Non-Euclidean metric using Geometric Algebra

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Abstract

The Geometric Algebra is a tool that can be used in different disciplines in Mathematics and Physics. In this paper, it will be used to show how the information of the Non-Euclidean metric in a curved space, can be included in the basis vectors of that space. Not needing any external (out of the metric) coordinate system and not needing to normalize or to make orthogonal the basis, to be able to operate in a simple manner. The different types of derivatives of these basis vectors will be shown. In a future revision, the Schwarzschild metric will be calculated just taking the derivatives of the basis vectors to obtain the geodesics in that space.

As Annex, future developments regarding GA are commented: rigid body dynamics, Electromagnetic field, hidden variables in quantum mechanics, specificities of time basis vector, 4π geometry (spin 1/2) and generalization of the Fourier Transform.

Keywords

Non-Euclidean metric, basis vectors, geometric algebra, geometric product, trivector, rigid body, electromagnetic field, hidden variables, time basis vector, spin, generalization of the Fourier Transform

1. Introduction

The Geometric Algebra is a very powerful tool used in Mathematics and Physics. In the chapter 2 and 3, it will be presented. If you are already familiarized with it, you can skip them.

In chapters 4,5 and 6, the way of operating with Euclidean and Non-Euclidean metrics in two and above number of dimensions will be explained. It will be shown that the basis vectors transmit the information of the metric and no normalization or orthogonalization is needed. And no projection, or external basis either.

How, to perform derivatives to the basis vectors is presented in chapter 7. An in the future, the Schwarzschild metric will be derived using these tools in chapter 8.

Special account must be taken to the Annex 1. In this chapter, future developments regarding GA are presented: rigid body dynamics, Electromagnetic field, hidden variables in quantum mechanics, specificities of time basis vector, 4π geometry (spin 1/2) and generalization of the Fourier Transform.

2. Geometric algebra. Geometric product

There are already several papers explaining what geometric algebra (GA) is [1][2][3][4]. So, I will make a presentation as simple as possible, trying to deliver the meanings and information needed. Although this means, that some formalism will be lost in the way. If you want a very formal explanation, please refer to previous referred papers. If you are already familiarized with geometric algebra, you can skip this chapter.

A vector **a** is an oriented segment. It has a length, that we call its modulus |a| and an oriented direction.

Figure 1. Representation of a vector.

Considering two vectors **a** and **b**, the geometric product is defined as:

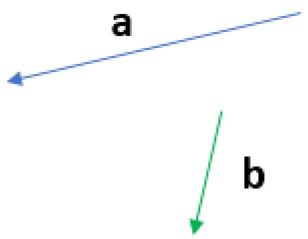


Figure 2. Representation of two vectors.

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a}\wedge\mathbf{b} \qquad (1)$$

Being the first element **a·b**, the inner or scalar product. And the second element **a^b** the so-called outer, exterior or wedge product.

The scalar product is a just a number that is the product of the modulus of the two vectors multiplied by the cosine of the angle that they form, that we call φ.

$$\mathbf{a} \cdot \mathbf{b} = |\mathrm{a}| \cdot |\mathrm{b}| \cdot \cos\varphi \qquad (2)$$

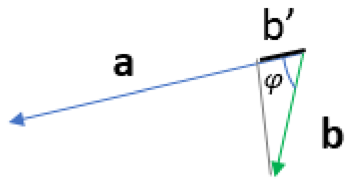


Figure 3. Representation of projection **b'** of vector **b** onto vector **a.**

This scalar represents the multiplication the modulus of one of the vectors (in this case **a**) by the length of the projection of the other vector towards the first one (b' that represents the projection of **b** into **a**).

By trigonometry we know:

$$b' = |b| \cdot \cos\varphi \qquad (3)$$

so:

$$|a| \cdot b' = |a| \cdot |b| \cdot \cos\varphi = \mathbf{a} \cdot \mathbf{b} \qquad (4)$$

The scalar product is commutative, this means, if we operate:

$$\mathbf{b} \cdot \mathbf{a} = |b| \cdot |a| \cdot \cos\varphi = |a| \cdot |b| \cdot \cos\varphi = \mathbf{a} \cdot \mathbf{b} \qquad (5)$$

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \qquad (6)$$

So, for scalar product, this is the message. It is just a number and the product is commutative.

The wedge product **a^b** is more complicated. But it is very important to get the meaning (more than the math) so you can follow the rest of the paper.

The first issue to comment is the modulus of the wedge product:

$$|\mathbf{a}\wedge\mathbf{b}| = |a| \cdot |b| \cdot \sin\varphi \qquad (7)$$

So, the modulus of the wedge product is the product of the modulus of the vectors multiplied by the sine of the angle they form, φ.

But what is **a^b?** Is it a number? Is it a vector? No, it is another entity, called a bivector. As the vector is an oriented segment, the bivector is an oriented surface or plane.

The same way that a vector can represent a magnitude that has a direction like speed, acceleration etc., the bivector can represent a magnitude related to a surface or a plane typically rotations. The bivector can be understood as the parallelogram formed by the two vectors, also as the plane formed by the two vectors, and also as a rotation in the plane formed by the two vectors.

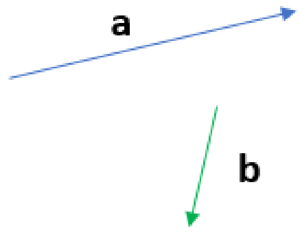


Figure 4. Representation of two vectors.

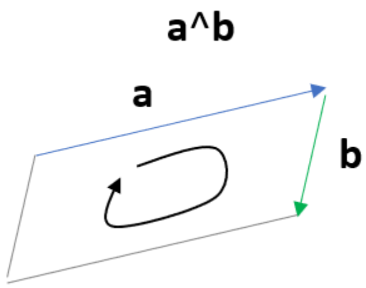


Figure 5. Representation of the bivector **a**^**b**.

In standard algebra, a surface is typically represented by the vector normal to the surface. In geometric algebra if you want to use or operate with a plane or a surface, you do not need to use the normal vector of the surface. You just use the wedge product of two vectors that are included in that surface. The result is a bivector (a different entity) that represents that surface (the same way that a vector represents one dimension, a bivector represents two dimensions).

The same way, in standard algebra, the rotation is represented by the rotation axis (a vector). In geometric algebra a rotation is represented by the plane that rotates (the plane perpendicular to the rotation axis). To define the plane of rotation, you perform the wedge product of two vectors that are in that plane. These two vectors will be perpendicular to the rotation axis. And as a result, you get a bivector that represents that plane.

It is also to be noted that the modulus of the wedge product corresponds to the surface of the parallelogram formed by the two vectors. As we can see here. We see that h is the height of the parallelogram.

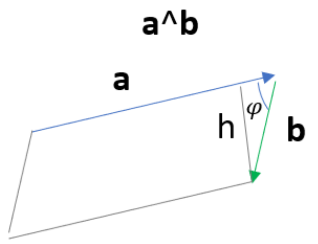


Figure 6. Representation of the height h of the parallelogram of the bivector **a^b**.

and

$$h = |\mathrm{b}| \cdot \sin\varphi \qquad (8)$$

and to calculate the surface:

$$\mathrm{Surface} = |\mathrm{a}| \cdot \mathrm{h} = |\mathrm{a}| \cdot |\mathrm{b}| \cdot \sin\varphi = |\mathbf{a}\wedge\mathbf{b}| \qquad (9)$$

Other issue to be commented is that the wedge product is anticommutative. This means:

$$\mathbf{a}\wedge\mathbf{b} = -\mathbf{b}\wedge\mathbf{a} \qquad (10)$$

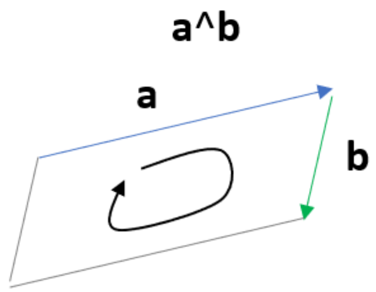


Figure 7. Representation of the bivector **a^b**.

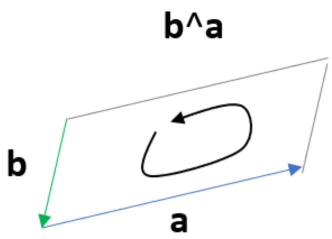


Figure 8. Representation of the bivector **b^a**

The modulus (the surface of the parallelogram) is the same, but the rotation sense is the opposite. We can see that the modulus is the same here:

$$|\mathbf{a}\wedge\mathbf{b}| = |\mathrm{a}| \cdot |\mathrm{b}| \cdot \sin\varphi = |\mathrm{b}| \cdot |\mathrm{a}| \cdot \sin\varphi = |\mathbf{b}\wedge\mathbf{a}| \qquad (11)$$

Another important point is that if two vectors have the same direction, the surface of the parallelogram is zero, so its wedge product is zero. One special case of this situation, is the wedge product of a vector by itself, which result is also zero (they have the same direction):

$$\mathbf{a}\wedge\mathbf{a} = 0 \qquad (12)$$

So, summing up, we have seen that the geometric product of two vectors, give the following result:

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a}\wedge\mathbf{b} = \text{scalar} + \text{bivector} \qquad (13)$$

Being the first element a number (a scalar) and the second element a bivector (as defined above). But can those to items coexist in the same sum? The answer is yes. Exactly the same as we do with complex numbers for example:

$$\mathrm{x} + \mathrm{iy} = \text{real number} + \text{imaginary number} \qquad (14)$$

Where we keep the sum of a real number and an imaginary number. As they are different entities, we can keep the sum of both (but not operate it), we just keep it indicated. The same applies with the geometric product. It gives two different entities that can coexist (a scalar and a bivector) that we cannot operate so we just keep it as indicated.

Due to the commutativity of the scalar product and the anticommutativity of the wedge product, we can get the following expressions:

$$\mathbf{ab} + \mathbf{ba} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a}\wedge\mathbf{b} + \mathbf{b} \cdot \mathbf{a} + \mathbf{b}\wedge\mathbf{a} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a}\wedge\mathbf{b} + \mathbf{a} \cdot \mathbf{b} - \mathbf{a}\wedge\mathbf{b} = \mathbf{2}(\mathbf{a} \cdot \mathbf{b}) \qquad (15)$$

So:

$$\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba}) \qquad (16)$$

The same way:

$$\mathbf{ab} - \mathbf{ba} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a}\wedge\mathbf{b} - \mathbf{b} \cdot \mathbf{a} - \mathbf{b}\wedge\mathbf{a} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a}\wedge\mathbf{b} - \mathbf{a} \cdot \mathbf{b} + \mathbf{a}\wedge\mathbf{b} = \mathbf{2}(\mathbf{a}\wedge\mathbf{b}) \qquad (17)$$

So:

$$\mathbf{a}\wedge\mathbf{b} = \frac{1}{2}(\mathbf{ab} - \mathbf{ba}) \qquad (18)$$

When the vectors are perpendicular, the geometric product and the wedge product gives the same result, because the scalar product is zero, as $\cos(\varphi)=\cos(\pi/2)=0$

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a}\wedge\mathbf{b} = |a| \cdot |b| \cdot \cos\varphi + \mathbf{a}\wedge\mathbf{b} = |a| \cdot |b| \cdot \cos(\frac{\pi}{2}) + \mathbf{a}\wedge\mathbf{b} = \mathbf{a}\wedge\mathbf{b} \quad (19)$$

As commented, when the vectors are colinear the wedge product is zero, as sin(φ)=sin(π/2)=0 . This means, when the vectors are colinear, the geometric product gives the same result as the scalar product:

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a}\wedge\mathbf{b} = \mathbf{a} \cdot \mathbf{b} + |a| \cdot |b| \cdot \sin\varphi = \mathbf{a} \cdot \mathbf{b} + |a| \cdot |b| \cdot \sin(0) = \mathbf{a} \cdot \mathbf{b} \quad (20)$$

A special case, is the geometric product of a vector by itself (which is colinear with itself):

$$\mathbf{a^2} = \mathbf{aa} = \mathbf{a} \cdot \mathbf{a} + \mathbf{a}\wedge\mathbf{a} = \mathbf{a} \cdot \mathbf{a} + 0 = |a|^{\mathbf{2}} \quad (21)$$

This means, as the wedge product is zero, the square of a vector is always a scalar. This scalar is the square of its modulus.

So, we can define a unitary vector as the vector itself divided by its modulus:

$$\frac{\mathbf{a}}{|a|} \quad (22)$$

Or we can define the inverse of a vector as:

$$\mathbf{a}^{-1} = \frac{\mathbf{a}}{|a|^{\mathbf{2}}} \quad (23)$$

We can check that:

$$\mathbf{aa}^{-1} = \mathbf{a}\frac{\mathbf{a}}{|a|^{\mathbf{2}}} = \frac{\mathbf{a^2}}{|a|^{\mathbf{2}}} = \frac{|a|^{\mathbf{2}}}{|a|^{\mathbf{2}}} = 1 \quad (24)$$

As we would expect for the inverse of whatever entity.

We will see that having the definition of the inverse of a vector is KEY for a lot of operations, that we will perform later. It lets us in a way, performs the “division” by vectors (not very formally speaking).

3. Multivectors. Trivector

Once we have seen that scalars and bivectors can be summed, leaving the sum indicated, we can go even further. The same way, that we can sum scalars and bivectors, we can sum scalars, vectors and bivectors (and even, higher grade vectors as we will see later). For example:

$$A = 5 + 3\mathbf{a} + 2\mathbf{b}\wedge\mathbf{c} \quad (25)$$

Here, A is called a multivector. It is an entity that has scalars, vectors and bivectors. Again, this can seem strange, but you can see it, as a polynomial for example:

$$7 + 2x + 9x^2 \qquad (26)$$

In a polynomial, there are different terms that you cannot sum directly but keep them in the sum as indicated, and there is no problem. A multivector is the same, you have scalars, vectors, bivectors (other grades, as trivectors) summed with no problem.
Now, I will take the opportunity to explain the trivector. The trivector is the wedge product of three vectors that are not in the same plane (this means, they are able to create a volume -not just a surface-).

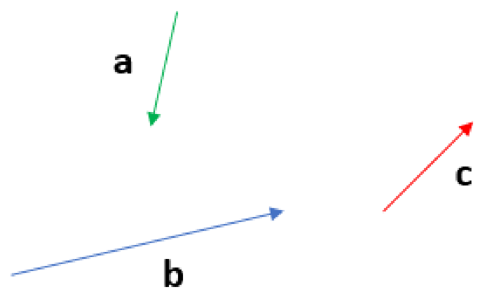


Figure 9. Representation of three vectors.

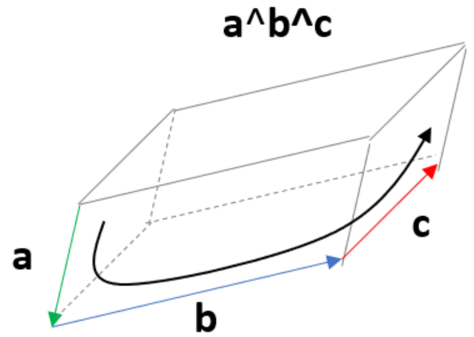


Figure 10. Representation of the trivector **a^b^c**.

We can see that the trivector represents a volume (a parallelepiped) created by the three vectors. Apart of the volume, it represents a rotation (or an orientation) regarding that volume. Again, if we make the following wedge product:

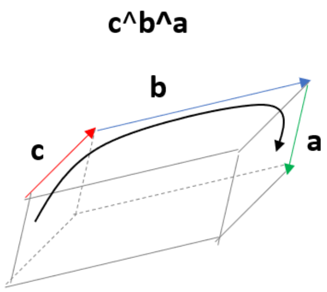


Figure 10. Representation of the trivector **c^b^a**.

We get the same volume but with opposite orientation. As we know that the wedge product is anticommutative, we can get this result, putting negative (or minus sign) every time we make a permutation of the product. This way:

$$\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c} = -\mathbf{a}\wedge\mathbf{c}\wedge\mathbf{b} = -(-\mathbf{c}\wedge\mathbf{a}\wedge\mathbf{b}) = \mathbf{c}\wedge\mathbf{a}\wedge\mathbf{b} = -\mathbf{c}\wedge\mathbf{b}\wedge\mathbf{a} \qquad (27)$$

Getting the expected result.

When the trivector is composed by unitary vectors, the wedge multiplication of the trivector by itself is -1 (the same as if it were *i*, the imaginary unit).

$$\begin{aligned}\left(\frac{\mathbf{a}}{|a|}\wedge\frac{\mathbf{b}}{|b|}\wedge\frac{\mathbf{c}}{|c|}\right)\wedge\left(\frac{\mathbf{a}}{|a|}\wedge\frac{\mathbf{b}}{|b|}\wedge\frac{\mathbf{c}}{|c|}\right) &= -\frac{\mathbf{a}}{|a|}\wedge\frac{\mathbf{b}}{|b|}\wedge\frac{\mathbf{c}}{|c|}\wedge\frac{\mathbf{a}}{|a|}\wedge\frac{\mathbf{c}}{|c|}\wedge\frac{\mathbf{b}}{|b|}\\ &= \frac{\mathbf{a}}{|a|}\wedge\frac{\mathbf{b}}{|b|}\wedge\frac{\mathbf{c}}{|c|}\wedge\frac{\mathbf{c}}{|c|}\wedge\frac{\mathbf{a}}{|a|}\wedge\frac{\mathbf{b}}{|b|} = \frac{\mathbf{a}}{|a|}\wedge\frac{\mathbf{b}}{|b|}\wedge\frac{\mathbf{c^2}}{|c|^2}\wedge\frac{\mathbf{a}}{|a|}\wedge\frac{\mathbf{b}}{|b|}\\ &= \frac{\mathbf{a}}{|a|}\wedge\frac{\mathbf{b}}{|b|}\wedge\frac{|c|^2}{|c|^2}\wedge\frac{\mathbf{a}}{|a|}\wedge\frac{\mathbf{b}}{|b|} = \frac{\mathbf{a}}{|a|}\wedge\frac{\mathbf{b}}{|b|}\wedge\frac{\mathbf{a}}{|a|}\wedge\frac{\mathbf{b}}{|b|} = -\frac{\mathbf{a}}{|a|}\wedge\frac{\mathbf{b}}{|b|}\wedge\frac{\mathbf{b}}{|b|}\wedge\frac{\mathbf{a}}{|a|}\\ &= -\frac{\mathbf{a}}{|a|}\wedge\frac{|b|^2}{|b|^2}\wedge\frac{\mathbf{a}}{|a|} = -\frac{\mathbf{a}}{|a|}\wedge\frac{\mathbf{a}}{|a|} = -\frac{|a|^2}{|a|^2} = -1 \qquad (28)\end{aligned}$$

As commented in the first chapter, when the vectors are perpendicular, the geometric product is the same as the wedge product. So, when the vectors are perpendicular and unitary (as in an orthonormal basis for example), we have (following the same steps as with above demonstration):

$$(\mathbf{e_1e_2e_3})^2 = (\mathbf{e_1e_2e_3})(\mathbf{e_1e_2e_3}) = \mathbf{e_1}\wedge\mathbf{e_2}\wedge\mathbf{e_3}\wedge\mathbf{e_1}\wedge\mathbf{e_2}\wedge\mathbf{e_3} = -1 \qquad (29)$$

This effect happens also with the bivectors (when they are orthonormal):

$$(\mathbf{e_1e_2})^{\mathbf{2}} = (\mathbf{e_1e_2})(\mathbf{e_1e_2}) = \mathbf{e_1}\wedge\mathbf{e_2}\wedge\mathbf{e_1}\wedge\mathbf{e_2} = -\mathbf{e_1}\wedge\mathbf{e_2}\wedge\mathbf{e_2}\wedge\mathbf{e_1} = -1 \qquad (30)$$

Again, if you want to learn about the beautiful geometric algebra in a proper manner, I recommend you the previous commented references [1][2][3][4].

4. Euclidean geometry in two dimensions

I will start with Euclidean geometry in two dimensions to try to clarify the concepts and the way of working as much as possible. In the next chapters we will us the same concepts for Non-Euclidean geometry and a higher number of dimensions.

In Euclidean geometry we can always find an orthonormal basis (perpendicular with unitary vectors) of vectors $\mathbf{e_1}$ and $\mathbf{e_2}$ such as:

$$\mathbf{e_1} \cdot \mathbf{e_1} = 1 \qquad (31)$$
$$\mathbf{e_2} \cdot \mathbf{e_2} = 1 \qquad (32)$$
$$\mathbf{e_1} \cdot \mathbf{e_2} = \mathbf{e_2} \cdot \mathbf{e_1} = 0 \qquad (33)$$

The representation of this metric, is the following tensor/matrix:

$$g_{\mu\nu} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (34)$$

Where:

$$g_{11} = \mathbf{e_1} \cdot \mathbf{e_1} = 1 \qquad (35)$$
$$g_{22} = \mathbf{e_2} \cdot \mathbf{e_2} = 1 \qquad (36)$$
$$g_{12} = \mathbf{e_1} \cdot \mathbf{e_2} = g_{21} = \mathbf{e_2} \cdot \mathbf{e_1} = 0 \qquad (37)$$

Coming back to the expressions (31)(32)(33) and the definition of geometric product (1), it is straightforward to calculate:

$$\mathbf{e_1 e_1} = \mathbf{e_1} \cdot \mathbf{e_1} + \mathbf{e_1}\hat{}\mathbf{e_1} = 1 + 0 = 1 \qquad (38)$$
$$\mathbf{e_2 e_2} = \mathbf{e_2} \cdot \mathbf{e_2} + \mathbf{e_2}\hat{}\mathbf{e_2} = 1 + 0 = 1 \qquad (39)$$
$$\mathbf{e_1 e_2} = \mathbf{e_1} \cdot \mathbf{e_2} + \mathbf{e_1}\hat{}\mathbf{e_2} = 0 + \mathbf{e_1}\hat{}\mathbf{e_2} = \mathbf{e_1}\hat{}\mathbf{e_2} \qquad (40)$$
$$\mathbf{e_2 e_1} = \mathbf{e_2} \cdot \mathbf{e_1} + \mathbf{e_2^{e_1}} = 0 + \mathbf{e_2}\hat{}\mathbf{e_1} = \mathbf{e_2}\hat{}\mathbf{e_1} = -\mathbf{e_1}\hat{}\mathbf{e_2} = -\mathbf{e_1 e_2} \qquad (41)$$

So, finally we get:

$$\mathbf{e_1 e_1} = 1 \qquad (42)$$
$$\mathbf{e_2 e_2} = 1 \qquad (43)$$
$$\mathbf{e_2 e_1} = -\mathbf{e_1 e_2} \qquad (44)$$

So, imagine you have two vectors **a** and **b**:

$$\mathbf{a} = a_1\mathbf{e_1} + a_2\mathbf{e_2} \qquad (45)$$
$$\mathbf{b} = b_1\mathbf{e_1} + b_2\mathbf{e_2} \qquad (46)$$

If we geometric multiply them:

$$\mathbf{ab} = (a_1\mathbf{e_1} + a_2\mathbf{e_2})(b_1\mathbf{e_1} + b_2\mathbf{e_2})$$
$$= a_1b_1\mathbf{e_1e_1} + a_1b_2\mathbf{e_1e_2} + a_2b_1\mathbf{e_2e_1} + a_2b_2\mathbf{e_2e_2} \quad (47)$$

Now, we only have to apply the rules that we have commented before, to get the result:

$$\mathbf{ab} = a_1b_1 \cdot 1 + a_1b_2\mathbf{e_1e_2} + a_2b_1(-\mathbf{e_1e_2}) + a_2b_2 \cdot 1 \qquad (48)$$
$$\mathbf{ab} = a_1b_1 + a_2b_2 + (a_1b_2 - a_2b_1)\,\mathbf{e_1e_2} \qquad (49)$$

And we know (1) that the geometric product is defined as:

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a}\hat{}\mathbf{b} \qquad (50)$$

Meaning that the scalar part, corresponds to the scalar product, and the bivector part to the wedge product:

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_{2_1} \qquad (51)$$
$$\mathbf{a}\hat{}\mathbf{b} = (a_1b_2 - a_2b_1)\,\mathbf{e_1e_2} \qquad (52)$$

What we have done here is pretty straightforward/simple but are exactly the same steps we will follow in the next chapters for Non-euclidean geometry.

If you have interest in how the “strange” wedge product can be related to matrix algebra, you can check it in Annex A2, for curiosity (not really necessary)

5. Non-Euclidean geometry in two dimensions

Now, we will continue with Non-Euclidean geometry in two dimensions. Remind that the goal is to demonstrate that it is not necessary any “external basis” to the geometry that is being studied or any rotation to an orthonormal basis to work with this geometry (the latter was used for example in [5].

You can use whatever basis vectors with their properties (inside the geometry itself, not necessary any projection or external dimensions-geometry) and geometric algebra will do the rest.

Imagine we have the following metric tensor/matrix in whichever non-orthogonal basis:

$$g_{\mu\nu} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \qquad (53)$$

This means:

$$\mathbf{e_1} \cdot \mathbf{e_1} = g_{11} \qquad (54)$$
$$\mathbf{e_2} \cdot \mathbf{e_2} = g_{22} \qquad (55)$$
$$\mathbf{e_1} \cdot \mathbf{e_2} = g_{12} = \mathbf{e_2} \cdot \mathbf{e_1} = g_{21} \qquad (56)$$

Using the definition of scalar product, we have:

$$g_{12} = g_{21} = \mathbf{e_1} \cdot \mathbf{e_2} = |\mathrm{e}_1| \cdot |\mathrm{e}_2| \cdot \cos\varphi \qquad (57)$$

$$\cos\varphi = \frac{g_{12}}{|\mathrm{e}_1| \cdot |\mathrm{e}_2|} \qquad (58)$$

This means the element g_{12} is related to the angle that both vectors form. In fact, when the vectors are orthogonal cos(φ)=cos(π/2)=0, so g_{12} is zero, in that case.

Using the definition of geometric product (and knowing that a wedge product of a vector by itself is always zero), we have:

$$\mathbf{e_1}\mathbf{e_1} = \mathbf{e_1} \cdot \mathbf{e_1} + \mathbf{e_1} {}^\wedge \mathbf{e_1} = g_{11} + 0 = g_{11} \qquad (59)$$
$$\mathbf{e_2}\mathbf{e_2} = \mathbf{e_2} \cdot \mathbf{e_2} + \mathbf{e_2} {}^\wedge \mathbf{e_2} = g_{22} + 0 = g_{22} \qquad (60)$$

Until now, everything very similar to Euclidean metric. Now, it is when the things start to be different:

$$\mathbf{e_1}\mathbf{e_2} = \mathbf{e_1} \cdot \mathbf{e_2} + \mathbf{e_1} {}^\wedge \mathbf{e_2} = g_{12} + \mathbf{e_1} {}^\wedge \mathbf{e_2} \qquad (61)$$
$$\mathbf{e_2}\mathbf{e_1} = \mathbf{e_2} \cdot \mathbf{e_1} + \mathbf{e_2^{e1}} = g_{21} + \mathbf{e_2} {}^\wedge \mathbf{e_1} \qquad (62)$$

We know that:

$$g_{21} = g_{12} \qquad (63)$$

And the wedge product is anticommutative so:

$$\mathbf{e_2}\wedge\mathbf{e_1} = -\mathbf{e_1}\wedge\mathbf{e_2} \qquad (64)$$

So, applying to the second equation:

$$\mathbf{e_2}\mathbf{e_1} = g_{21} + \mathbf{e_2}\wedge\mathbf{e_1} = g_{12} - \mathbf{e_1}\wedge\mathbf{e_2} \qquad (65)$$

In parallel we can isolate $\mathbf{e_1}\wedge\mathbf{e_2}$ from the first equation:

$$\mathbf{e_1}\wedge\mathbf{e_2} = \mathbf{e_1}\mathbf{e_2} - g_{12} \qquad (66)$$

Inverting signs in the equation:

$$-\mathbf{e_1}\wedge\mathbf{e_2} = g_{12} - \mathbf{e_1}\mathbf{e_2} \qquad (67)$$

And now, substituting we have:

$$\mathbf{e_2}\mathbf{e_1} = g_{12} - \mathbf{e_1}\wedge\mathbf{e_2} = 2g_{21} - \mathbf{e_1}\mathbf{e_2} \qquad (68)$$

So, summing up, the equations for the geometric products of the basis vectors are:

$$\mathbf{e_1}\mathbf{e_1} = g_{11} \qquad (69)$$
$$\mathbf{e_2}\mathbf{e_2} = g_{22} \qquad (70)$$
$$\mathbf{e_2}\mathbf{e_1} = 2g_{21} - \mathbf{e_1}\mathbf{e_2} \qquad (71)$$

And with this definition, you can operate with whichever vector or multivector you want. And it is not necessary any change of basis to convert it in orthonormal for example or any external basis using other geometry or dimensions.

Example:

$$\mathbf{a} = a_1\mathbf{e_1} + a_2\mathbf{e_2} \qquad (72)$$
$$\mathbf{b} = b_1\mathbf{e_1} + b_2\mathbf{e_2} \qquad (73)$$

If we geometric multiply them:

$$\mathbf{ab} = (a_1\mathbf{e_1} + a_2\mathbf{e_2})(b_1\mathbf{e_1} + b_2\mathbf{e_2})$$
$$= a_1b_1\mathbf{e_1}\mathbf{e_1} + a_1b_2\mathbf{e_1}\mathbf{e_2} + a_2b_1\mathbf{e_2}\mathbf{e_1} + a_2b_2\mathbf{e_2}\mathbf{e_2} \quad (74)$$

Now, we just substitute the geometric products of the basis, according previous equations. Beware of the third element, it is not anticommutative any more, we have to apply the equation (71).

$$\mathbf{ab} = a_1b_1g_{11} + a_1b_2\mathbf{e_1}\mathbf{e_2} + a_2b_1(2g_{21} - \mathbf{e_1}\mathbf{e_2}) + a_2b_2g_{22} \qquad (75)$$

$$\mathbf{ab} = a_1b_1g_{11} + a_2b_2g_{22} + a_1b_2\mathbf{e_1}\mathbf{e_2} + 2a_2b_1g_{21} - a_2b_1\mathbf{e_1}\mathbf{e_2} \qquad (76)$$

$$\mathbf{ab} = a_1b_1g_{11} + a_2b_2g_{22} + 2a_2b_1g_{21} + (a_1b_2 - a_2b_1)\mathbf{e_1}\mathbf{e_2} \qquad (77)$$

Of course, this seems complicated. But the message here is just the following. You can forget about the Non-Euclidean metric during all the operations. You can indicate all the geometric products of the basis vectors as usual. Only, when you have finished all the operations (leaving indicated the geometric product of the basis vectors), you can resolve the values, making the final operations with the basis vectors. Just remembering that you cannot reverse the geometric products of different basic vectors $\mathbf{e_1e_2} \neq \mathbf{-e_2e_1}$ but you have to apply the equation:

$$\mathbf{e_2 e_1} = 2g_{21} - \mathbf{e_1 e_2} \quad (78)$$

And just continue, operating.

Now, moving forward, let's calculate the modulus of **a** following the steps correctly:

$$\mathbf{a} = a_1\mathbf{e_1} + a_2\mathbf{e_2} \quad (79)$$

$$\begin{aligned} |\mathrm{a}|^{\mathbf{2}} = \mathbf{aa} = (a_1\mathbf{e_1} + a_2\mathbf{e_2})(a_1\mathbf{e_1} + a_2\mathbf{e_2}) &= \\ &= a_1a_1\mathbf{e_1e_1} + a_1a_2\mathbf{e_1e_2} + a_2a_1\mathbf{e_2e_1} + a_2a_2\mathbf{e_2e_2} = \\ &= a_1^2g_{11} + a_1a_2\mathbf{e_1e_2} + a_2a_1(2g_{21} - \mathbf{e_1e_2}) + a_2^2g_{22} = \\ &= a_1^2g_{11} + a_2^2g_{22} + a_1a_2\mathbf{e_1e_2} + a_2a_1(2g_{21} - \mathbf{e_1e_2}) = \\ &= a_1^2g_{11} + a_2^2g_{22} + 2g_{21}a_2a_1 + a_1a_2\mathbf{e_1e_2} - a_2a_1\mathbf{e_1e_2} = \\ &= a_1^2g_{11} + a_2^2g_{22} + 2g_{21}a_2a_1 \quad (80) \end{aligned}$$

So,

$$|\mathrm{a}| = \sqrt{a_1^2g_{11} + a_2^2g_{22} + 2g_{21}a_2a_1} \quad (81)$$

Following the same with **b**, we would have:

$$|\mathrm{b}|^{\mathbf{2}} = b_1^2g_{11} + b_2^2g_{22} + 2g_{21}b_2b_1 \quad (82)$$

$$|\mathrm{b}| = \sqrt{b_1^2g_{11} + b_2^2g_{22} + 2g_{21}b_2b_1} \quad (83)$$

And what is the modulus of **ab**? Now, it is pretty simple. In general, the square of the modulus of a multivector is the geometric product of the multivector by its conjugate (the same multivector reversing all the geometric products inside it). This is:

$$|\mathrm{ab}|^2 = \mathbf{ab}\widetilde{(\mathbf{ab})} = \mathbf{abba} \quad (84)$$

So, directly:

$$|\mathrm{ab}|^2 = \mathbf{abba} = \mathbf{a}|\mathrm{b}|^{\mathbf{2}}\mathbf{a} = |\mathrm{b}|^{\mathbf{2}}\mathbf{aa} = |\mathrm{b}|^{\mathbf{2}}|\mathrm{a}|^{\mathbf{2}} = |\mathrm{a}|^{\mathbf{2}}|\mathrm{b}|^{\mathbf{2}} \quad (85)$$

So:

$$|\mathrm{ab}| = |\mathrm{a}||\mathrm{b}| \quad (86)$$

This equation is KEY. It seems simple but remember that we are working in a Non-Euclidean metric. This means, the modulus of the geometric product of two vectors is the same as the product of the modulus of the vectors (independently of the metric, of the basis, if it is or not orthogonal or orthonormal etc…). And even, it is independent of the angle the vectors form. It does not matter, always the modulus of the geometric product is equal to the product of the modulus of the vectors, independently of the basis, the metric and of the angle they form. This property (but related only to Euclidean spaces with non-orthogonal vectors) was commented also in [6].

You can see the importance of this now. First, we calculate the modulus of the basis vectors (this is straightforward):

$$|e_1|^2 = \mathbf{e_1 e_1} = g_{11} \quad (87)$$
$$|e_2|^2 = \mathbf{e_2 e_2} = g_{22} \quad (88)$$

$$|e_1| = \sqrt{g_{11}} \quad (89)$$
$$|e_2| = \sqrt{g_{22}} \quad (90)$$

But now, if we calculate the modulus of $\mathbf{e_1 e_2}$:

$$|e_1 e_2|^2 = \mathbf{e_1 e_2}(\widetilde{\mathbf{e_1 e_2}}) = \mathbf{e_1 e_2 e_2 e_1} = |e_1|^2 |e_2|^2 = g_{11} g_{22} \quad (91)$$
$$|e_1 e_2| = |e_1||e_2| = \sqrt{g_{11}}\sqrt{g_{22}} \quad (92)$$

The important thing here is that it does not depend on g_{12}, it does not depend on the angle or the relation between $\mathbf{e_1}$ and $\mathbf{e_2}$. It only depends on $\mathbf{e_1}$ and $\mathbf{e_2}$, not in the relation between both. This means, it does not matter if the basis is orthogonal or not, the modulus of the geometric product only depends on the modulus of each vector.

So where does g_{12} appear? It only appears when we have to reverse the two basis vectors to perform any operation. Imagine we need to calculate:

$$\mathbf{e_1 e_2 e_1} = \mathbf{e_1}(2g_{21} - \mathbf{e_1 e_2}) = 2g_{21}\mathbf{e_1} - \mathbf{e_1 e_1 e_2} = 2g_{21}\mathbf{e_1} - |e_1|^2 \mathbf{e_2}$$
$$= 2g_{21}\mathbf{e_1} - g_{11}\mathbf{e_2} \quad (93)$$

But, if what we want is to calculate the modulus of $\mathbf{e_1 e_2 e_1}$ we get:

$$|\mathbf{e_1 e_2 e_1}|^2 = \mathbf{e_1 e_2 e_1} \widetilde{\mathbf{e_1 e_2 e_1}} = \mathbf{e_1 e_2 e_1 e_1 e_2 e_1} = \mathbf{e_1 e_2}|e_1|^2 \mathbf{e_2 e_1} = |e_1|^2 \mathbf{e_1 e_2 e_2 e_1}$$
$$= |e_1|^2 \mathbf{e_1} |e_2|^2 \mathbf{e_1} = |e_1|^2 |e_1|^2 |e_2|^2 \quad (94)$$

So:

$$|\mathbf{e_1 e_2 e_1}| = |e_1|^2 |e_2| \quad (95)$$

As expected. Modulus of the geometric product equals the product of the modulus (and the metric or the relation between vectors does not affect at all).

6. Non-Euclidean geometry in more than two dimensions

The way of working is exactly the same as in two dimensions. Only the metric tensor will change. In two dimensions was:

$$g_{\mu\nu} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \quad (96)$$

For three dimensions is, just to add the metrics with the third dimension:

$$g_{\mu\nu} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \qquad (97)$$

The operations will be performed exactly the same as they were performed for two dimensions.

For space-time algebra (four dimensions, being time one of them), we will use the nomenclature from 1 to 3 for space dimensions and the dimension of time, will be 0.

$$g_{\mu\nu} = \begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{bmatrix} \qquad (98)$$

I take the opportunity here to comment an advantage of geometric algebra compared to other algebras, that is not usually commented.

In a pure geometric algebra work, it is not really necessary to define or to know in how many dimensions you are working upfront. Normally this is done (defining a geometry), but it should not be strictly necessary. As geometric algebra work is based in the operations of the basis vectors, you can work in a discipline (physics, mathematics) in three dimensions and add later a fourth dimension not changing anything of the work you have done already (this is impossible with matrices for example). Just adding a new vector (which wedge product is different to zero with the other existing vectors) you have a new dimension, being perfectly valid all the work already done (but just that vector was not present in those equations). Again, this is impossible (even a nightmare) with matrices for example.

7. Derivatives of basis vectors using geometric algebra

Before proceeding with the derivatives, I will just make a comment. Normally in the literature of non-Euclidean metric, the basis vectors are written with subscripts e_j (named covariant) and the coefficients of the coordinates are written with superscripts as in e^j for example (named contravariant).

Here, the difference will just be that the basis vectors will be bold and subscripted as in $\mathbf{e_j}$ (as all the vectors in this paper) and the coefficients of the coordinates will not be bold, but also will be subscripted as e_j. This is done to remark the fact, that it is not necessary to talk about contravariant or covariant any more in GA. The basis vectors have the information of the metric and you can work just operating them (including their coefficients or coordinates) getting the correct results. No need to make this discrimination any more.

Returning to the work, the first thing we are going to show is that the partial derivative of a unitary vector (in this case, we have put a general basis vector $\mathbf{e_k}$ but works with any unitary vector) with respect to whatever variable is zero (in this case, we have used a general coefficient e_j but works with any variable) .

This means, the partial derivative of a vector divided by its own modulus with respect to whatever variable is zero, as we see here:

$$\frac{\partial}{\partial e_j}\left(\frac{\boldsymbol{e_k}}{\|e_k\|}\right) = \frac{\partial \boldsymbol{e_k}}{\partial e_j}\frac{1}{\|e_k\|} - \boldsymbol{e_k}\frac{1}{\|e_k\|^2}\frac{\partial \|e_k\|}{\partial e_j} = \frac{\partial \|e_k\|}{\partial e_j}\frac{\boldsymbol{e_k}}{\|e_k\|}\frac{1}{\|e_k\|} - \boldsymbol{e_k}\frac{1}{\|e_k\|^2}\frac{\partial \|e_k\|}{\partial e_j}$$
$$= \frac{\partial \|e_k\|}{\partial e_j}\frac{\boldsymbol{e_k}}{\|e_k\|^2} - \frac{\boldsymbol{e_k}}{\|e_k\|^2}\frac{\partial \|e_k\|}{\partial e_j} = 0 \qquad (99)$$

Following the same concept, we can obtain the derivative of the geometric product of unitary vectors as:

$$\frac{\partial}{\partial e_i}\left(\frac{\boldsymbol{e_k}}{\|e_k\|}\frac{\boldsymbol{e_j}}{\|e_j\|}\right) = \frac{\partial}{\partial e_i}\left(\frac{\boldsymbol{e_k}}{\|e_k\|}\right)\frac{\boldsymbol{e_j}}{\|e_j\|} + \frac{\boldsymbol{e_k}}{\|e_k\|}\frac{\partial}{\partial e_i}\left(\frac{\boldsymbol{e_j}}{\|e_j\|}\right) = 0 \quad (100)$$

Getting always the result zero.

The next step is to calculate the partial derivative of a basis vector -general, not unitary- which respect to whatever variable.

To do that, we calculate first the partial derivative of the geometric product of a basis vector by itself:

$$\frac{\partial}{\partial e_j}(\boldsymbol{e_k}\boldsymbol{e_k}) = \frac{\partial \|e_k\|^2}{\partial e_j} = 2\|e_k\|\frac{\partial \|e_k\|}{\partial e_j} \quad (101)$$

$$\frac{\partial}{\partial e_j}(\boldsymbol{e_k}\boldsymbol{e_k}) = \frac{\partial \boldsymbol{e_k}}{\partial e_j}\boldsymbol{e_k} + \boldsymbol{e_k}\frac{\partial \boldsymbol{e_k}}{\partial e_j} = 2\frac{\partial \boldsymbol{e_k}}{\partial e_j}\cdot \boldsymbol{e_k} \quad (102)$$

And we obtain that the result is two times the dot product of the derivative of the vector by the vector.

Now, we can calculate the geometric product of a basis vector by its inverse.

$$\frac{\partial}{\partial e_j}(\boldsymbol{e_k}\boldsymbol{e_k}^{-1}) = \frac{\partial}{\partial e_j}(1) = 0 \quad (103)$$

And we get the result zero. But we can calculate the derivative another way:

$$\frac{\partial}{\partial e_j}(\boldsymbol{e_k}\boldsymbol{e_k}^{-1}) = \frac{\partial \boldsymbol{e_k}}{\partial e_j}\boldsymbol{e_k}^{-1} - \boldsymbol{e_k}\frac{1}{\|e_k\|^2}\frac{\partial \boldsymbol{e_k}}{\partial e_j} = \frac{\partial \boldsymbol{e_k}}{\partial e_j}\frac{\boldsymbol{e_k}}{\|e_k\|^2} - \boldsymbol{e_k}\frac{1}{\|e_k\|^2}\frac{\partial \boldsymbol{e_k}}{\partial e_j}$$
$$= \frac{1}{\|e_k\|^2}\left(\frac{\partial \boldsymbol{e_k}}{\partial e_j}\boldsymbol{e_k} - \boldsymbol{e_k}\frac{\partial \boldsymbol{e_k}}{\partial e_j}\right) = \frac{1}{\|e_k\|^2}\left(\frac{\partial \boldsymbol{e_k}^{e_k}}{\partial e_j}\right) = 0 \quad (104)$$

$$\frac{\partial \boldsymbol{e_k}}{\partial e_j}{}^\wedge \boldsymbol{e_k} = \boldsymbol{0} \quad (105)$$

Getting the result that the wedge product of the derivative of the vector by the vector is zero.

So now, we can operate and get:

$$\frac{\partial}{\partial e_j}(\boldsymbol{e_k}\boldsymbol{e_k}) = 2\frac{\partial \boldsymbol{e_k}}{\partial e_j}\cdot \boldsymbol{e_k} = 2\frac{\partial \boldsymbol{e_k}}{\partial e_j}\cdot \boldsymbol{e_k} + 2(0) = 2\frac{\partial \boldsymbol{e_k}}{\partial e_j}\cdot \boldsymbol{e_k} + 2\left(\frac{\partial \boldsymbol{e_k}^{e_k}}{\partial e_j}\right)$$
$$= 2\frac{\partial \boldsymbol{e_k}}{\partial e_j}\boldsymbol{e_k} \quad (106)$$

$$\frac{\partial}{\partial e_j}(\boldsymbol{e_k}\boldsymbol{e_k}) = 2\frac{\partial \boldsymbol{e_k}}{\partial e_j}\boldsymbol{e_k} = 2\|e_k\|\frac{\partial \|e_k\|}{\partial e_j} \quad (107)$$

$$2\frac{\partial \boldsymbol{e_k}}{\partial e_j}\boldsymbol{e_k} = 2\|e_k\|\frac{\partial \|e_k\|}{\partial e_j} \quad (108)$$

$$\frac{\partial \boldsymbol{e_k}}{\partial e_j}\boldsymbol{e_k} = \|e_k\|\frac{\partial \|e_k\|}{\partial e_j} \qquad (109)$$

$$\frac{\partial \boldsymbol{e_k}}{\partial e_j}\boldsymbol{e_k}\boldsymbol{e_k} = \|e_k\|\frac{\partial \|e_k\|}{\partial e_j}\boldsymbol{e_k} \qquad (110)$$

$$\frac{\partial \boldsymbol{e_k}}{\partial e_j}\|e_k\|^2 = \|e_k\|\frac{\partial \|e_k\|}{\partial e_j}\boldsymbol{e_k} \qquad (111)$$

$$\frac{\partial \boldsymbol{e_k}}{\partial e_j} = \frac{\partial \|e_k\|}{\partial e_j}\frac{\boldsymbol{e_k}}{\|e_k\|} \qquad (112)$$

So, in the end, we have obtained that the derivative of a basis vector is the same as the derivative of its modulus multiplied by the unitary basis vector.

As the derivative of the unitary vector is zero, we can make a second derivative very easily as:

$$\begin{aligned}\frac{\partial}{\partial e_i}\left(\frac{\partial \boldsymbol{e_k}}{\partial e_j}\right) = \frac{\partial}{\partial e_i}\left(\frac{\partial \|e_k\|}{\partial e_j}\frac{\boldsymbol{e_k}}{\|e_k\|}\right) = \frac{\partial^2 \|e_k\|}{\partial e_i \partial e_j}\frac{\boldsymbol{e_k}}{\|e_k\|} + \frac{\partial \|e_k\|}{\partial e_j}\frac{\partial}{\partial e_i}\left(\frac{\boldsymbol{e_k}}{\|e_k\|}\right) \\ = \frac{\partial^2 \|e_k\|}{\partial e_i \partial e_j}\frac{\boldsymbol{e_k}}{\|e_k\|} + 0 = \frac{\partial^2 \|e_k\|}{\partial e_i \partial e_j}\frac{\boldsymbol{e_k}}{\|e_k\|} \qquad (113)\end{aligned}$$

Getting that it is equal to the second derivative of its modulus multiplied by the unitary vector. This way, the chain rule does not get infinite but, it only has one term even if we take subsequent derivatives.

We can go even further, and instead of making the partial derivative with respect to a co-ordinate, we can make the derivative directly with respect to a vector (in this case, a basis vector but it could be whatever vector).

We will use the following definition

$$\frac{d\boldsymbol{e_k}}{d\boldsymbol{e_j}} = \frac{d\boldsymbol{e_k}(\boldsymbol{r})}{d\boldsymbol{e_j}} = \lim_{h\to 0}\frac{\boldsymbol{e_k}(\boldsymbol{r}+h\boldsymbol{e_j}) - \boldsymbol{e_k}(\boldsymbol{r})}{h\boldsymbol{e_j}} \qquad (114)$$

Being the vector **r** the one that defines the position where the vector basis is located (or the position where the derivative is taken). And $\mathbf{e_j}$ is the direction towards the derivative is taken in the vector $\mathbf{e_k}(\mathbf{r})$

Using the definition of derivative and the equation (23) we can do the following (because in GA we have the inverse operation for a vector):

$$\begin{aligned}\frac{d\boldsymbol{e_k}}{d\boldsymbol{e_j}} = \lim_{h\to 0}\frac{\boldsymbol{e_k}(\boldsymbol{r}+h\boldsymbol{e_j}) - \boldsymbol{e_k}(\boldsymbol{r})}{h\boldsymbol{e_j}} = \lim_{h\to 0}\frac{\boldsymbol{e_k}(\boldsymbol{r}+h\boldsymbol{e_j}) - \boldsymbol{e_k}(\boldsymbol{r})}{h}\frac{1}{\boldsymbol{e_j}} \\ = \lim_{h\to 0}\frac{\boldsymbol{e_k}(\boldsymbol{r}+h\boldsymbol{e_j}) - \boldsymbol{e_k}(\boldsymbol{r})}{h}\boldsymbol{e_j}^{-1} \\ = \lim_{h\to 0}\frac{\boldsymbol{e_k}(\boldsymbol{r}+h\boldsymbol{e_j}) - \boldsymbol{e_k}(\boldsymbol{r})}{h}\frac{\boldsymbol{e_j}}{\|e_j\|^2} \qquad (115)\end{aligned}$$

The first element:

$$\lim_{h\to 0}\frac{\boldsymbol{e_k}(\boldsymbol{r}+h\boldsymbol{e_j})-\boldsymbol{e_k}(\boldsymbol{r})}{h} \qquad (116)$$

Is by definition a directional derivative [7]. So, using its relationship with the gradient [8]:

$$\frac{d\boldsymbol{e_k}}{d\boldsymbol{e_j}}=\lim_{h\to 0}\frac{\boldsymbol{e_k}(\boldsymbol{r}+h\boldsymbol{e_j})-\boldsymbol{e_k}(\boldsymbol{r})}{h}\frac{\boldsymbol{e_j}}{\|e_j\|^2}=(\boldsymbol{\nabla e_k}\cdot\boldsymbol{e_j})\frac{\boldsymbol{e_j}}{\|e_j\|^2} \qquad (117)$$

Using the definition of gradient, we have:

$$\frac{\partial\boldsymbol{e_k}}{\partial\boldsymbol{e_j}}=(\boldsymbol{\nabla e_k}\cdot\boldsymbol{e_j})\frac{\boldsymbol{e_j}}{\|e_j\|^2}=\left(\sum_{i=0}^{3}\frac{\partial\boldsymbol{e_k}}{\partial e_i}\boldsymbol{e_i}\cdot\boldsymbol{e_j}\right)\frac{\boldsymbol{e_j}}{\|e_j\|^2} \qquad (118)$$

Knowing that the dot products of the basis vectors is the metric g_{ij}:

$$\frac{\partial\boldsymbol{e_k}}{\partial\boldsymbol{e_j}}=\left(\sum_{i=0}^{3}\frac{\partial\boldsymbol{e_k}}{\partial e_i}\boldsymbol{e_i}\cdot\boldsymbol{e_j}\right)\frac{\boldsymbol{e_j}}{\|e_j\|^2}=\left(\sum_{i=0}^{3}\frac{\partial\boldsymbol{e_k}}{\partial e_i}g_{ij}\right)\frac{\boldsymbol{e_j}}{\|e_j\|^2} \qquad (119)$$

Now, applying the equation (112) to the partial derivative, we have:

$$\frac{\partial\boldsymbol{e_k}}{\partial\boldsymbol{e_j}}=\left(\sum_{i=0}^{3}\frac{\partial\boldsymbol{e_k}}{\partial e_i}g_{ij}\right)\frac{\boldsymbol{e_j}}{\|e_j\|^2}=\left(\sum_{i=0}^{3}\frac{\partial\|e_k\|}{\partial e_i}\frac{\boldsymbol{e_k}}{\|e_k\|}g_{ij}\right)\frac{\boldsymbol{e_j}}{\|e_j\|^2} \qquad (120)$$

As j is independent of the summation in i, we can reorder and introduce all the elements in the summation:

$$\begin{aligned}\frac{\partial\boldsymbol{e_k}}{\partial\boldsymbol{e_j}}&=\left(\sum_{i=0}^{3}\frac{\partial\|e_k\|}{\partial e_i}\frac{\boldsymbol{e_k}}{\|e_k\|}g_{ij}\right)\frac{\boldsymbol{e_j}}{\|e_j\|^2}=\sum_{i=0}^{3}\frac{g_{ij}}{\|e_j\|}\frac{\partial\|e_k\|}{\partial e_i}\frac{\boldsymbol{e_k}}{\|e_k\|}\frac{\boldsymbol{e_j}}{\|e_j\|}\\&=\sum_{i=0}^{3}\frac{g_{ij}}{\sqrt{g_{jj}}}\frac{\partial\|e_k\|}{\partial e_i}\frac{\boldsymbol{e_k}}{\|e_k\|}\frac{\boldsymbol{e_j}}{\|e_j\|} \qquad (121)\end{aligned}$$

We can see that the derivative is in fact a bivector. As we can see in [7] when the directional derivative is taken to a vector field, the result is a tensor field. A tensor can be defined as an operator that transform vectors. In GA, a bivector is one (of the multiple ways) that a tensor can be represented. Because its geometric multiplication by a vector can create other vectors (apart from other entities).

In the annex A.1.5, it is defined an alternative definition of gradient that instead of having the basis vectors in its definition, has the inverse of them. In that case, above formula would include another divisor term (g_{ii}). In a future study, it will be confirmed which one is correct (see next chapter).

8. Derivation of Schwarzschild metric using geometric algebra

The original idea of the paper was to calculate the Schwarzschild metric using the commented definition of derivatives. We can use them to take the derivatives of the elements of the length vector (that is composed by basis vectors) to calculate the geodesics and therefore, the metric.

Finally, I leave this for another revision of the paper, as the calculation is not as straight forward as expected, and some things need to be rechecked. I prefer to publish now all the info I have reducing the risks of not publishing anything in the end.

I will come back with this point.

A1. Annex 1. Crazy ideas using GA for the future

A1.1 Rigid body

You can see a presentation of a rigid body

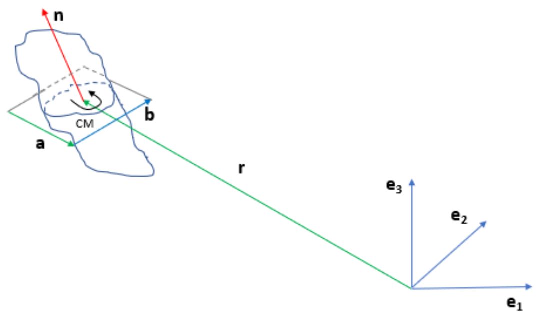


Figure 11. Vector and bivector involved in rigid body position (vector **r**) and orientation (bivector **a^b**).

The green vector **r** is the classical position vector of the center of masses. Normally, we would define also a vector **n** that represents the orientation of the body. In GA, this does not work like that. What we could define is a bivector **a^b** attached to a plane of the body which orientation will tell us, how the body is oriented.

This means, a multivector that has a vector (for position) and a bivector (for orientation) can define the positioning of a solid body. And in three dimensions, still we have the scalar and the trivector free for other uses.

So, in general, the estate vector of a solid body will have the form:

$$R = r_0 + r_1\boldsymbol{e_1} + r_2\boldsymbol{e_2} + r_3\boldsymbol{e_3} + r_{12}\boldsymbol{e_1}\boldsymbol{e_2} + r_{13}\boldsymbol{e_1}\boldsymbol{e_3} + r_{23}\boldsymbol{e_2}\boldsymbol{e_3} + r_{123}\boldsymbol{e_1}\boldsymbol{e_2}\boldsymbol{e_3} \quad (122)$$

Being r_i (except i=0) the coordinates of the position and the r_{ij} the coordinates of the orientation. r_0 and r_{123} keep free for this purpose (we will see possible interpretations for EM and QM for example). In this case, the suffix 0 of r_0 just represents the scalar (it is not to be confused with the time suffix 0).

If we take the derivative of the state vector with respect to proper time, we get:

$$\frac{dR}{d\tau} = \frac{dr_0}{d\tau} + \frac{dr_1}{d\tau}\boldsymbol{e_1} + \frac{dr_2}{d\tau}\boldsymbol{e_2} + \frac{dr_3}{d\tau}\boldsymbol{e_3} + \frac{dr_{12}}{d\tau}\boldsymbol{e_1e_2} + \frac{dr_{13}}{d\tau}\boldsymbol{e_1e_3} + \frac{dr_{23}}{d\tau}\boldsymbol{e_2e_3} + \frac{dr_{123}}{d\tau}\boldsymbol{e_1e_2e_3} \qquad (123)$$

Calling u the velocities (the derivatives of with respect to time):

$$U = u_0 + u_1\boldsymbol{e_1} + u_2\boldsymbol{e_2} + u_3\boldsymbol{e_3} + u_{12}\boldsymbol{e_1e_2} + u_{13}\boldsymbol{e_1e_3} + u_{23}\boldsymbol{e_2e_3} + u_{123}\boldsymbol{e_1e_2e_3} \qquad (124)$$

This time the u_i (except i=0) are the components of the linear speed of the of the center of mass and the u_{ij} are the components of the rotational speed of the rigid body (represented by a bivector). This bivector tells us the plane of rotation (instead of the axis as in classical geometry) and the direction of rotation (**a**^**b** or **b**^**a**). Again, u_0 represents the scalar (not the time suffix 0).

The scalar and the trivector could have different meanings to be commented later.

A1.2 Electromagnetism

There are different references regarding EM in GA [1][4][9][10]. But in all of them the electric field and the magnetic field are differentiated because their basis vectors are different. In STA both fields are bivectors (but the electric field has the time component in it and the magnetic just space vectors).

This differentiation is really the same as if one is a vector and the other a bivector (no time vectors are really necessary). I will explain why later.

This means, one of the possible ways to represent the electromagnetic multivector would be as:

$$F = E + \boldsymbol{e_{123}}B \qquad (125)$$

Being $\boldsymbol{e_{123}}$ the trivector (also called in the literature the pseudoscalar or I):

$$\boldsymbol{e_{123}} = \boldsymbol{e_1e_2e_3} \qquad (126)$$

So:

$$F = E + \boldsymbol{e_1 e_2 e_3} B \quad (127)$$

In this equation, E is the electric field vector:

$$E = E_1 \boldsymbol{e_1} + E_2 \boldsymbol{e_2} + E_3 \boldsymbol{e_3} \quad (128)$$

and $\mathbf{e}_{123}$B, is a bivector:

$$\boldsymbol{e_{123}} B = \boldsymbol{e_1 e_2 e_3} B_1 \boldsymbol{e_1} + \boldsymbol{e_1 e_2 e_3} B_2 \boldsymbol{e_2} + \boldsymbol{e_1 e_2 e_3} B_3 \boldsymbol{e_3} e$$
$$= B_1 \boldsymbol{e_2 e_3} - B_2 \boldsymbol{e_1 e_3} + B_3 \boldsymbol{e_1 e_2} \quad (129)$$

So, the electromagnetic field is a multivector composed by a vector (electric) and a bivector (magnetic).

$$F = F_1 \boldsymbol{e_1} + F_2 \boldsymbol{e_2} + F_3 \boldsymbol{e_3} + F_{12} \boldsymbol{e_1 e_2} + F_{13} \boldsymbol{e_1 e_3} + F_{23} \boldsymbol{e_2 e_3} \quad (130)$$

Being:

$$F_1 = E_1 \quad (131)$$
$$F_2 = E_2 \quad (132)$$
$$F_3 = E_3 \quad (133)$$

$$F_{12} = B_3 \quad (134)$$
$$F_{13} = -B_2 \quad (135)$$
$$F_{23} = B_1 \quad (136)$$

As we can check here:

$$F = F_1 \boldsymbol{e_1} + F_2 \boldsymbol{e_2} + F_3 \boldsymbol{e_3} + B_3 \boldsymbol{e_1 e_2} + (-B_2) \boldsymbol{e_1 e_3} + B_1 \boldsymbol{e_2 e_3} \quad (137)$$

$$F = F_1 \boldsymbol{e_1} + F_2 \boldsymbol{e_2} + F_3 \boldsymbol{e_3} + B_3 \boldsymbol{e_1 e_2} - B_2 \boldsymbol{e_1 e_3} + B_1 \boldsymbol{e_2 e_3} \quad (138)$$

Again, could complete the multivector with a scalar and a trivector. The possible meanings of these new components will be commented later:

$$F = F_0 + F_1 \boldsymbol{e_1} + F_2 \boldsymbol{e_2} + F_3 \boldsymbol{e_3} + F_{12} \boldsymbol{e_1 e_2} + F_{13} \boldsymbol{e_1 e_3} + F_{23} \boldsymbol{e_2 e_3} + F_{123} \boldsymbol{e_1 e_2 e_3} \quad (139)$$

A1.3 Hidden variables

One of the issues that could be addressed using this definition of the electromagnetic field are the hidden variables in QM.

The way is the following. Simplifying, in standard geometry, it is considered that the electric field interacts with the charge of the particle and in the other side the magnetic field interacts with the speed of the charge (the cross product of B and u -in GA, this would be the wedge product-).

The issue as that we only see these interactions, because they are the only ones that affect permanently the speed or direction of the particle.

If there are other interactions that just change the orientation of the particle or create oscillatory movements, this would have not been important for us -or would be really invisible- so we would not take care about them.

So, coming back to the electromagnetic field (in its general form):

$$F = F_0 + F_1\boldsymbol{e_1} + F_2\boldsymbol{e_2} + F_3\boldsymbol{e_3} + F_{12}\boldsymbol{e_1}\boldsymbol{e_2} + F_{13}\boldsymbol{e_1}\boldsymbol{e_3} + F_{23}\boldsymbol{e_2}\boldsymbol{e_3} + F_{123}\boldsymbol{e_1}\boldsymbol{e_2}\boldsymbol{e_3} \quad (140)$$

And coming back to the speed (in its generalized form including, linear speed, rotation, trivector and scalar):

$$U = u_0 + u_1\boldsymbol{e_1} + u_2\boldsymbol{e_2} + u_3\boldsymbol{e_3} + u_{12}\boldsymbol{e_1}\boldsymbol{e_2} + u_{13}\boldsymbol{e_1}\boldsymbol{e_3} + u_{23}\boldsymbol{e_2}\boldsymbol{e_3} + u_{123}\boldsymbol{e_1}\boldsymbol{e_2}\boldsymbol{e_3} \quad (141)$$

Now, if we make a complete geometric product between them (and the charge also, that would be included as a scalar -but it is another thing that could be studied-), we will see the following.

We will see that the multiplications between vectors will give scalars and bivectors. This means will affect, the rotation (and the scalar whatever it's represents), but not the speed or its direction. So, this effect will be in general invisible for us. But it will anyhow affect the orientation of the particle. And it should be included in every study leading orientation of spin etc. like in the Stern-Gerlach experiment [11].

This means, normally the experiments regarding entanglement are thought to function like this. First, we measure a magnitude (spin for example) in a particle. So, now we can know which the result in a second particle -that is entangled with the first one- will be, because somehow the first particle has interacted with the second.

But it is much logical to understand it like the following. There is a field which effects, normally do not affect us (so it keeps invisible in general, for our purposes). It does not affect the speed or the direction of particles, but yes, its orientation for example, which we normally do not measure, so we do not care about. When we finally measure the orientation (or spin) of a particle, this field is affecting the result. So, when we measure the second one, the same field is affecting. For us, seems something random as we are not taking the field into account. But the reality is that the first particle is not affecting the second. The first particle is just giving us info regarding the invisible field, that will affect the same way also the second particle. Not magic, just a field we normally do not take into account (as the trivector for example in F, F_{123}, as it does not affect speed or direction of the particles, just orientation). This has been commented in different ways in several papers already [12][13].

Continuing with the interactions of F with u, multiplications between scalars and vectors will give vectors, so they will affect the linear speed. This is the traditional case of the electric filed for example.

Multiplications between vector and bivectors can give trivectors (I will explain later possible meanings), but also vectors. This means, they can affect the linear speed. This is the case of the magnetic field (bivector) multiplied by the linear speed (vector).

But it is also the case of the electric field (vector) multiplied by the rotational speed (bivector). So why we do not see this effect. The reason is that this change in speed is oscillatory. The mean value is zero. The reason is because the rotational speed (or the orientation of its plane) is continuously changing due the other interactions. So, any interaction with it, will have a random answer, with mean value zero. This is the case of the zitterbewegung proposed by [14][15].

This way we can try to understand the apparently random effects in particles and also, that sometimes these random effects have predictable results. But it is not magic, or any interaction at distance, it is only that there are a lot of interactions that are invisible for us continuously (that are random in appearance) but are really-cause effect interactions that could be calculated with the complete information. They are provoked by the elements of the generalized electromagnetic field F, normally not considered, as the trivector (as they just affect orientations, not speed or directions of the particles), when acting to the generalized speed vector (that has linear speed, rotational speed and trivector).
in different areas of space) or other interactions.

A1.4 Basis vectors and measurement units

Another thing to be studied, is the following. Until now, the basis vectors, were vectors that could not be inverted, or its multiplication had no added value (if the result was one or a unit vector). Now, that the basis vector s can be used to give the information of the metric for whatever interactions, it is important that all of them are included in the equations. I mean, if we have a magnitude like the acceleration that has the following measurement units:

$$a = \frac{m}{s^2} \qquad (142)$$

Normally it is represented only with one vector that points to the direction of the magnitude (in this case acceleration). Something like:

$$a = \frac{m}{s^2}\boldsymbol{e_1} \quad (143)$$

The issue is, that if we want to transmit the metric of the geometry, we should use all the necessary basis vectors corresponding to the units to include the metric correctly. This means, something like:

$$a = \frac{m}{s^2}\frac{\boldsymbol{e_1}}{\boldsymbol{e_0}^2} = \frac{m}{s^2}\frac{\boldsymbol{e_1}}{|e_0|^2} \qquad (144)$$

We can see, that in the end, the acceleration has only one space vector as expected, but the information of the metric has been included also by the time vector (in this case).

It is expected that if the real measurement units are defined and used in disciplines like EM, QM etc. they could be used in a non-Eucliden space just being very careful with this point.

A1.5 Alternative definition of gradient

The definition of gradient (in this case, for example, for three dimensions) is [8]:

$$\nabla f = \sum_{i=1}^{3}\frac{\partial f}{\partial e_i}\boldsymbol{e_i} \qquad (145)$$

The issue is that as the partial derivative with respect to the variable is dividing, it would have more sense, the following definition, with the basis vector dividing (to be coherent with the measurement units -if the basis vector has them-:

$$\nabla f = \sum_{i=1}^{3}\frac{\partial f}{\partial e_i}\frac{1}{\boldsymbol{e_i}} \qquad (146)$$

The issue is that in most disciplines, this does not matter at all. As the basis vector is unitary (and the division for vectors is normally not even defined). But, this could be important when the basis vector transmits information, as the metric for example. It is very important to introduce this information in the correct manner. We can see the difference here:

$$\nabla f = \sum_{i=1}^{3}\frac{\partial f}{\partial e_i}\frac{1}{\boldsymbol{e_i}} = \sum_{i=1}^{3}\frac{\partial f}{\partial e_i}\boldsymbol{e_i}^{-1} = \sum_{i=1}^{3}\frac{\partial f}{\partial e_i}\frac{\boldsymbol{e_i}}{|e_i|^2} \qquad (147)$$

You can see, that is the same as the definition of the gradient when it is used in general coordinates [8].

Same comment as in previous point (A1.4), if the basis vectors are used wisely and are in their correct positions (and all of them, depending on which the measurement units are) the metric will be implicitly included in the calculations, not needing a special or "external" geometry to operate with them.

A1.6 Time in GA

Another topic I wanted to comment is regarding the basis vector of time in GA. Normally, time is considered as a fourth dimension with a special basis vector that has negative signature (or at least opposite to the other space basis vectors).

This is, something like:

$$\mathbf{e_0}^{\mathbf{2}} = -|\mathrm{e}_0|^{\mathbf{2}} \quad (148)$$

While the space vectors are

$$\mathbf{e}_{\boldsymbol{i}}^{\mathbf{2}} = |\mathrm{e}_i|^{\mathbf{2}} \quad (149)$$

Being i, the space coordinates from 1 to 3. Or it could be with signatures reversed but always different in space and time.

Also, you can check that in the normalized non-Euclidean metrics, the time basis vectors are always the inverse of the multiplication of the other basis vectors. As we can see for example in Schwarzschild metric [16].

$$g = -\left(1-\frac{r_s}{r}\right)c^2dt^2 + \left(1-\frac{r_s}{r}\right)^{-1}dr^2 + r^2d\theta^2 + r^2sin^2\theta d\varphi^2 \quad (150)$$

As commented, we can see on side, that the time has a negative signature compared with the space basis vectors.

On the other hand, we can see that if we disregard the coefficients related to the spherical system of coordinates -and to the harmonization of units- and we consider only the coefficients that really appear because of the non-Eucliden metric caused by the gravitation field, we have:

For space:

$$\left(1-\frac{r_s}{r}\right)^{-1} \cdot 1 \cdot 1 = \left(1-\frac{r_s}{r}\right)^{-1} \quad (151)$$

And for time:

$$-\left(1-\frac{r_s}{r}\right) \quad (152)$$

This means, the metric of time is inverse and with different sign than the combination of the space basis vectors (the multiplications of their non-Euclidean coefficients).

So, there is a solution for the basis vector of the time, not adding a new basis vector to the system of coordinates. But to define the time vector as derived by the space vectors this way:

$$\boldsymbol{e_0} = \frac{1}{\boldsymbol{e_1 e_2 e_3}} = (\boldsymbol{e_1 e_2 e_3})^{-1} = \boldsymbol{e_3}^{-1}\boldsymbol{e_2}^{-1}\boldsymbol{e_1}^{-1} = \frac{\boldsymbol{e_3}}{|e_3|^2}\frac{\boldsymbol{e_2}}{|e_2|^2}\frac{\boldsymbol{e_1}}{|e_1|^2}$$

$$= -\frac{\boldsymbol{e_1}}{|e_1|^2}\frac{\boldsymbol{e_2}}{|e_2|^2}\frac{\boldsymbol{e_3}}{|e_3|^2} \quad (153)$$

This means, the basis vector of time, is not independent. It is generated by the combination of the space vectors. This does not mean, that there is not a dimension of time. The degree of freedom of time (its multiplying parameter, its coefficient) is still free, what it is depending on the space vectors is its basis vector.

The advantage of the GA is that three space vectors create a universe with 8 degrees of freedom (or dimensions if you want to call it): scalar, 3 space vectors, three space bivectors and one trivector. It happens that the trivector behaves exactly as the basis vector of time we need. It has a negative signature (its multiplication by itself is negative) and its metrics is the opposite as the other space vectors.

So, this means in our universe, the basis vectors with odd grade (one and three) creates what we normally call dimensions (space and time). The even grade basis vectors (scalars and bivectors) are to be studied. But the scalar, is just an escalation factor, for any interaction, that probably is hidden (or not necessary in our calculations), or the opposite, it is key to understand the distortions of gravity (it has different values depending on the metric for example).

A1.7 Spin as trivector

The last point to be commented is the spin of a particle. We have seen in a.1.1 the generalized speed of an object:

$$U = u_0 + u_1\boldsymbol{e_1} + u_2\boldsymbol{e_2} + u_3\boldsymbol{e_3} + u_{12}\boldsymbol{e_1 e_2} + u_{13}\boldsymbol{e_1 e_3} + u_{23}\boldsymbol{e_2 e_3} + u_{123}\boldsymbol{e_1 e_2 e_3} \quad (154)$$

Being the u_i the linear speed and the u_{ij} the rotational speed (and its orientation- normally called axis, in GA plane of rotation perpendicular to the axis-).

And the meaning of the trivector u_{123} as something unknown. We can consider it as a rotation in two axes at the same time (a rotation in two planes at the same time).

And if consider the generalized multivector of the electromagnetic interaction:

$$F = F_0 + F_1\boldsymbol{e_1} + F_2\boldsymbol{e_2} + F_3\boldsymbol{e_3} + F_{12}\boldsymbol{e_1 e_2} + F_{13}\boldsymbol{e_1 e_3} + F_{23}\boldsymbol{e_2 e_3} + F_{123}\boldsymbol{e_1 e_2 e_3} \quad (155)$$

We can see the electric vector F_i will interact with the trivector, acting on the bivectors (in the orientation and rotational speed). And the magnetic field F_{ij} will interact with the trivector resulting in a vector (in the linear speed). But as it is expected that the orientation of the trivector is changing continuously, this would create an oscillatory movement with mean value zero (not detectable, it would be seen as a random movement, noise).

Just to close the issue of the spin, we can try to explain the 4π geometry (the ½ spin) in opposition to the 2π general geometry, using the trivector (as this rotation in two axes at the same time).

We can see it using a die. In standard 2π geometry, to rotate a die to get to the same position will need 4 rotations (that represents a standard complete 2π rotation):

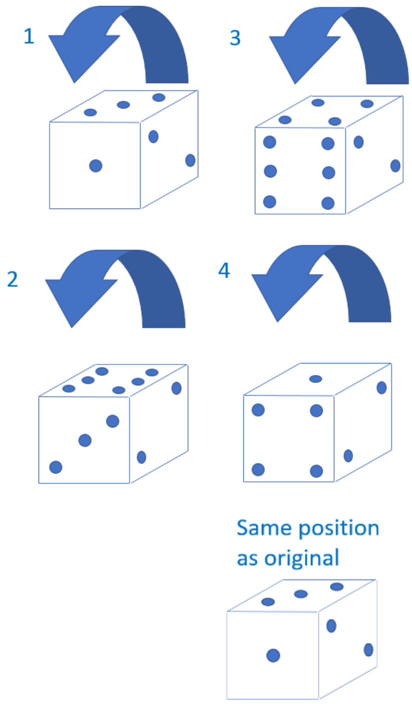
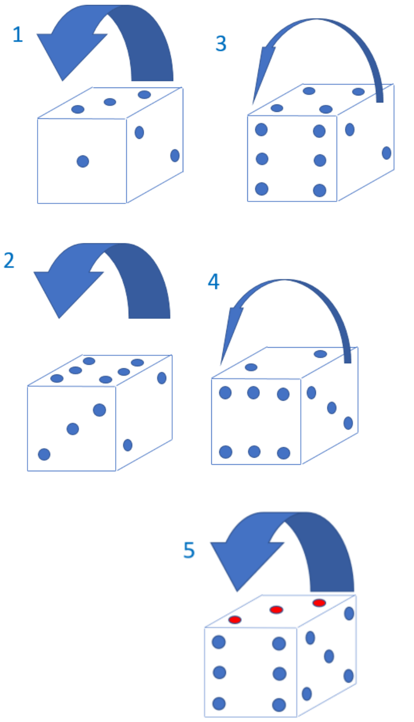
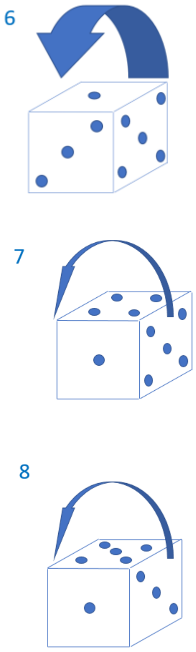


Figure 12. The die returns to its original position after 4 movements (2π rotation).

But if we mix the rotation axes, we will need 8 movements -as you will see in the pictures below-, that represent two times 2π (this means, 4π geometry or ½ spin). Here we change the axes in sequence, but it is expected that a trivector rotation, changes the axes somehow simultaneously! even if we are not able to imagine it (the same as we cannot imagine 4 dimensions, but we can understand how they work). The issue is that we are so accustomed to our world of rigid bodies with only one axis of rotation, that we cannot imagine that particles could be imagined for example as a kind of slime that could have different rotation axes at the same time (not breaking itself).



We can see in 5, that the upper face is the same as the original, but the orientation of the die is not the same.



Same position as original

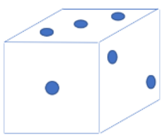


Figure 13. The die returns to its original position after 8 movements (4π rotation). This is because the axis of rotation is changing during the rotation. In the figure example changes in a sequent manner, in a trivector rotation the change of axis occurs simultaneously to the rotation.

A1.8 Generalized Fourier transform using GA

If we see the definition of the Fourier transform [17], it has the imaginary unit in its definition.

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\omega} dx \quad (156)$$

The Laplace transform [18] is similar but instead of using a pure imaginary number it uses a complex number (real+imaginary):

$$F(\sigma + i\omega) = \int_{0}^{\infty} f(t) e^{-(\sigma + i\omega)t} dt \quad (157)$$

Now, that we know that in GA the imaginary unit can be a bivector (or even a trivector) as seen in (29) and (30), we can represent it in GA as:

$$F(\sigma + e_1 e_2 \omega) = \int_{0}^{\infty} f(t) e^{-(\sigma + e_1 e_2 \omega)t} dt \quad (158)$$

Having the same result because as $(e_1e_2)^2$=-1, is indistinguishable to the imaginary unit i^2=-1, as can it can be seen in (30).

The point is that now, we can generalize even more the Fourier/Laplace transform if we include the complete multivector (instead of only scalar and bivector), this way:

$$F(\sigma + e_1\sigma_1 + e_2\sigma_2 + e_1 e_2 \omega) = \int_{0}^{\infty} f(t) e^{-(\sigma + e_1\sigma_1 + e_2\sigma_2 + e_1 e_2 \omega)t} dt \quad (159)$$

The issue is that as here e_1^2= e_2^2=1, these elements create hyperbolic functions (sinh and cosh) instead of sin and cos (that are created by $(e_1e_2)^2$=-1). These added functions could have influence in several disciplines, as signal theory, or number theory. You can check the paper [19] where the Fourier transform is used to check primality of a number. Probably generalizing this transform could add new insights in this field.

Annex 2. Calculation of wedge product using matrices

A2. 1 Wedge product using matrices in 2 dimensions

If we have the following two vectors in two dimensions:

$$\boldsymbol{a} = a_1 \boldsymbol{e_1} + a_2 \boldsymbol{e_2} \quad (160)$$

$$\boldsymbol{b} = b_1 \boldsymbol{e_1} + b_2 \boldsymbol{e_2} \quad (161)$$

In (52), we have calculated that:

$$\mathbf{a}\wedge\mathbf{b} = (a_1 b_2 - a_2 b_1)\ \mathbf{e_1 e_2} \quad (162)$$

So, in an orthonormal basis, the modulus of the wedge product is:

$$|\mathbf{a}\wedge\mathbf{b}| = (a_1 b_2 - a_2 b_1)\ |\mathbf{e_1 e_2}| = a_1 b_2 - a_2 b_1 \quad (163)$$

So, if you want to calculate the wedge product using matrices, calculate the following determinant.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - b_1 a_2 \quad (164)$$

You can see that we obtain the desired result.

A2. 2 Wedge product in 3 dimensions

In three dimensions, if we have the following vectors:

$$\boldsymbol{a} = a_1\boldsymbol{e_1} + a_2\boldsymbol{e_2} + a_3\boldsymbol{e_3} \quad (165)$$
$$\boldsymbol{b} = b_1\boldsymbol{e_1} + b_2\boldsymbol{e_2} + b_3\boldsymbol{e_3} \quad (166)$$

We can calculate the wedge product in an orthonormal basis as:

$$\begin{aligned}
\boldsymbol{a}\wedge\boldsymbol{b} &= \frac{1}{2}(\boldsymbol{ab} - \boldsymbol{ba}) \\
&= \frac{1}{2}[(a_1\boldsymbol{e_1} + a_2\boldsymbol{e_2} + a_3\boldsymbol{e_3})(b_1\boldsymbol{e_1} + b_2\boldsymbol{e_2} + b_3\boldsymbol{e_3}) \\
&\quad - (b_1\boldsymbol{e_1} + b_2\boldsymbol{e_2} + b_3\boldsymbol{e_3})(a_1\boldsymbol{e_1} + a_2\boldsymbol{e_2} + a_3\boldsymbol{e_3})] \\
&= \frac{1}{2}[a_1 b_1\boldsymbol{e_1}^2 + a_1 b_2\boldsymbol{e_1 e_2} + a_1 b_3\boldsymbol{e_1 e_3} + a_2 b_1\boldsymbol{e_2 e_1} + a_2 b_2\boldsymbol{e_2}^2 \\
&\quad + a_2 b_3\boldsymbol{e_2 e_3} + a_3 b_1\boldsymbol{e_3 e_1} + a_3 b_2\boldsymbol{e_3 e_2} + a_3 b_3\boldsymbol{e_3}^2 \\
&\quad - (a_1 b_1\boldsymbol{e_1}^2 + a_1 b_2\boldsymbol{e_2 e_1} + a_1 b_3\boldsymbol{e_3 e_1} + a_2 b_1\boldsymbol{e_1 e_2} + a_2 b_2\boldsymbol{e_2}^2 \\
&\quad + a_2 b_3\boldsymbol{e_3 e_2} + a_3 b_1\boldsymbol{e_1 e_3} + a_3 b_2\boldsymbol{e_2 e_3} + a_3 b_3\boldsymbol{e_3}^2)] \\
&= \frac{1}{2}[2a_1 b_2\boldsymbol{e_1 e_2} + 2a_1 b_3\boldsymbol{e_1 e_3} + 2a_2 b_1\boldsymbol{e_2 e_1} \\
&\quad + 2a_2 b_3\boldsymbol{e_2 e_3} + 2a_3 b_1\boldsymbol{e_3 e_1} + 2a_3 b_2\boldsymbol{e_3 e_2}] \\
&= (a_1 b_2 - a_2 b_1)\boldsymbol{e_1 e_2} + (a_1 b_3 - a_3 b_1)\boldsymbol{e_1 e_3} \\
&\quad + (a_2 b_3 - a_3 b_2)\boldsymbol{e_2 e_3} \quad (167)
\end{aligned}$$

If you want to calculate it using matrices, calculate the following determinant and you will get the same result (remember that $\mathbf{e_3 e_1} = -\mathbf{e_1 e_3}$ when orthonormal basis):

$$\boldsymbol{a}\wedge\boldsymbol{b} = \begin{vmatrix} a_1 & b_1 & \boldsymbol{e_2 e_3} \\ a_2 & b_2 & \boldsymbol{e_3 e_1} \\ a_3 & b_3 & \boldsymbol{e_1 e_2} \end{vmatrix} \quad (168)$$

It is necessary to make a complete mathematical study of all the equivalences between standard mathematics in different branches and geometric algebra. Of course, the main comparison should be done with Gibbs algebra, and matrices but also Riemann metrics, quantum algebra etc. This way the translation from previous disciplines to geometric algebra will be almost immediate.

9. Conclusions

We have seen in the paper, how to use of geometric algebra to be able to work in a non-Euclidean geometry. The difference with other geometries is that you can include all the information regarding the metric in the basis vectors themselves. It is not needed to rotate them to get an orthonormal basis or to use an external basis.

You can forget about the geometry, performing all the calculations/operations you need as if you are in an Euclidean geometry and just in the end when you resolve the operations involving the metric vectors themselves, the information of the metric appears naturally in the results.

The definitions of derivatives involving these basis vectors would allow us also to calculate non-Euclidean metrics as the Schwarzschild metric (exercise not finished at this stage).

To finish, it has been discussed also the possibilities of using geometric algebra in other disciplines, leading to really surprising results, that could key to understand our view in these areas. Examples are: rigid body and particle state vectors, electromagnetism, hidden variables, definition of the time basis vector as dependent of the space basis vectors, relation between the trivector and the spin, generalization of the gradient and of the Fourier transform.

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