

Assuming $c < rad^2(abc)$, The abc Conjecture is True

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Abstract

In this paper, assuming that $c < rad^2(abc)$ is true, we give a proof of the abc conjecture by proposing the expression of the constant $K(\epsilon)$, then we approve that $\forall \epsilon > 0$, for a, b, c positive integers relatively prime with $c = a + b$, we have $c < K(\epsilon).rad(abc)^{1+\epsilon}$. Some numerical examples are given.

Résumé

Assumant que la conjecture $c < rad^2(abc)$ est vraie, on donne une démonstration de la conjecture abc en proposant la constante $K(\epsilon)$. On approuve alors pour tout $\epsilon > 0$, et pour tout triplet (a, b, c) avec $c = a + b$ et a, b, c des entiers positifs relativement premiers, on a $c < K(\epsilon)rad^{1+\epsilon}(abc)$. Des exemples numériques sont présentés.

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*To the memory of my Father who taught me arithmetic
To my wife Wahida and my children Sinda and Mohamed Mazen*

1 Introduction and notations

Let a positive integer $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as :

$$a = \prod_i a_i^{\alpha_i} = rad(a). \prod_i a_i^{\alpha_i - 1} \quad (1)$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \implies a = \mu_a . rad(a) \quad (2)$$

The abc conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Esterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the abc conjecture is given below:

Conjecture 1.1. (abc Conjecture): Let a, b, c positive integers relatively prime with $c = a + b$, then for each $\epsilon > 0$, there exists a constant $K(\epsilon)$ such that :

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \quad (3)$$

$K(\epsilon)$ depending only of ϵ .

The idea to try to write a paper about this conjecture was born after the publication in November 2018, of an article in *Quanta* magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [2]. The difficulty to find a proof of the *abc* conjecture is due to the incomprehensibility how the prime factors are organized in c giving a, b with $c = a + b$. So, I will give a simple proof that can be understood by undergraduate students.

We know that numerically, $\frac{\text{Log } c}{\text{Log}(\text{rad}(abc))} \leq 1.629912$ [1]. A conjecture was proposed that $c < \text{rad}^2(abc)$ [3]. It is the key to resolve the *abc* conjecture. In my paper, I assume that the conjecture $c < \text{rad}^2(abc)$ holds, I propose an elementary proof of the *abc* conjecture. The paper is organized as follows: in the second section, we give the proof of the *abc* conjecture. The numerical examples are discussed in sections three and four.

2 The Proof of the *abc* conjecture

We note $R = \text{rad}(abc)$ in the case $c = a + b$ or $R = \text{rad}(ac)$ in the case $c = a + 1$. We assume that $c < R^2$ is true. We recall the following proposition [4]:

Proposition 2.1. Let $\epsilon \rightarrow K(\epsilon)$ the application verifying the *abc* conjecture, then:

$$\lim_{\epsilon \rightarrow 0} K(\epsilon) = +\infty \quad (4)$$

2.1 Case : $\epsilon \geq 1$

Using the result that $c < R^2$, we have $\forall \epsilon \geq 1$:

$$c < R^2 \leq R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad \text{with } K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \quad \epsilon \geq 1 \quad (5)$$

We verify easily that $K(\epsilon) > 1$ for $\epsilon \geq 1$ and $K(\epsilon)$ verifies the proposition (4). Then the *abc* conjecture is true.

2.2 Case: $\epsilon < 1$

2.2.1 Case: $c < R$

In this case, we can write :

$$c < R < R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad \text{with } K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \quad \epsilon < 1 \quad (6)$$

here also $K(\epsilon) > 1$ for $\epsilon < 1$ and it verifies the proposition (4). Hence the *abc* conjecture is true.

2.2.2 Case: $c > R$

In this case, we confirm that :

$$c < K(\epsilon).R^{1+\epsilon}, \quad \text{with } K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, 0 < \epsilon < 1 \quad (7)$$

If not, then $\exists \epsilon_0 \in]0, 1[$, so that the triples (a, b, c) checking $c > R$ and:

$$c < R^{1+\epsilon_0}.K(\epsilon_0) \quad (8)$$

are in finite number. As we have $c > R$:

$$\begin{aligned} c < R^{1+\epsilon_0}.K(\epsilon_0) &\implies c < c^{1+\epsilon_0}.K(\epsilon_0) \implies 1 < c^{\epsilon_0}.K(\epsilon_0) \\ &\implies c > \frac{1}{K(\epsilon_0)^{1/\epsilon_0}} \implies c > 1 \end{aligned} \quad (9)$$

As $K(\epsilon_0) > 1$, we obtain $c > 1$. We deduce that it exists an infinity of triples (a, b, c) verifying (8), hence the contradiction. Then the proof of the abc conjecture is finished. We obtain that $\forall \epsilon > 0$, $c = a + b$ with a, b, c relatively coprime:

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \quad \text{with } K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)} \quad \epsilon > 0 \quad (10)$$

and the constant $K(\epsilon)$ verifies the proposition (4).

Q.E.D

In the two following sections, we are going to verify some numerical examples.

3 Examples : Case $c = a + 1$

3.1 Example 1

The example is given by:

$$1 + 5 \times 127 \times (2 \times 3 \times 7)^3 = 19^6 \quad (11)$$

$a = 5 \times 127 \times (2 \times 3 \times 7)^3 = 47\,045\,880 \implies \mu_a = 2 \times 3 \times 7 = 42$ and $rad(a) = 2 \times 3 \times 5 \times 7 \times 127$, in this example, $\mu_a < rad(a)$.

$c = 19^6 = 47\,045\,881 \implies rad(c) = 19$. Then $rad(ac) = rad(ac) = 2 \times 3 \times 5 \times 7 \times 19 \times 127 = 506\,730$.

We have $c > rad(ac)$ but $rad^2(ac) = 506\,730^2 = 256\,775\,292\,900 > c = 47\,045\,881$.

3.1.1 Case $\epsilon = 0.01$

$c < K(\epsilon).rad(ac)^{1+\epsilon} \implies 47\,045\,881 \stackrel{?}{<} e^{10000}.506\,730^{1.01}$. The expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{\frac{1}{0.0001}} = e^{10000} = 8.7477777149120053120152473488653e + 4342 \quad (12)$$

We deduce that $c \ll K(0.01).506\,730^{1.01}$ and the equation (10) is verified.

3.1.2 Case $\epsilon = 0.1$

$K(0.1) = e^{\frac{1}{0.01}} = e^{100} = 2.6879363309671754205917012128876e + 43 \implies c < K(0.1) \times 506\,730^{1.01}$, and the equation (10) is verified.

3.1.3 Case $\epsilon = 1$

$K(1) = e \implies c = 47\,045\,881 < e.rad^2(ac) = 697\,987\,143\,184,212$ and the equation (10) is verified.

3.1.4 Case $\epsilon = 100$

$$K(100) = e^{0.0001} \implies c = 47\,045\,881 \stackrel{?}{<} e^{0.0001} \cdot 506\,730^{101} = 1.5222350248607608781853142687284e + 576$$

and the equation (10) is verified.

3.2 Example 2

We give here the example 2 from <https://nita.j.users.lmno.cnrs.fr>:

$$3^7 \times 7^5 \times 13^5 \times 17 \times 1831 + 1 = 2^{30} \times 5^2 \times 127 \times 353 \quad (13)$$

$a = 3^7 \times 7^5 \times 13^5 \times 17 \times 1831 = 424\,808\,316\,456\,140\,799 \Rightarrow rad(a) = 3 \times 7 \times 13 \times 17 \times 1831 = 849\,7671 \implies \mu_a > rad(a)$,

$b = 1, rad(c) = 2 \times 5 \times 127 \times 353$ Then $rad(ac) = 849\,767 \times 448\,310 = 3\,809\,590\,886\,010 < c$. $rad^2(ac) = 14\,512\,982\,718\,770\,456\,813\,720\,100 > c$. For example, we take $\epsilon = 0.5$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{1/0.25} = e^4 = 54.59800313096579789056 \quad (14)$$

Let us verify (10):

$$c \stackrel{?}{<} K(\epsilon).rad(ac)^{1+\epsilon} \implies c = 424808316456140800 \stackrel{?}{<} K(0.5) \times (3\,809\,590\,886\,010)^{1.5} \implies 424808316456140800 < 405970304762905691174.98260818045 \quad (15)$$

Hence (10) is verified.

4 Examples : Case $c = a + b$ **4.1 Example 1**

We give here the example of Eric Reyssat [1], it is given by:

$$3^{10} \times 109 + 2 = 23^5 = 6436343 \quad (16)$$

$a = 3^{10} \cdot 109 \Rightarrow \mu_a = 3^9 = 19683$ and $rad(a) = 3 \times 109$,

$b = 2 \Rightarrow \mu_b = 1$ and $rad(b) = 2$,

$c = 23^5 = 6436343 \Rightarrow rad(c) = 23$. Then $rad(abc) = 2 \times 3 \times 109 \times 23 = 15042$. For example, we take $\epsilon = 0.01$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{999.99} = 8.7477777149120053120152473488653e + 4342 \quad (17)$$

Let us verify (10):

$$c \stackrel{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \implies c = 6436343 \stackrel{?}{<} K(0.01) \times (3 \times 109 \times 2 \times 23)^{1.01} \implies \\ 6436343 \ll K(0.01) \times 15042^{1.01} \quad (18)$$

Hence (10) is verified.

4.2 Example 2

The example of Nitaj about the ABC conjecture [1] is:

$$a = 11^{16}.13^2.79 = 613\,474\,843\,408\,551\,921\,511 \Rightarrow rad(a) = 11.13.79 \quad (19)$$

$$b = 7^2.41^2.311^3 = 2\,477\,678\,547\,239 \Rightarrow rad(b) = 7.41.311 \quad (20)$$

$$c = 2.3^3.5^{23}.953 = 613\,474\,845\,886\,230\,468\,750 \Rightarrow rad(c) = 2.3.5.953 \quad (21)$$

$$rad(abc) = 2.3.5.7.11.13.41.79.311.953 = 28\,828\,335\,646\,110 \quad (22)$$

4.2.1 Case 1

we take $\epsilon = 100$ we have:

$$c \stackrel{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \implies \\ 613\,474\,845\,886\,230\,468\,750 \stackrel{?}{<} e^{0.0001} \cdot (2.3.5.7.11.13.41.79.311.953)^{101} \implies \\ 613\,474\,845\,886\,230\,468\,750 < 2.7657949971494838920022381186039e + 1359$$

then (10) is verified.

4.2.2 Case 2 $\epsilon = 0.5$

We take $\epsilon = 0.5$, then:

$$c \stackrel{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \implies \quad (23)$$

$$613\,474\,845\,886\,230\,468\,750 \stackrel{?}{<} e^4 \cdot (2.3.5.7.11.13.41.79.311.953)^{1.5} \implies$$

$$613\,474\,845\,886\,230\,468\,750 < 8\,450\,961\,319\,227\,998\,887\,403.9993 \quad (24)$$

We obtain that (10) is verified.

4.2.3 Case 3 $\epsilon = 1$

We take $\epsilon = 1$, then

$$c \stackrel{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \implies$$

$$613\,474\,845\,886\,230\,468\,750 \stackrel{?}{<} e \cdot (2.3.5.7.11.13.41.79.311.953)^2 \implies$$

$$613\,474\,845\,886\,230\,468\,750 < 831\,072\,936\,124\,776\,471\,158\,132\,100 \times e \quad (25)$$

We obtain that (10) is verified.

4.3 Example 3

It is of Ralf Bonse about the ABC conjecture [3] :

$$2543^4.182587.2802983.85813163 + 2^{15}.3^{77}.11.173 = 5^{56}.245983 \quad (26)$$

$$a = 2543^4.182587.2802983.85813163$$

$$b = 2^{15}.3^{77}.11.173$$

$$c = 5^{56}.245983$$

$$\text{rad}(abc) = 2.3.5.11.173.2543.182587.245983.2802983.85813163$$

$$\text{rad}(abc) = 1.5683959920004546031461002610848e + 33 \quad (27)$$

4.3.1 Case 1 $\epsilon = 10$

For example, we take $\epsilon = 10$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{0.01} = 1.007815740428295674320461741677$$

Let us verify (10):

$$\begin{aligned} c &\stackrel{?}{<} K(\epsilon).\text{rad}(abc)^{1+\epsilon} \Rightarrow c = 5^{56}.245983 \stackrel{?}{<} \\ e^{0.01} &.(2.3.5.11.173.2543.182587.245983.2802983.85813163)^{11} \\ \Rightarrow &3.4136998783296235160378273576498e + 44 < \\ &1.423620059649490817600812092572e + 365 \end{aligned} \quad (28)$$

The equation (10) is verified.

4.3.2 Case 2 $\epsilon = 0.4$

We take $\epsilon = 0.4 \Rightarrow K(\epsilon) = 12.18247347425151215912625669608$, then: The

$$\begin{aligned} c &\stackrel{?}{<} K(\epsilon).\text{rad}(abc)^{1+\epsilon} \Rightarrow c = 5^{56}.245983 \stackrel{?}{<} \\ e^{6.25} &.(2.3.5.11.173.2543.182587.245983.2802983.85813163)^{1.4} \\ \Rightarrow &3.4136998783296235160378273576498e + 44 < \\ &3.6255465680011453642792720569685e + 47 \end{aligned} \quad (29)$$

And the equation (10) is verified.

Ouf, end of the mystery!

5 Conclusion

Assuming $c < R^2$ is true, we have given an elementary proof of the *abc* conjecture, confirmed by some numerical examples. We can announce the important theorem:

Theorem 5.1. *Let a, b, c positive integers relatively prime with $c = a + b$, assuming $c < \text{rad}^2(abc)$ holds, then for each $\epsilon > 0$, there exists $K(\epsilon)$ such that :*

$$c < K(\epsilon).\text{rad}(abc)^{1+\epsilon} \quad (30)$$

where $K(\epsilon)$ is a constant depending of ϵ proposed as :

$$K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \epsilon > 0$$

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