Asymptotic Safety, Black-Hole Cosmology and the Universe as a Gravitating Vacuum State *

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Abstract

A model of the Universe as a dynamical homogeneous anisotropic self-gravitating fluid, consistent with Kantowski-Sachs homogeneous anisotropic cosmology and Black-Hole cosmology, is developed. Renormalization Group (RG) improved black-hole solutions resulting from Asymptotic Safety in Quantum Gravity are constructed which explicitly *remove* the singularities at $t = 0$. Two temporal horizons at $t_- \simeq t_P$ (Planck time) and $t_+ \simeq t_H$ (Hubble time) are found. For times below the Planck time $t < t_P$, and above the Hubble time $t > t_H$, the components of the Kantowski-Sachs metric exhibit a key sign *change*, so the roles of the spatial z and temporal coordinates t are *exchanged*, and one recovers a *repulsive* inflationary de Sitter-like core around $z = 0$, and a Schwarzschild-like metric in the exterior region $z > R_H = 2G_oM$. Therefore, in this fashion one has found a dynamical Universe *inside* a Black Hole whose Schwarzschild radius coincides with the Hubble radius $r_s = 2G_oM = R_H$. For these reasons we conclude by arguing that our Universe could be seen as Gravitating Vacuum State inside a Black-Hole.

Keywords: Black-Hole Cosmology; Asymptotic Safety; Kantowski-Sachs metric; Cosmological Constant.

*Dedicated to the loving memory of Frank (Tony) Smith, a brilliant, and kind American gentleman

1 Introduction : Asymptotic Safety and RG improvement of the Schwarzschild Black Hole

The problem of dark energy and the solution to the cosmological constant problem is one of the most challenging problems facing Cosmology today. There are a vast numerable proposals for its solution. Most recently, the authors [?] have challenged the key assumption made in the discovery of dark energy, and argued why such assumption is an error. They presented some serious evidence for the luminosity evolution in SN (supernova) cosmology. They concluded by saying that future direction of SN cosmology, therefore, should investigate this systematic bias before proceeding to the details of the dark energy.

These recent findings by [?] propelled us to extend our work in [?] and model our Universe as Gravitating Vacuum State inside a Black-Hole. In order to do so, we rely on Kantowski-Sachs homogeneous anisotropic cosmology [?] and Black-Hole cosmology [?], and Asymptotic Safety in Quantum Gravity [?], [?]. Testing Asymptotic Safety at the conceptual level requires the ability to construct approximations of the gravitational renormalization group (RG) flow beyond the realm of perturbation theory. A very powerful framework for carrying out such computations is the Wetterich-Morris functional renormalization group equation (FRGE) for the gravitational effective average action Γ_k [?]

$$k\frac{\partial\Gamma_k[g,\bar{g}]}{\partial k} \;=\; \frac{1}{2}\; Tr\;[\;\frac{k\partial_k\mathcal{R}_k}{\Gamma^{(2)}+\mathcal{R}_k}] \tag{1}$$

where k is the RG mass scale. The construction of the FRGE uses the background field formalism, splitting the metric $g_{\mu\nu}$ into a fixed background $\bar{g}_{\mu\nu}$ and fluctuations $h_{\mu\nu}$. The Hessian $\Gamma_k^{(2)}$ is the second functional derivative of Γ_k with respect to the fluctuation field at a fixed background. The infrared regulator $\mathcal{R}_k$ provides a scale-dependent mass term suppressing fluctuactions with momenta $p^2 < k^2$, while integrating out those with $p^2 > k^2$. The functional trace (matrix-valued operator trace) Tr stands for summation over internal indices, and integration over spacetime and momenta. It appears with positive sign for bosonic fields; a negative sign for fermionic ones, Grassmann odd fields (ghosts), and a factor of two for complex fields.

A priori one may then expect that resulting RG flow may actually depend strongly on the choice of background. As it was explicitly demonstrated in [?], this is *not* the case if the flow is computed via early-time heat-kernel methods so that the background merely serves as a book-keeping device for disentangling the flow of different coupling constants. The authors [?] presented a ten-step algorithm to systematically compute the expansion of such flow equations in a given background quantity specified by the approximation scheme. The method is based on off-diagonal heat-kernel techniques and can be implemented on a computer algebra system, opening access to complex computations in, e.g., Gravity or Yang-Mills theory. In a first illustrative example, they re-derived the gravitational beta-functions of the Einstein-Hilbert truncation, demonstrating their background-independence. It is thus clear that such a computational scheme, enabling to deal with the general operator structures appearing in the FRGE without resorting to a simplifying choice of background, is highly desirable.

The arguably simplest approximation of the gravitational RG flow is obtained from projecting the FRGE onto the Einstein-Hilbert action approximating Γ_k by [?], [?]

$$\Gamma_k \;=\; \int d^4x \,\sqrt{|g|}\;\frac{1}{16\pi G(k)}\;[\;R(g_{\mu\nu})\;-2\;\Lambda(k)\;]\;+\;\cdots \tag{2}$$

where the ellipsis $\cdots$ denote the gauge fixing and ghost terms. This ansatz comprises two scale-dependent coupling constants, Newton's constant G_k and a cosmological constant Λ_k. The scale-dependence of these couplings is conveniently expressed in terms of their dimensionless counterparts $\lambda_k \equiv \Lambda_k k^{-2}$; $g_k \equiv G_k k^2$, and captured by the beta functions

$$\beta_g(g_k,\lambda_k) \;=\; k\partial_k g_k,\quad \beta_\lambda(g_k,\lambda_k) \;=\; k\partial_k\lambda_k \tag{3}$$

Eq-(1) yields a system of coupled differential equations determining the scale-dependence of $G(k), \Lambda(k)$. The interacting (non-Gaussian) ultra-violet (UV) fixed points are determined by the conditions $\beta_g(g_*,\lambda_*) = 0; \beta_\lambda(g_*,\lambda_*) = 0$, with $g_* \neq 0; \lambda_* \neq 0$ and are postulated to correspond to a conformal invariant field theory.

The Renormalization group flow of the gravitational coupling and cosmological constant in Asymptotic Safety was studied by [?]. The scale dependence of $G(k)$ and $\Lambda(k)$ was found to be

$$G(k^2) \;=\; \frac{G_o}{1\;+\;g_*^{-1}\;G_o\;k^2},\quad \Lambda(k) \;=\; \Lambda_o \;+\; \frac{b\;G(k)}{4}\;k^4,\quad \Lambda_o > 0,\quad b > 0 \tag{4}$$

In $D = 4$, the dimensionless gravitational coupling has a nontrivial fixed point $g = G(k)k^2 \to g_*$ in the $k \to \infty$ limit, and the dimensionless variable $\lambda = \Lambda(k)k^{-2}$ has also a nontrivial ultraviolet fixed point $\lambda_* \neq 0$ [?]. The interacting (non-Gaussian) fixed points $g_k = G(k)\;k^2$, and $\lambda_k = \Lambda(k)\;k^{-2}$ in the ultraviolet limit $k \to \infty$ turned out to be, respectively, [?]

$$g_* \;=\; 0.707,\quad \lambda_* \;=\; 0.193,\quad b \;=\; 4\frac{\lambda_*}{g_*}, \tag{5}$$

G_o and Λ_o are the present day value of the Newtonian gravitational coupling and the cosmological constant. The infrared limits are $\Lambda(k \to 0) = \Lambda_0 > 0,\; G(k \to 0) = G_o = G_N$. Whereas the ultraviolet limits are $\Lambda(k = \infty) = \infty; G(k = \infty) = 0$.

The results in eq-(5) have been used by several authors, see [?], [?] and references therein, to construct a renormalization group (RG) improvement of the Schwarzschild Black-Hole Spacetime by recurring to the correspondence $k^2 \to k^2(r)$, which is based in constructing a judicious monotonically decreasing function $k^2 = k^2(r)$, and which in turn allows to replace $G(k^2) \to G(r)$.

Let us start with the renormalization-group improved Schwarzschild black-hole metric [?]

$$(ds)^2 \;=\; -\;(1-\frac{2G(r)M_o}{r})(dt)^2 \;+\;(1-\frac{2G(r)M_o}{r})^{-1}(dr)^2 \;+\; r^2(d\Omega_2)^2 \quad (5)$$

based on the Renormalization group flow of $G(r)$ in the Asymptotic Safety program [?]. The metric (5) is *not* a solution of the vacuum field equations but instead is a solution to the modified Einstein equations $G^\mu_\nu = 8\pi G(r) T^\mu_\nu$ where the *running* Newtonian coupling $G(r)$ and an effective stress energy tensor

$$T^\mu_\nu \;\equiv\; diag\;(-\rho(r),\; p_r(r),\; p_\theta(r),\; p_\varphi(r)) \quad (6)$$

appears in the right hand side. The components of T^μ_ν associated to the modified Einstein equations $G^\mu_\nu = 8\pi G(r) T^\mu_\nu$ are respectively given by

$$\rho \;=\; -\;p_r \;=\; \frac{M}{4\pi r^2 G(r)}\;\frac{dG(r)}{dr},\;\; p_\theta \;=\; p_\varphi \;=\; -\;\frac{M}{8\pi r G(r)}\;\frac{d^2G(r)}{dr^2} \quad (7)$$

The energy-momentum tensor is in this case an $effective$ stress energy tensor resulting from vacuum polarizations effects of the quantum gravitational field [?] (like a quantum-gravitational self-energy). As explained by [?], the quantum system is self-sustaining: a small variation of the Newton's constant triggers a ripple effect, consisting of successive back-reactions of the semi-classical background spacetime which, in turn, provokes further variations of the Newton's coupling and so forth.

As a result, the sequence of RG improvements is completely determined by a series of recursive relations. In the limiting case, after choosing the following monotonically decreasing function $k^2 = k^2(r) = \xi G_o \rho(r) = \frac{\xi G_o M}{4\pi r^2 G(r)} \frac{dG(r)}{dr}$, where ξ is a positive constant, and upon substituting $k^2 = k^2[\rho(r)]$ into the right-hand side of the running gravitational coupling $G(k^2(r))$ in eq-(4), leads to the differential equation for the sought-after functional form of $G(r)$

$$G(r) \;=\; \frac{G_o}{1\;+\;g_*^{-1}\;G_o\;k^2[\rho(r)]},\;\; k^2[\rho(r)] \;=\; \xi\;G_o\;\rho(r) \;=\; \frac{\xi G_o M}{4\pi r^2 G(r)}\;\frac{dG(r)}{dr} \quad (8)$$

The solution to the differential equation is [?]

$$G(r) \;=\; G_o\;(1-e^{-r^3/r_s l_{cr}^2});\;\; r_s \;=\; 2G_oM,\;\; l_{cr} \;=\; \sqrt{\frac{3\xi}{8\pi g_*}}\;L_{Planck} \quad (9)$$

leading to a Dymnikova-type of metric [?] in eq-(5). We shall choose $\xi = (8\pi g_*/3) \Rightarrow l_{cr} = L_P$, where L_P is the PLanck length scale. A simple inspection reveals that there is *no* singularity at $r = 0$. An expansion of the exponential gives for very small values of $r:\; 1-(2G(r)M)/r) \simeq 1-(2G_oMr^2/2G_oML_P^2) = 1-(r^2/L_P^2)$, and one recovers a *repulsive* de Sitter core around the origin $r = 0$.

Hence, the key result of [?] is that if the gravitational renormalization group (RG) flow attains a non-trivial fixed point at high energies, the back-reaction effects produced by the running Newton's coupling leads to an iterated sequence of recurrence relations which converges to a "renormalized" black-hole spacetime of the Dymnikova-type, which is free of singularities.

In particular, a repulsive de Sitter behavior has also been been found in [?], and also in the gravastar (gravitational vacuum star) picture, proposed by [?] where the gravastar has an effective phase transition at/near where the event horizon is expected to form, and the interior is replaced by a de Sitter condensate. Based on these ideas, and the RG-improved black-hole solutions resulting from Asymptotic Safety, we shall proceed with our proposal that our Universe could be seen as Gravitating Vacuum State inside a Black-Hole.

2 Renormalization-Group-Improved Kantowski-Sachs-like Metrics

Kantowski-Sachs metric and Schwarzschild Black Hole Interior

Adopting the units $c = 1$, and after replacing the radial variable r for the spatial coordinate z (for reasons explained below), the Kantowski-Sachs metric associated with the *interior* region of Schwarzschild black-hole is given by

$$(ds)^2 \; = \; - \; \left(\frac{2G_oM}{t} \; - \; 1 \right)^{-1} (dt)^2 + \left(\frac{2G_oM}{t} \; - \; 1 \right) (dz)^2 \; + t^2 (d\Omega_2)^2 \quad (10)$$

Such metric is a solution of the Einstein vacuum field equations and was analyzed in full detail by the authors [?]. As it is well known to the experts inside the black-hole horizon region the roles of r and t are exchanged. The Kantowski-Sachs metrics [?] are associated with spatially homogeneous *anisotropic* relativistic cosmological models [?].

The black hole mass parameter M in (10) assumes the role now of a characteristic *time* for the existence of universes inside the interior Schwarzschild solution, as may be inferred from the cosmological interpretation of the interior metric [?]. For example, in Black-Hole Cosmology [?] one sets M to coincide with the mass of the Universe enclosed inside the Hubble horizon radius R_H, and which also coincides with the Schwarzschild radius $2G_oM$. Hence, the characteristic time will be set equal to the Hubble horizon time $t_H \equiv 2G_oM$. An interesting numerical cincidence is that the uniform density over a spherical ball of radius R_H given by $M/(4\pi/3)R_H^3 = \frac{3}{8\pi G_o R_H^2}$ coincides precisely with the observed critical density (also vacuum density) of our universe. For more details of Black Hole cosmology see [?].

Given the metric (10), the scalar Kretschmann invariant polynomial is

$$K = R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} = 48 \; \frac{(G_oM)^2}{t^6}$$

showing that a curvature singularity occurs at $t = 0$. This is our main motivation to recur to Asymptotic Safety in Quantum Gravity in oder to provide a Renormalization-Group improved version of the Kantowski-Sachs metric (10), and show that there is *no* longer a singularity at $t = 0$. Furthermore, we shall also include a running cosmological "constant" $\Lambda(t)$, besides a running gravitational coupling $G(t)$, when we evaluate the variable vacuum energy density, and which asymptotically should tend to the observed (extremely small) vacuum energy density $10^{-122} M_P^4$.

Amongst the most interesting features of the metric (10) that were found by [?] are :

(i) The issue of proper distances. The proper distance between two particles at rest separated by a constant z, decreases along the z-direction as coordinate time flows from $t = 0$ to $t = 2G_o M$, and increases as coordinate time flows backwards from $t = 2G_o M$ to $t = 0$.

Whereas the angular proper distance between two simultaneous events along a spatial trajectory with $dz = 0$ and $\theta = \pi/2$, increases as t varies from $t = 0$ to $t_H = 2G_o M$, and decreases when the temporal coordinate runs backwards from $t_H = 2G_o M$ to $t = 0$. The proper time always runs $forward$.

(ii) The null and timelike geodesics were rigorously examined. In the null geodesic case, when $\theta = (\pi/2)$ and $dz = 0$, they noted that these are *not* circular orbits, as the z coordinate can *no* longer be considered as a radial coordinate.

(iii) Another surprising result, considering the interior point of view, is that the trajectories of particles at rest are geodesics, contrary to the exterior region where particles at rest are necessarily accelerated. This fact is due to the non-static character of the interior geometry.

(iv) The Kruskal diagram for the interior region of the Schwarzschild spacetime reveals this unexpected behavior : A hypothetical test particle starts its movement at the event E_1, i.e., at $t = 0$, and arrives at $t_H = 2GM$ and $z = -\infty$, at event E_2. It re-enters into the interior region at event E_3, corresponding to $t_H = 2G_o M$ and $z = +\infty$, ending up in the spacelike singularity at $t = 0$, at event E_4.

As viewed from an interior observer the test particle exits the interior region, at $t_H = 2G_o M$ and $z = -\infty$, to reappear instantaneously at $t = 2G_o M$, at the positive side of the z axis (at $z = \infty$). According to the point of view of the interior observer, *no* coordinate time has elapsed during the test particles excursion in the exterior region. This analysis is analogous to the one outlined in [?].

(v) Another curious feature was that all infalling null or timelike particles enter into the interior at different places $z = \pm\infty$, but simultaneously at $t_H = 2G_o M$. For further details we refer to [?].

Having described the most interesting features of the metric (10) found by [?] let us proceed with the Renormalization Group-improved Kantowski-Sachs metric associated with the interior of a black-hole. It is given by

$$(ds)^2 \ = \ - \ \left(\frac{2G(t)M}{t} \ - \ 1 \right)^{-1} (dt)^2 \ + \ \left(\frac{2G(t)M_o}{t} \ - \ 1 \right) (dz)^2 \ + \ t^2 (d\Omega_2)^2 \tag{11}$$

The modified Einstein equations are $G^\mu_\nu \ = \ 8\pi G(t) T^\mu_\nu$, where as before, the *running* Newtonian coupling $G(t)$, and the effective stress energy tensor due to vacuum polarizations effects of the quantum gravitational field [**?**] appear in the right hand side. Dirac proposed long ago the possibility of the temporal variation of the fundamental constants.

The energy-momentum tensor corresponding to the modified Einstein equations is

$$T^\mu_\nu \ \equiv \ diag \ (-\rho(t), \ p_z(t), \ p_\theta(t), \ p_\varphi(t)) \tag{12}$$

and whose components are respectively given by

$$\rho \ = \ - \ p_z \ = \ \frac{M}{4\pi \ t^2 \ G(t)} \ \frac{dG(t)}{dt}, \ \ p_\theta \ = \ p_\varphi \ = \ - \ \frac{M}{8\pi \ t \ G(t)} \ \frac{d^2G(t)}{dt^2} \tag{13}$$

After choosing the monotonically decreasing function of time

$$k^2 \ = \ k^2(t) \ = \ \xi \ G_o \ \rho(t) \ = \ \xi \ G_o M \ \frac{(dG(t)/dt)}{4\pi \ t^2 G(t)} \tag{14}$$

the running gravitational coupling $G(t)$ obtained in the dynamical renormalization of the Kantowski-Scachs-like metric (10) is given by the solution to the differential equation

$$G(t) \ = \ \frac{G_o}{1 \ + \ g_*^{-1} \ G_o \ k^2[G(t)]}, \ \ k^2[G(t)] \ = \ \xi \ G_o \ \rho(t) \ = \ \frac{\xi G_o M}{4\pi t^2 G(t)} \ \frac{dG(t)}{dt} \tag{15}$$

The solution to the above differential equation has the same functional form as before

$$G(t) \ = \ G_o \ (1 - e^{-t^3/t_s t^2_{cr}}); \ \ t_s \ = \ 2G_o M = t_H, \ \ t_{cr} \ = \ \sqrt{\frac{3\xi}{8\pi g_*}} \ t_{Planck} \tag{16}$$

We shall set again $\xi \ = \ (8\pi g_*/3) \ \Rightarrow \ t_{cr} \ = \ t_P$, Planck's time. Note also that the expression $G(t)$ (16) has the following correspondence (in natural units $\hbar = c = 1$)

$$r \ \leftrightarrow \ t, \ \ l_{cr} = L_P \ \leftrightarrow \ t_P, \ \ r_s = 2G_o M \ \leftrightarrow \ t_H \tag{17}$$

with the prior solution $G(r)$ of eq-(9). The Schwarzschild radius r_s (black hole horizon) corresponds now to the cosmological horizon R_H (Hubble radius), and the Planck scale L_P corresponds to the Planck time t_P.

Once again, we find that when $G(t)$ is given by eq-(16) there is *no* singularity at $t = 0$. Given that $t_{cr} = t_P$; $t_s = 2G_o M = t_H$, a simple expansion of the exponential for very small values of t gives $(2G(t)M/t)-1 \simeq (t^2/t_P^2)-1$, leading to no singularity of the metric at $t=0$. There are two temporal horizons, $t_- \simeq t_P$; $t_+ \simeq t_H$ around the Planck and Hubble time, respectively. Taking the trace of $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G(t)T_{\mu\nu}$ yields the scalar curvature R in terms of the trace of the stress energy tensor $T = -\rho_r + p_r + p_\theta + p_\phi$, which in turn gives

$$R(t) \ \simeq \ \frac{1}{t_P^2} \left(A \ - \ \frac{Bt^3}{t_P^2 t_H} \right) \ e^{-(t^3/t_P^2 t_H)} \tag{18}$$

where A, B are numerical factors. At $t = 0 \Rightarrow R \simeq \frac{1}{t_P^2} =$ finite. Therefore, one has achieved in eq-(11) a dynamical regularization of the Kantowski-Sachs metric : there is *no* singularity at $t = 0$. The scalar curvature vanishes in the $t = \infty$ limit.

Inserting the solution (16) found for $G(t)$ into eq-(14), $k^2(t)$ becomes

$$k^2(t) \ = \ \xi \ G_o M \ \frac{3}{4\pi t_P^2 t_H} \ \frac{exp(-t^3/t_P^2 t_H)}{(\ 1 \ - \ exp(-t^3/t_P^2 t_H)\ )} \tag{19}$$

with $G_o = t_P^2$, and $2G_oM = t_H$. And the variable energy density is

$$\rho(t) \ = \ M \ \frac{(dG(t)/dt)}{4\pi \ t^2 G(t)} \ = \ \frac{3M}{4\pi t_P^2 t_H} \ \frac{exp(-t^3/t_P^2 t_H)}{(\ 1 \ - \ exp(-t^3/t_P^2 t_H)\ )} \tag{20}$$

The density *blows* up at $t = 0$, and is zero at $t = \infty$. However the Ricci tensor and the scalar curvature (Einstein tensor) are *finite* at $t = 0$. The reason being that when $G(t = 0) = 0; \rho(t = 0) = \infty$, their product $G(t = 0)\rho(t = 0) \simeq \frac{3}{t_P^2}$ is *finite*.

To sum up : one has attained a dynamical regularization of the Kantowski-Sachs metric (11) associated with a black hole interior and that there is *no* singularity at $t = 0$. Two temporal horizons at $t_- \simeq t_P$ and $t_+ \simeq t_H$ are found. For times below the Planck scale $t < t_P$, and above the Hubble time $t > t_H$, the components of the Kantowski-Sachs metric (11) exhibit a key sign *change*, so the roles of the spatial z and temporal coordinates t are *exchanged*, and one recovers a *repulsive* inflationary de Sitter-like core around $z = 0$, and a Schwarzschild-like metric in the exterior region $z > R_H = 2G_oM$. Therefore, truly one has a dynamical Universe *inside* a Black Hole horizon $r_s = 2G_oM = R_H$. If one views the presence of the temporal horizons at t_P and t_H as signals of a phase transition, then our findings are compatible with the recent work by [**?**] about a new quantum phase of the Universe before inflation and its cosmological and dark energy implications.

Concluding, we have modeled our Universe as a homogeneous *anisotropic* self-gravitating fluid consistent with the Kantowski-Sachs homogeneous anisotropic cosmology and Black-Hole cosmology. If one wishes, one can repeat the whole calculations and *include* the running cosmological constant $\Lambda(k^2(t))$ if one desires to identify the running cosmological constant with the running vacuum energy density. Below we shall include the running cosmological constant in order to suitably *modify* the Kantowski-Sachs-like metric (10). This will *change* the expression for $k^2(t)$ in (18), and in turn, lead to a very different expression for $\rho(t)$ than the one provided by eq-(19).

Inclusion of the Running Cosmological Constant

Let us introduce the running cosmological constant $\Lambda(t)$ into the following RG improved and modified Kantowski-Sachs metric

$$(ds)^2 \;=\; \left(\frac{2G(t)M}{t} \;+\; \frac{\Lambda(t)}{3}t^2 \;-\; 1\right)^{-1} (dt)^2 + \left(\frac{2G(t)M_o}{t} \;+\; \frac{\Lambda(t)}{3}t^2 \;-\; 1\right)(dz)^2$$

$$+\; t^2\;(d\Omega_2)^2 \tag{20}$$

Given the effective stress energy tensor associated with a self-gravitating anisotropic fluid Universe

$$T_\mu^\nu \;=\; diag\;(-\;\rho(t),\; p_z(t),\; p_\theta(t),\; p_\phi(t)) \tag{21}$$

the modified Einstein equations with a running cosmological and gravitational constant

$$R_\mu^\nu \;-\frac{1}{2}\;\delta_\mu^\nu\;R \;+\; \Lambda(t)\;\delta_\mu^\nu \;=\; 8\pi G(t)\;T_\mu^\nu \tag{22}$$

and corresponding to the metric (20) become

$$\Lambda(t) \;-\; \frac{2M}{t^2}\;\frac{d}{dt}\;\left(\;G(t)\;+\;\frac{\Lambda(t)t^3}{6M}\;\right) \;=\; -\;8\pi G(t)\;\rho(t) \tag{23a}$$

$$\Lambda(t) \;-\; \frac{2M}{t^2}\;\frac{d}{dt}\;\left(\;G(t)\;+\;\frac{\Lambda(t)t^3}{6M}\;\right) \;=\; 8\pi G(t)\;p_z(t) \tag{23b}$$

$$\Lambda(t) \;-\; \frac{M}{t}\;\frac{d^2}{dt^2}\;\left(\;G(t)\;+\;\frac{\Lambda(t)t^3}{6M}\;\right) \;=\; 8\pi G(t)\;p_\theta(t) \tag{23c}$$

$$\Lambda(t) \;-\; \frac{M}{t}\;\frac{d^2}{dt^2}\;\left(\;G(t)\;+\;\frac{\Lambda(t)t^3}{6M}\;\right) \;=\; 8\pi G(t)\;p_\phi(t) \tag{23d}$$

From eqs-(23) one can read-off the expressions for the density and pressure

$$\rho(t) \;=\; -\;p_z(t) \;=\; \frac{M}{4\pi\;t^2\;G(t)}\;\frac{d}{dt}\;\left(\;G(t)\;+\;\frac{\Lambda(t)t^3}{6M}\;\right) \;-\; \frac{\Lambda(t)}{8\pi G(t)} \;=$$

$$\frac{M}{4\pi t^2 G(t)}\,\frac{dG(t)}{dt} \;+\; \frac{1}{24\pi}\;\frac{t}{G(t)}\;\frac{d\Lambda(t)}{dt} \tag{27}$$

$$p_{\theta\theta} \;=\; p_{\phi\phi} \;=\; \frac{\Lambda(t)}{8\pi G(t)} \;-\; \frac{M}{8\pi\;t\;G(t)}\;\frac{d^2}{dt^2}\;\left(\;G(t)\;+\;\frac{\Lambda(t)t^3}{6M}\;\right) \tag{25}$$

The relation $\rho(t) = -p_z(t)$ bears the same form as the dark energy equation of state $\rho = -p$. The conservation of energy $\nabla_\nu(8\pi G(t) T^\nu_\mu - \Lambda(t)\delta^\nu_\mu) = 0$ follows directly from the Bianchi identities and leads to the relation between $\rho = -p_z$ and the tangential pressure components of the anisotropic fluid (Universe) $p_\theta = p_\phi$

$$\frac{d}{dt}\left(G(t)\rho(t) + \frac{\Lambda(t)}{8\pi} \right) + \frac{2\ G(t)\ (\ \rho(t) + p_\theta(t)\ )}{t} = 0 \tag{26}$$

The $k^2 \leftrightarrow \rho(t)$ relation is postulated to be of the same form as before $k^2 = \xi G_o \rho$ (ξ is a positive numerical constant) but where now $\rho(t)$ is given by eq-(24)

$$k^2(t) = k^2[\rho(t)] = k^2[G(t); \Lambda(t)] = \xi\ G_o\ \rho(t) =$$
$$\xi\ G_o \left(\frac{M}{4\pi\ t^2\ G(t)} \frac{d}{dt} [\ G(t) + \frac{\Lambda(t) t^3}{6M}\ ] - \frac{\Lambda(t)}{8\pi G(t)} \right) =$$
$$\xi\ G_o \left(\frac{M}{4\pi t^2 G(t)} \frac{dG(t)}{dt} + \frac{1}{24\pi} \frac{t}{G(t)} \frac{d\Lambda(t)}{dt} \right) \tag{27}$$

Upon substituting the above expression (27) for $k^2[\rho(t)] = k^2[G(t); \Lambda(t)]$ (given in terms of $G(t), \Lambda(t)$ and their first order derivatives) into the right hand side of the running gravitational coupling

$$G(t) = \frac{G_o}{1 + g_*^{-1}\ G_o\ k^2[\rho(t)]} = \frac{G_o}{1 + g_*^{-1}\ G_o\ k^2[G(t); \Lambda(t)]} \tag{28a}$$

it furnishes one differential equation involving $G(t)$ and $\Lambda(t)$

$$G(t) + \xi \frac{G_o^2}{g_*} \left(\frac{M}{4\pi t^2} \frac{dG(t)}{dt} + \frac{t}{24\pi} \frac{d\Lambda(t)}{dt} \right) - G_o = 0 \tag{28b}$$

The second differential equation is obtained from the running cosmological constant

$$\Lambda(t) = \Lambda_o + \frac{b}{4}\ G(t)\ k^4[\rho(t)] = \Lambda_o + \frac{b}{4}\ G(t)\ k^4[G(t); \Lambda(t)] \tag{29a}$$

where k^4 is the square of the expression $k^2[G(t); \Lambda(t)]$ displayed in eq-(27)

$$\Lambda(t) - \Lambda_o - \frac{b}{4}\ \xi^2\ \frac{G_o^2}{G(t)} \left(\frac{M}{4\pi t^2} \frac{dG(t)}{dt} + \frac{t}{24\pi} \frac{d\Lambda(t)}{dt} \right)^2 = 0 \tag{29b}$$

The differential equations (28b,29b) comprise a very complicated coupled system of two first-order non-linear differential equations whose origin stems from the *running* gravitational coupling, and cosmological constant, combined with the RG improvement of the Einstein field equations with a cosmological constant, and associated with the Kantowski-Sachs-like metric of eq-(20). By eliminating one of the functions, the two $first$-order nonlinear differential equations (NLDE) can be reduced to a single *second*-order NLDE. Eliminating $\Lambda(t)$ from eqs-(28b, 29b) yields the following second order NLDE for $G(t)$:

$$\frac{24\pi g_*}{\xi G_o^2 t}\ [\ G_o - G(t)\ +\ \frac{\xi G_o^2 M}{4\pi g_* t^2}\ \frac{dG(t)}{dt}\ ]\ +$$

$$\frac{b\xi^2 G_o^2}{4}\ \frac{(dG(t)/dt)}{G(t)^2}\ \left(\ \frac{M}{4\pi t^2}\frac{dG(t)}{dt}\ +\ \frac{g_*}{\xi G_o^2 G(t)}\ (\ G_o - G(t)\ +\ \frac{\xi G_o^2 M}{4\pi g_* t^2}\ \frac{dG(t)}{dt}\ )\ \right)^2$$

$$-\ \frac{b\xi^2 G_o^2}{4G(t)}\ \frac{d}{dt}\ \left(\ \frac{M}{4\pi t^2}\frac{dG(t)}{dt}\ +\ \frac{g_*}{\xi G_o^2 G(t)}\ (\ G_o - G(t)\ +\ \frac{\xi G_o^2 M}{4\pi g_* t^2}\ \frac{dG(t)}{dt}\ )\ \right)^2\ =\ 0 \tag{30}$$

Since the NLDE eq-(30) is of second order one requires to impose boundary conditions on $G(t)$ and $(dG(t)/dt)$. The choice of boundary conditions must be consistent with the FRGE flow solutions (4) which require that for very late times $t \to \infty$: $G(t) \to G_o$; $\Lambda(t) \to \Lambda_o$. Therefore, the solutions to the coupled system of differential equations (28b,29b) must obey the boundary conditions :

In the asymptotic $t \to \infty$ limit one should have : $G(t) \to G_o$; $\Lambda(t) \to \Lambda_o$; $\frac{dG(t)}{dt} \to 0$; $\frac{d\Lambda(t)}{dt} \to 0$. And, in this way $\rho_{vac}(t) \to \frac{\Lambda_o}{8\pi G_o} \simeq 10^{-122} M_P^4$ one recovers the observed vacuum energy density.

And, when $t \to 0$: $\Lambda(t)$ and $(d\Lambda/dt) \to \infty$; $G(t)$ and $(dG/dt) \to 0$; while $\rho_{vac} \to \infty$. Having solved the complicated system of differential equations for $G(t), \Lambda(t)$, the temporal behavior of the *running* vacuum energy density is given by $\rho_{vac}(t) = \frac{\Lambda(t)}{8\pi G(t)}$. It will be extremely small $\frac{\Lambda_o}{8\pi G_o} \sim 10^{-122} M_P^4$ at $t \to \infty$, and it blows up at $t = 0$. Due to the regularization effects of the running $G(t), \Lambda(t)$, the curvature scalar R, Ricci tensor $R_{\mu\nu}$, and the products $G(t)\rho(t); G(t)\rho_{vac}(t)$ are $finite$ at $t = 0$. It is beyond the scope of this work and my computational abilities to analytically and numerically solve eq-(30). For these reasons we can only provide physical reasonings rather than solving eq-(30). We believe that the consequences of the findings in this work deserve further investigation.

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