Why Finite Mathematics Is More Fundamental Than Classical One

Felix M. Lev

Artwork Conversion Software Inc., 509 N. Sepulveda Blvd, Manhattan Beach, CA 90266, USA (Email: felixlev314@gmail.com)

Abstract

In our previous publications we have proved that quantum theory based on finite mathematics is more fundamental than standard quantum theory, and, as a consequence, finite mathematics is more fundamental than classical one. The goal of the present paper is to explain without formulas why those conclusions are natural.

Keywords: classical mathematics, finite mathematics, quantum theory

1 Introduction

The history of mankind undoubtedly shows that classical mathematics (involving such notions as infinitely small/large, continuity etc.) has demonstrated its power in many areas of science. Nevertheless, this mathematics cannot be a part of the ultimate theory describing nature on the very fundamental level.

One of the reasons follows. The notions of infinitely small/large, continuity etc. were proposed by Newton and Leibniz more than 300 years ago. At that time people did not know about existence of atoms and elementary particles and believed that any body can be divided by an arbitrarily large number of arbitrarily small parts. However, now it is obvious that standard division has only a limited applicability because when we reach the level of atoms and elementary particles standard division loses its meaning. For example, a glass of water contains approximately 10^{25} molecules. We can divide this water by ten, million, etc. but when we reach the level of atoms and elementary particles further division operation loses its usual meaning and we cannot obtain arbitrarily small parts. In nature there are no continuous curves and surfaces. For example, if we draw a line on a sheet of paper and look at this line by a microscope then we will see that the line is strongly discontinuous because it consists of atoms. So, as far as application of mathematics to physics is concerned, classical mathematics is only an approximation which in many cases works with very high accuracy but the ultimate quantum theory cannot be based on classical mathematics.

Another reason is that, in spite of great efforts of many great mathematicians (Cantor, Fraenkel, Gödel, Hilbert, Kronecker, Russell, Zermelo and others), the problem of foundation of classical mathematics has not been solved and probably it cannot be solved in principle. For example, Gödel's incompleteness theorems state that no system of axioms can ensure that all facts about natural numbers can be proven and the system of axioms in classical mathematics cannot demonstrate its own consistency.

Several alternatives to classical mathematics have been proposed, for example constructive mathematics, finitistic mathematics and others. The main goal of those alternatives is to find a treatment of infinity which will solve the problem of foundation of mathematics. For example, while in finitistic mathematics all natural numbers are accepted as existing (i.e. this mathematics deals with the infinite number of objects), the set of all natural numbers is not considered to exist as a mathematical object.

On the other hand, finite mathematics (considering e.g. finite rings and fields) deals only with a finite number of objects, and, as a consequence, there are no foundational problems in this mathematics because every statement can be directly verified by using only a finite number of steps. Nevertheless, a belief of the overwhelming majority of scientists is that finite mathematics is something inferior what is used only in special applications.

In mathematics there is no definition of the notion that one branch of mathematics is more fundamental than the other if the branches are treated only as abstract sciences. For example, classical and finite mathematics are treated as independent branches of mathematics each of which deals with its own problems. However, one can pose a problem what mathematics is more pertinent for applications. Since quantum theory is the most general physics theory (i.e. all other physics theories are special cases of quantum one), the problem what mathematics is the most fundamental is the problem of physics, not mathematics. Since no version of constructive and finitistic mathematics is used in quantum theory, the remaining problem is what version of quantum theory is more fundamental: quantum theory based on classical mathematical or quantum theory based on finite mathematics.

For investigating this problem one should first define a notion when theory A is more general than theory B. Following our Ref. [1], we propose the following

Definition: *Let theory A contain a finite parameter and theory B be obtained from theory A in the formal limit when the parameter goes to zero or infinity. Suppose that with any desired accuracy theory A can reproduce any result of theory B by choosing a value of the parameter. On the contrary, when the limit is already taken then one cannot return back to theory A and theory B cannot reproduce all results of theory A. Then theory A is more general than theory B and theory B is a special degenerate case of theory A.*

Well-known examples in physics are that: a) classical theory is a special degenerate case of quantum one in the formal limit $\hbar \to 0$ where $\hbar$ is the Planck constant; b) nonrelativistic theory is a special degenerate case of relativistic one in the formal limit $c \to \infty$ where c is the speed of light; c) Poincare invariant theory is a special degenerate case of de Sitter invariant theories in the formal limit $R \to \infty$ where R is the parameter defining contraction from the de Sitter Lie algebras to the Poincare Lie algebra. In his famous paper "Missed Opportunities" [2] Dyson notes that b) and c) follow not only from physical but also from pure mathematical considerations. Poincare group is more symmetric than Galilei one and the transition from the former to the latter at $c \to \infty$ is called contraction. Analogously de Sitter groups are more symmetric than Poincare one and the transition from the former to the latter at $R \to \infty$ also is called contraction. At the same time, since de Sitter groups are semisimple they have a maximum possible symmetry and cannot be obtained from more symmetric groups by contraction.

However, as argued in our publications (see e.g. Ref. [3]), symmetry on quantum level should be treated not from the point of view of a symmetry group but from the point of view of commutation relations in the symmetry Lie algebra. Then, as shown in Ref. [4], the properties a), b) and c) take place because quantum symmetry Lie algebra is more general than classical one, Poincare Lie algebra is more general than Galilei one and de Sitter Lie algebras are more general than Poincare one.

In our publications (see e.g. Refs. [1, 3, 5, 6]) we discussed an approach called Finite Quantum Theory (FQT) where quantum theory is based not on classical but on finite mathematics. Physical states in FQT are elements of a linear space over a finite field or ring, operators of physical quantities are linear operators in this space and symmetry is defined by a Lie algebra over a finite field or ring. It has been rigorously proved in Ref. [1] that FQT is more general than standard quantum theory and the latter is a special degenerate case of the former in the formal limit when the characteristic p of the field or ring in FQT goes to infinity. Therefore, as follows from the above remarks

Main Statement: Even classical mathematics itself is a special degenerate case of finite mathematics in the formal limit $p \to \infty$.

The goal of the present paper is to explain without formulas why this conclusion is natural. In Sec. 2 we argue that standard arithmetic operations are ambiguous and only operations modulo a number are unambiguously defined. Then in Sec. 3 we describe steps necessary for proving **Main Statement**. Finally, Sec. 4 is discussion.

2 Remarks on arithmetic

Although the present paper deals with a pure mathematical problem, we believe that for illustration it is important to discuss philosophical aspects of such a simple problem as operations with natural numbers.

In the 20s of the 20th century the Viennese circle of philosophers under the leadership of Schlick developed an approach called logical positivism which contains verification principle: *A proposition is only cognitively meaningful if it can be definitively and conclusively determined to be either true or false* (see e.g. Refs. [7]). On the other hand, as noted by Grayling [8], *"The general laws of science are not, even in principle, verifiable, if verifying means furnishing conclusive proof of their truth. They can be strongly supported by repeated experiments and accumulated evidence but* *they cannot be verified completely"*. Popper proposed the concept of falsificationism [9]: *If no cases where a claim is false can be found, then the hypothesis is accepted as provisionally true.*

According to the principles of quantum theory, there should be no statements accepted without proof and based on belief in their correctness (i.e. axioms). The theory should contain only those statements that can be verified, at least in principle, where by "verified" physicists mean experiments involving only a finite number of steps. So the philosophy of quantum theory is similar to verificationism, not falsificationism. Note that Popper was a strong opponent of the philosophy of quantum theory and supported Einstein in his dispute with Bohr.

The verification principle does not work in standard classical mathematics. For example, it cannot be determined whether the statement that $a + b = b + a$ for all natural numbers a and b is true or false. According to falsificationism, this statement is provisionally true until one has found some numbers a and b for which $a + b \neq b + a$. There exist different theories of arithmetic (e.g. Peano arithmetic or Robinson arithmetic) aiming to solve foundational problems of standard arithmetic. However, those theories are not used in applications.

From the point of view of verificationism and principles of quantum theory, classical mathematics is not well defined not only because it contains an infinite number of numbers. For example, let us pose a problem whether 10+20 equals 30. Then one should describe an experiment which gives the answer to this problem. Any computing device can operate only with a finite number of bits and can perform calculations only modulo some number p. Say $p = 40$, then the experiment will confirm that 10+20=30 while if $p = 25$ then one will get 10+20=5.

So the statements that 10+20=30 and even that $2 \cdot 2 = 4$ *are ambiguous because they do not contain information on how they should be verified.* On the other hands, the statements

$$10 + 20 = 30\,(mod\,40),\;\; 10 + 20 = 5\,(mod\,25),$$

$$2 \cdot 2 = 4\,(mod\,5),\;\; 2 \cdot 2 = 2\,(mod\,2)$$

are well defined because they do contain such an information. So, from the point of view of verificationism and principles of quantum theory, only operations modulo a number are well defined.

We believe the following observation is very important: although classical mathematics (including its constructive version) is a part of our everyday life, people typically do not realize that *classical mathematics is implicitly based on the assumption that one can have any desired amount of resources.* In other words, standard operations with natural numbers are implicitly treated as limits of operations modulo p when $p \to \infty$. Usually in mathematics, legitimacy of every limit is thoroughly investigated, but in the simplest case of standard operations with natural numbers it is not even mentioned that those operations can be treated as limits of operations modulo p. In real life such limits even might not exist if, for example, the Universe contains a finite number of elementary particles.

Classical mathematics proceeds from standard arithmetic which does not contain operations modulo a number while finite mathematics necessarily involves such operations. As already noted, the goal of the present paper is to give a simple explanation why, regardless of philosophical preferences, finite mathematics is more fundamental than classical one.

3 Explanation of the main statement

Classical mathematics starts from natural numbers but here only addition and multiplication are always possible. In order to make addition invertible we introduce negative integers and get the ring of integers Z. However, if instead of all natural numbers we consider only a set R_p of p numbers 0, 1, 2, ... $p-1$ where addition and multiplication are defined as usual but modulo p then we get a ring without adding new elements. In our opinion the notation Z/p for R_p is not quite adequate because it may give a wrong impression that finite mathematics starts from the infinite set Z and that Z is more general than R_p. However, although Z has more elements than R_p, Z cannot be more general than R_p because Z does not contain operations modulo a number.

In classical mathematics the ring Z is the starting point for introducing rational and real numbers. In turn this gives rise to the set theory where infinite sets with different cardinalities are possible. The problem of actual infinity is discussed in a vast literature. The *technique* of classical mathematics does not involve actual infinities and here infinities are understood only as limits. However, the *basis* of classical mathematics does involve actual infinities. In particular, by analogy with the situation in standard arithmetic discussed in the preceding section, it is not even posed a problem whether Z can be treated as a limit of finite sets, and from the very beginning Z is treated as actual and not potential infinity.

As proved in Ref. [1], the result of any finite combination of additions, subtractions and multiplications in Z can be reproduced in R_p if p is chosen to be sufficiently large. This means that Z can be treated as a limit of R_p when $p \to \infty$. When the limit is already taken, one cannot return back from Z to R_p, and in Z it is not possible to reproduce all results in R_p because in Z there are no operations modulo a number. According to **Definition** in Sec. 1 this means that

Statement 1: The ring R_p is more general than Z, and Z is a special degenerate case of R_p.

This example demonstrates that once we involve infinity and replace R_p by Z then we automatically obtain a degenerate theory because in Z there are no operations modulo a number.

Although Z is a degenerate case of R_p, applications of classical mathematics involve extensions of Z to the fields of rational and real numbers. When p is prime then R_p becomes the Galois field F_p and its possible extensions can be only finite fields containing p^k elements where k is a natural number. Then a question arises whether finite mathematics can reproduce all results obtained by applications of classical mathematics. For example, if p is prime then the results in F_p considerably differ from those in the set Q of rational numbers even when p is very large. In particular, $1/2$ in F_p is a very large number $(p+1)/2$. Since quantum theory is the most general physical theory, a problem arises whether standard quantum theory based on classical mathematics is most general or is a special degenerate case of a quantum theory based on finite mathematics.

As noted in Sec. 1, de Sitter invariant quantum theory is more general than Poincare invariant quantum theory. In the former, quantum states are described by representations of the de Sitter algebras. According to principles of quantum theory, from the ten linearly independent operators defining such representations one should construct a maximal set S of mutually commuting operators defining independent physical quantities and construct a basis in the representation space such that the basis elements are eigenvectors of the operators from S. In Secs. 4.1 and 8.2 of Ref. [6] we have shown that

Statement 2: For the de Sitter algebras there exist sets S and representations such that basis vectors in the representation spaces are eigenvectors of the operators from S with eigenvalues belonging to Z. Such representations reproduce standard representations of the Poincare algebra in the formal limit $R \to \infty$. Therefore the remaining problem is whether or not quantum theory based on finite mathematics can be a generalization of standard quantum theory where states are described by elements of a separable complex Hilbert spaces H.

Let $(e_1, e_2, ...)$ be a basis of H normalized such that the norm of each e_j is an integer. The known fact in the theory of Hilbert spaces is that *with any desired accuracy each element of H can be approximated by a finite linear combination of the basis elements with rational coefficients* because the set of such linear combinations is dense in H.

The next observation is that spaces in quantum theory are projective, i.e. for any complex number $c \neq 0$ and any element $x \in H$, x and cx describe the same state. This follows from the physical fact that not the probability itself but only ratios of probabilities have a physical meaning. In view of this property, the linear combination approximating the element $x \in H$ can be multiplied by a common denominator of all the rational coefficients in this combination. As a result, we have

Statement 3: Each element of H can be approximated by a finite linear combination with the coefficients $a_j + ib_j$ where all the numbers a_j and b_j belong to Z.

In the literature it is also considered a version of quantum theory based not on real but on p-adic numbers (see e.g. the review paper [10] and references therein). Both, the sets of real and p-adic numbers are the completions of the set of rational numbers but with respect to different metrics. Therefore the set of rational numbers is dense in both, in the set of real numbers and in the set of p-adic numbers $\mathcal{Q}_p$. In the p-adic case, the Hilbert space analog of H is the space of complex-valued functions $L^2(\mathcal{Q}_p)$ and therefore there is an analog of *Statement 3*.

We conclude that Hilbert spaces in standard quantum theory contain a big redundancy of elements. Indeed, although formally the description of states in standard quantum theory involves rational and real numbers, such numbers play only an auxiliary role because with any desired accuracy each state can be described by using only integers. Therefore, as follows from **Definition** in Sec. 1 and *Statements 1-3*,

- Standard quantum theory based on classical mathematics is a special degenerate case of quantum theory based on finite mathematics.
- **Main Statement** formulated in Sec. 1 is valid.

4 Discussion

As noted in our publications, while the elements of Z can be naturally depicted as points on a straight line, the elements of R_p can be naturally depicted as points on a circumference (see e.g. Fig. 6.2 in Ref. [6]). Then the above construction has a known historical analogy. For many years people believed that the Earth was flat and infinite, and only after a long period of time they realized that it was finite and curved. It is difficult to notice the curvature dealing only with distances much less than the radius of the curvature. Analogously one might think that the set of numbers describing nature in our Universe has a "curvature" defined by a very large number p but we do not notice it dealing only with numbers much less than p.

As noted in Sec. 1, the fact that finite mathematics is more fundamental than classical one agrees with the general trend that when a theory contains a finite parameter then in the formal limit when the parameter goes to zero or infinity one gets a less general degenerated theory. In particular, as noted in the preceding section, introducing infinity automatically implies transition to a degenerate theory because in this case operations modulo a number are lost. Therefore *even from the pure mathematical point of view* the notion of infinity cannot be fundamental, and theories involving infinities can be only approximations of more general theories. The famous Kronecker expression is: "God made the integers, all else is the work of man". However, in view of the above discussion, it is reasonable to reformulate this expression as "God made not all integers but only a finite subset of them while infinity, infinitely small/large, continuity etc. is the work of man".

Following our previous publications, we have explained in the preceding section that quantum theory based on finite mathematics is more fundamental than standard quantum theory and therefore classical mathematics is a special degenerate case of finite one in the formal limit $p \to \infty$. The fact that at the present stage of the Universe p is a huge number explains why in many cases classical mathematics describes natural phenomena with a very high accuracy. At the same time, as shown in Refs. [3, 5, 6], the explanation of several phenomena can be given only in the theory where p is finite.

One of the examples is that in our approach gravity is a manifestation of the fact that p is finite. In Ref. [6] we have derived the approximate expression for the gravitational constant which depends on p as $1/lnp$. By comparing this expression with the experimental value we get that lnp is of the order of 10^{80} or more, i.e. p is a huge number of the order of $exp(10^{80})$ or more. However, since lnp is "only" of the order of 10^{80} or more, the existence of p is observable while in the formal limit $p \to \infty$ gravity disappears.

As noted in Sec. 1, the problem of foundation of classical mathematics has not been solved, and probably it cannot be solved in principle. Let us recall that classical mathematics does not involve operations modulo a number. The philosophy of great mathematicians working on foundation of classical mathematics was usually based on macroscopic experience in which the notions of infinitely small, infinitely large, continuity and standard division are natural. However, as noted in Sec. 1, those notions contradict the existence of elementary particles and are not natural in quantum theory. The illusion of continuity arises when one neglects the discrete structure of matter.

However, since in applications classical mathematics is a special degenerate case of finite one, foundational problems of classical mathematics are important only when it is treated as an abstract science. The technique of classical mathematics is very powerful and in many cases (but not all of them) describes reality with a high accuracy.

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