration consisting of the three semicircles is called a collinear arbelos and denoted by (α, β, γ) [2, 4, 5, 6].

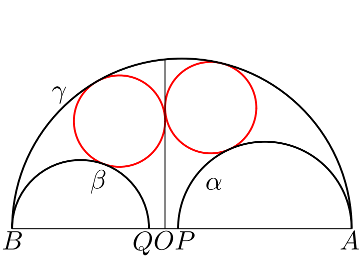

Figure 3.

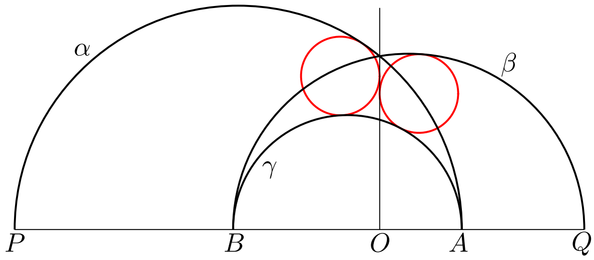

Figure 4.

We use a rectangular coordinate system with origin O such that the points A and B have coordinates $(2a, 0)$ and $(-2b, 0)$, respectively, where we assume that the three semicircles lie on the region $y \geq 0$. Let $(2p, 0)$ and $(2q, 0)$ be the coordinates of the points P and Q, respectively. Notice that the axis coincides with the radical axis of α and β and the two points P and Q lie between A and B or lie in the order P, B, A, Q, which are equivalent to $-b < p < a$ and $p < -b$, respectively (see Figures 3 and 4).

Let $s = |AQ|/2$ and $t = |BP|/2$. Since $ap + bq = 0$, we have

$$ta = sb \quad and \quad tq + sp = 0. \tag{1}$$

Circles of radius $r_{\mathrm{A}} = st/(s+t)$ are called Archimedean circles of (α, β, γ). If $P = O$, then $Q = O$ and (α, β, γ) and its Archimedean circles coincide with the ordinary arbelos mentioned in the opening sentence and its Archimedean circles. Figures 3 and 4 show typical Archimedean circles of (α, β, γ).

3. Generalization

In this section we generalize Theorem 1. From now on we consider a collinear arbelos (α, β, γ). We now redefine the point S as the point of intersection of the line AI and α for a collinear arbelos (α, β, γ).

If two congruent circles have an internal common tangent passing through the point I, one of which touches one of α and γ internally and touches the other externally, and the other circle touches one of α' and γ' internally and touches the other externally, then we call the two congruent circles a *congruent pair* or *congruent pair associated with* I and call the common tangent passing through I the I*-tangent*. Figure 2 shows that the circle δ_1 and δ_2 are congruent pair associated with I with I-tangent IS'.

Let $v = a - \sqrt{a(a+b)}$. Then $-b < v < 0$, since $v - (-b) = a + b - \sqrt{a(a+b)} > 0$. Therefore the point of coordinates $(2v, 0)$ lies between B and O. Theorem 1 is generalized as follows.

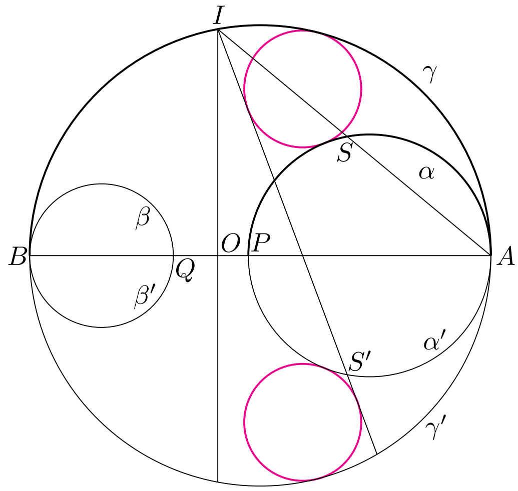

Figure 5: A congruent pair associated with I in the case $v < p < a$.

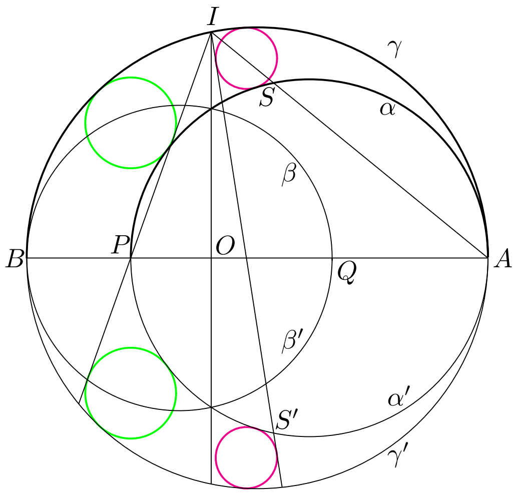

Figure 6: $p = v$.

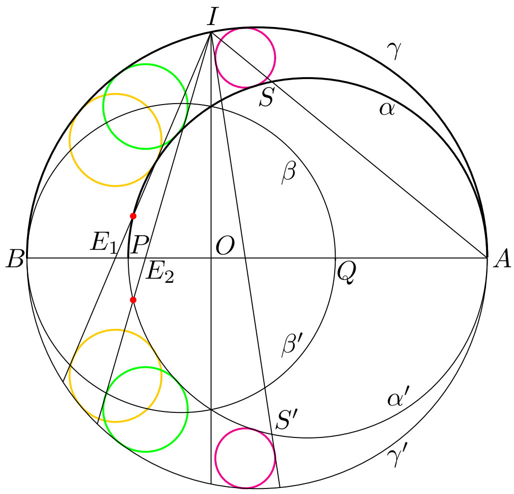

Figure 7: $-b < p < v$.

Theorem 2. *The following statements are true for the collinear arbelos.*
(i) *There is a congruent pair associated with I with I-tangent IS' in any case. The common radius equals*

$$\frac{4a(a-p)|b+p|}{(2a+b-p)^2}. \tag{2}$$

(ii) *If $v < p < a$, there is only one congruent pair associated with I stated in* (i).
(iii) *If $p = v$, there are exactly two congruent pairs associated with I, one of which is stated in* (i). *The remaining congruent pair consists of circles of radius*

$$\frac{2b\sqrt{a(a+b)}}{(\sqrt{a}+\sqrt{a+b})^2} \tag{3}$$

with I-tangent IP.
(iv) *If $-b < p < v$ or $p < -b$, there are exactly three congruent pairs associated with I, one of which is stated in* (i).

Proof. Let (x_s, y_s) be the coordinates of S. Firstly we assume $-b < p < a$. Then we get $2a : (2a - x_s) = (a + b) : (a - p)$, since A is the internal center of similitude of γ and α. Therefore by (1), we have

$$x_s = \frac{2a(b+p)}{a+b} = \frac{2at}{a+ta/s} = \frac{2st}{s+t} = 2r_{\mathrm{A}}.$$

Also by the same similarity, we have

$$y_s = 2\sqrt{ab}(a-r_{\mathrm{A}})/a = 2(a-r_{\mathrm{A}})\sqrt{\frac{b}{a}}.$$

We consider a congruent pair associated with I, whose common radius is r, and have centers of coordinates $(e, \pm f)$. Since the I-tangent passes through the midpoint of the segment joining the two centers, it has an equation

$$2\sqrt{ab}(x-e)+ey=0, \tag{4}$$

where recall that I has coordinates $(0, 2\sqrt{ab})$. The distances from the centers of the congruent circles to the centers of α, γ and the I-tangent equal $a-p+r$, $a+b-r$ and r, respectively. Therefore we get the following three equations.

$$(a+p-e)^2+f^2=(a-p+r)^2, \tag{5}$$

$$(a-b-e)^2+f^2=(a+b-r)^2, \tag{6}$$

$$\frac{|ef|}{\sqrt{4ab+e^2}}=r. \tag{7}$$

Conversely from real numbers e, f and $r>0$ satisfying the three equations, we get a congruent pair associated with I with centers of coordinates $(e, \pm f)$ and common radius r. Eliminating f from (5) and (6), and also from (5) and (7), we have

$$b(e+r)+p(e-r)-2a(b+p-r)=0, \tag{8}$$

$$4abr^2+e^2((e-2a)(e-2p)-2(a-p)r)=0. \tag{9}$$

Solving (8) and (9) for e and r, we get

$$e=2a \quad \textit{and} \quad r=0, \tag{10}$$

or

$$e=\frac{2a(b+p)}{2a+b-p} \quad \textit{and} \quad r=\frac{4a(a-p)(b+p)}{(2a+b-p)^2}, \tag{11}$$

or

$$e=\frac{2\left(b(p-a)\mp\sqrt{w}\right)}{2a+b-p} \quad \textit{and} \quad r=\frac{2(b+p)\left((a+b)(2a-p)\pm\sqrt{w}\right)}{(2a+b-p)^2}, \tag{12}$$

where $w=b(a+b)(p^2-a(b+2p))=b(a+b)(p-v)(p-a-\sqrt{a(a+b)})$. Notice that $w\ge 0$ implies $r>0$ in (12), since

$$(a+b)^2(2a-p)^2-w=a(a+b)(2a+b-p)^2>0. \tag{13}$$

While $w\ge 0$ if and only if $-b<p\le v$. Therefore we get no congruent pairs, one congruent pairs or two congruent pairs associated with I given by (12), according as $v<p<a$, $p=v$ or $-b<p<v$.

We secondly assume $p < -b$ and consider a congruent pair associated with I, whose common radius is r and have centers of coordinates $(e, \pm f)$. Then we get

$$(x_s, y_s) = \left(-2r_{\mathrm{A}}, 2(a + r_{\mathrm{A}})\sqrt{\frac{b}{a}}\right)$$

similarly. Since the distances from the centers of the congruent circles to the centers of α, γ and the I-tangent equal $a - p - r$, $a + b + r$ and r, respectively, the relations between e, f, r, a, b and p are obtained by changing the signs of r in (5), (6), (7). Therefore changing the signs of r in (10), (11) and (12), we get

$$e = 2a \quad and \quad r = 0, \tag{14}$$

or

$$e = \frac{2a(b + p)}{2a + b - p} \quad and \quad r = \frac{4a(a - p)(-b - p)}{(2a + b - p)^2}, \tag{15}$$

or

$$e = \frac{2\left(b(p - a) \mp \sqrt{w}\right)}{2a + b - p} \quad and \quad r = \frac{2(b + p)\left(-(a + b)(2a - p) \mp \sqrt{w}\right)}{(2a + b - p)^2}. \tag{16}$$

Since $p < -b$ implies $w > 0$ and $r > 0$ by (13) in (16), it gives two congruent pairs associated with I.

We now prove (i). If we consider the congruent pairs associated with I given by (11) or (15), then in any case $e = 2a(b + p)/(2a + b - p)$, and we have

$$2\sqrt{ab}(x_s - e) + e(-y_s) = \frac{-8\sqrt{ab}(b + p)(ap + bq)}{(2a + b - p)(a + b + p - q)} = 0,$$

where notice that $s = a - q$ and $t = p + b$ if $-b < p$, and $s = q - a$ and $t = -b - p$ if $p < -b$. Therefore the point S' lies on the I-tangent expressed by (4). This proves (i).

If $v < p < a$, we get only one congruent pair associated with I given by (11) (see Figure 5). This proves (ii). If $p = v$, then (12) gives one congruent pair associated with I. Substituting $p = v$ in (12), we get

$$e = \frac{2(-ab + bp)}{2a + b - p} = 2(a - \sqrt{a(a + b)}) = 2p \quad \text{and} \quad r = \frac{2b\sqrt{a(a + b)}}{(\sqrt{a} + \sqrt{a + b})^2}.$$

Hence the I-tangent coincides with the line IP and the common radius is given by (3) (see Figure 6). This proves (iii). If $-b < p < v$, then (12) gives two congruent pairs associated with I (see Figure 7). If $p < -b$, (16) also gives two congruent pairs associated with I (see Figure 8). This proves (iv). □

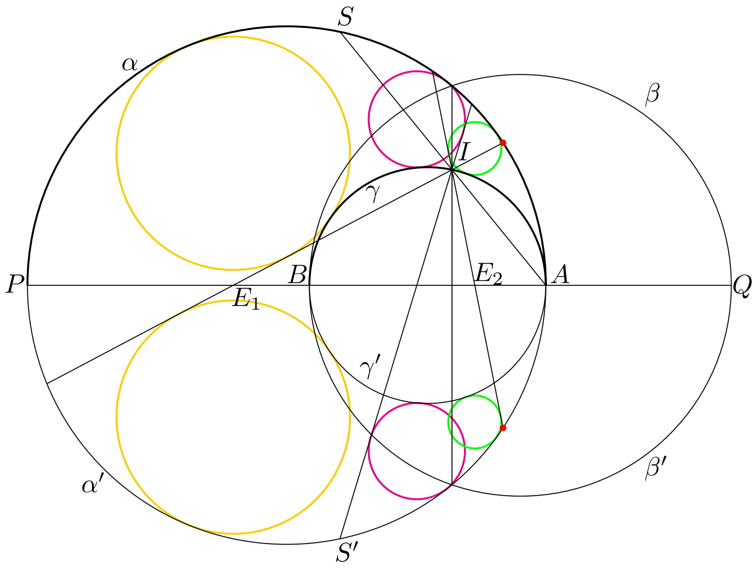

Figure 8: $p < -b$.

4. Two congruent pairs relevant to each other

Let us consider two congruent pairs associated with I. If one of the points of intersection of α and one of their I-tangents coincides with the reflection of the point of intersection of α' and the other I-tangent, the two congruent pairs are said to be *relevant to each other*. In Figure 2, the point A can be regarded as a trivial congruent pair of radius 0 associated with I with I-tangent IS, which are regarded as overlapping two point circles. This is also suggested by (10) and (14). Therefore Figure 2 shows the two congruent pairs with I-tangents IS and IS' relevant to each other.

We assume the case (iv) in Theorem 2. Let E_1 and E_2 be the points of coordinates

$$\left(\frac{2(b(p-a)-\sqrt{w})}{2a+b-p}, 0\right) \text{ and } \left(\frac{2(b(p-a)+\sqrt{w})}{2a+b-p}, 0\right),$$

respectively. The lines E_1I and E_2I are the I-tangents of the two congruent pairs given by (12) and (16). One of the points of intersection of E_1I and α has coordinates

$$\left(-\frac{2a(b+p)}{a-p}, \frac{2\sqrt{aw/b}}{a-p}\right).$$

On the other hand, the point of intersection of E_2I and α' has coordinates

$$\left(-\frac{2a(b+p)}{a-p}, -\frac{2\sqrt{aw/b}}{a-p}\right).$$

Therefore the two congruent pairs with I-tangents E_1I and E_2I are relevant to each other (see Figures 7 and 8, where the two points of intersection are colored in red). Notice that the case (ii) is obtained if the two points and P coincide. Since $P = P'$ in this case, the congruent pair is relevant to itself. Including the trivial congruent pairs and the concept of the relevancy, we restate Theorem 2 as follows.

Theorem 3. *The following statements are true for the collinear arbelos.*
(i) *There are two congruent pairs associated with I relevant to each other in any case. One is the trivial pair consisting of the point circle A with I-tangent IS. The other consist of circles of radius given by* (2) *with I-tangent IS'.*
(ii) *If $v < p < a$, there are exactly two congruent pairs associated with I relevant to each other stated in* (i)*.*
(iii) *If $p = v$, there are exactly three congruent pairs associated with I, two of which are the pairs relevant to each other stated in* (i)*. The remaining congruent pair is relevant to itself and consists of circles of radius given by* (3) *with I-tangent IP.*
(iv) *If $-b < p < v$ or $p < -b$, there are exactly four congruent pairs associated with I, two of which are the pairs relevant to each other stated in* (i)*. The remaining two pairs are also relevant to each other.*

5. The point S

Theorem 2 gives a new characterization of the point S. In this section we consider properties of the point S for the collinear arbelos. We summarize the characterizations of this point:

(i) The point of intersection of AI and α.
(ii) The point of tangency of α and the external common tangent of α and β.
(iii) The point on α of x-coordinate $2r_{\mathrm{A}}$ (resp. $-2r_{\mathrm{A}}$) if $-b < p$ (resp. $p < -b$).
(iv) The point such that the line joining I and the reflection of this point in AB is the I-tangent of a congruent pair associated with I of common radius (2).

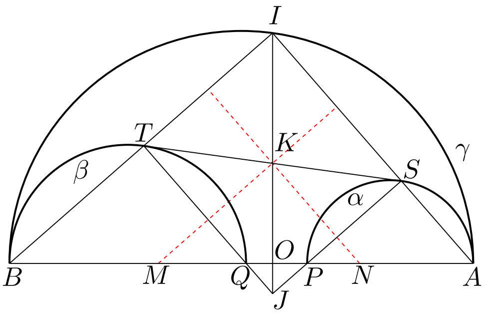

Figure 9.

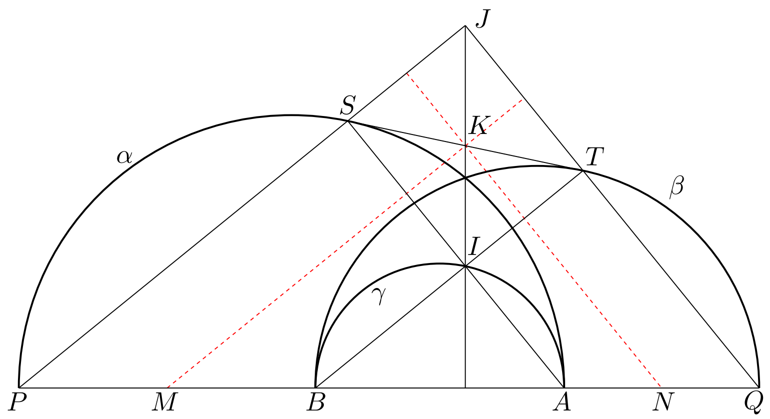

Figure 10.

Let T be the point of intersection of BI and β, and let J be the point of intersection of the lines PS and QT (see Figures 9 and 10). Then $SITJ$ is a rectangle. Since the distances from S and T to the axis are the same and equals to $2r_{\mathrm{A}}$, ST and IJ bisect each other. Let K be the midpoint of ST. Since K lies on the axis and J lies on the line IK, J lies on the axis. If M is the midpoint of BP, then the line KM is the perpendicular bisector of IS and JT. Similarly if N is the midpoint of AQ, then KN is the perpendicular bisector of IT and JS.

References

[1] Akita (秋田義蕃) ed., Sampō Tenzan Tebikigusa Furoku (算法点竄手引草附録) 1834, Tohoku University Digital Collection.
[2] H. Okumura, Ootoba's Archimedean circles of the collinear arbelos, Sangaku J. Math., **4** (2020) pp.31-35.
[3] H. Okumura, A note on the arbelos in Wasan geometry, a problem in Sampō Tenzan Tebikigusa Furoku, Sangaku J. Math., **3** (2019) 12–14.
[4] H. Okumura, Lamoenian circles of the collinear arbelos, KoG, **17** (2013) 9–13.
[5] H. Okumura, Archimedean circles of the collinear arbelos and the skewed arbelos, J. Geom. Graph., **17** (2013) 31–52.
[6] H. Okumura, Ubiquitous Archimedean circles of the collinear arbelos, KoG, **16** (2012) 17–20.