General Order Differentials and Division by Zero Calculus

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Abstract: In this paper, we will give several examples that in the general order n differentials of functions we find the division by zero and by applying the division by zero calculus, we can find the good formulas for $n = 0$. This viewpoint is new and curious at this moment for some general situation. Therefore, as prototype examples, we would like to discuss this property.

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1 Introduction and division by zero calculus

As a typical example, we recall the formula; for the function

$$y = \log x,$$

we have, for general order n derivatives,

$$y^{(n)} = (-1)^{n-1} \frac{(n-1)!}{x^n}. \tag{1.1}$$

How will be the case for $n = 0$ in this formula? We will expect that for $n = 0$, $y = \log x$. However, in this case $(-1)!$ diverges as $\Gamma(0)$. In this short paper, we will show that this curious property may be interpretated by the division by zero, precisely by the division by zero calculus.

We will recall the definition of the division by zero calculus. For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n, \tag{1.2}$$

we **define** the value and any order derivatives of the function f at the singular point a by

$$f^{(n)}(a) = n!C_n. \tag{1.3}$$

For the correspondence (1.3) for the function $f(z)$, we will call it **the division by zero calculus**.

In addition, we will refer to the naturality of the division by zero calculus.

Recall the Cauchy integral formula for an analytic function $f(z)$; for an analytic function $f(z)$ around $z = a$ and for a smooth simple Jordan closed curve $\gamma(a)$ enclosing one time the point a, we have

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{\gamma(a)} \frac{f(z)}{(z-a)^{n+1}} dz. \tag{1.4}$$

Even when the function $f(z)$ has any singularity at the point a, we assume that this formula is valid as the division by zero calculus. We define the values of the functions $f(z)$ and $f^{(n)}(z)$ at the singular point $z = a$ with the Cauchy integral.

The division by zero calculus opens a new world since Aristotele-Euclid. See, in particular, [1] and also the references for recent related results.

In particular, in (1.4), of course, $f^{(0)}(a) = f(a)$, however, the situation is not so in (1.1).

2 Interpretation by the division by zero calculus

By using the identity $(n-1)! = \Gamma(n)$ and we obtain, around $n = 0$, by considering an analytic function in n for $\Gamma(n)$

$$\Gamma(n) = \frac{1}{n} - \gamma + \frac{1}{12}(6\gamma^2 + \pi^2)n + ...$$

By the expansion

$$x^{-n} = \exp(-n \log x) = 1 - n \log x + \frac{1}{2} n^2 (\log x)^2 + \cdots,$$

we obtain the result, by the division by zero calculus

$$y^{(0)} = \log x + \gamma.$$

Here, the Euler constant γ appears in an extra way as in an integral constant.

For

$$y = \arctan x,$$

we have the formula

$$y^{(n)} = (n-1)! \cos^n y \sin n \left(y + \frac{\pi}{2} \right). \tag{2.1}$$

From the expansion

$$\cos^n y \sin n \left(y + \frac{\pi}{2} \right) = \left(y + \frac{\pi}{2} \right) n + ()n^2 + \cdots,$$

we have

$$y + \frac{\pi}{2} = \arctan x + \frac{\pi}{2}. \tag{2.2}$$

We consider the function

$$y = a \arctan \frac{x}{a}.$$

Then, for $x > 0$

$$y^{(n)} = (-1)^{n-1} a \frac{(n-1)!}{(a^2 + x^2)^{(n/2)}} \sin \left(n \arctan \frac{a}{x} \right). \tag{2.3}$$

For $x < 0$

$$y^{(n)} = -a \frac{(n-1)!}{(a^2 + x^2)^{(n/2)}} \sin \left(n \arctan \frac{a}{x} \right). \tag{2.4}$$

From the expansion

$$a^{-n}\sin bn = (1 - n\log a + n^2() + \cdots)\left(bn - \frac{b^3n^3}{3!} + \cdots\right),$$

we have

$$y^{(n)} = -a\arctan\frac{a}{x} = a\left(\arctan\frac{x}{a} \pm \frac{\pi}{2}\right).$$

For the function

$$y = \arctan\frac{x\sin\alpha}{1 - x\cos\alpha},$$

we have

$$y^{(n)} = \frac{(n-1)!}{\sin^n\alpha}\sin n(\alpha + y)\sin^n(\alpha + y). \tag{2.5}$$

For the function

$$y = \arctan\frac{x\sin\alpha}{1 + x\cos\alpha},$$

we have

$$y^{(n)} = (-1)^{n-1}\frac{(n-1)!}{\sin^n\alpha}\sin n(\alpha - y)\sin^n(\alpha - y). \tag{2.6}$$

For these functions, from the expansion

$$a^{-n}\sin bn\sin^n(cn) = (1 - n\log a + n^2() + \cdots)\left(bn - \frac{b^3n^3}{3!} + \cdots\right)$$

$$\cdot(1 - n\log c + n^2() + \cdots),$$

we obtain

$$y + \alpha,$$

and

$$y - \alpha,$$

respectively.

3 Interesting examples

In connection with the problem, we will give interesting examples.

For the function

$$y = \frac{ax+b}{cx+d},$$

we have, in general

$$y^{(n)} = (-1)^{n-1} n! \frac{(ad-bc)c^{n-1}}{(cx+d)^{n+1}}.$$

For $n = 0$, however, $y^{(0)} \neq y$.

For the function

$$y = x^3 \log \frac{x}{a},$$

we have, in general,

$$y^{(n)} = (-1)^{n-4} \frac{6(n-4)!}{x^{n-3}}. \tag{3.1}$$

For the case $n = 0$, by the expansion of $\Gamma(n-3)$ at $n = 0$

$$\Gamma(n-3) = -\frac{1}{6n} + \frac{1}{36}(6\gamma - 11) + ()n + \cdots,$$

we have

$$x^3 \log x + \frac{x^3}{60}(6\gamma - 11).$$

For the function

$$y_n = x^{n-1} \log x,$$

we have

$$y_n^{(n)} = \frac{(n-1)!}{x}.$$

Then, for $n = 0$, we have

$$-\frac{\gamma}{x}$$

and it is not y_0. However, they are valid for $n > 0$.

4 Integral formulas

In general order n derivative representations of functions, when we consider negative orders, we have integral formulas for some case.

For example, in (1.1), when we use the expansions

$$\Gamma(n) = -\frac{1}{n+1} + (\gamma - 1) + ()(n+1) + \cdots$$

and

$$\frac{1}{x^n} = x - (n+1)x \log x + ()(n+1)^2 + \cdots,$$

we have the formula

$$\frac{1}{2}x^2 \log x + \frac{1}{4}(3 - 2\gamma)x^2.$$

In (3.1), from the expansions

$$\Gamma(n-3) = \frac{1}{24(n+1)} + \frac{1}{288}(25 - 12\gamma) + ()(n+1) + \cdots,$$

we obtain

$$\frac{1}{4}x^4 \log x - \frac{3}{144}(25 - 12\gamma)x^4.$$

5 Conclusion

Why division by zero for zero order representations for some general differential order representations of functions does happen?

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