A remark on a golden arbelos in Wasan geometry

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Abstract. We consider a problem in Wasan geometry involving a golden arbelos.

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1. Introduction

We consider the arbelos appeared in Wasan geometry, and consider an arbelos formed by three semicircles α, β and γ with diameters AO, BO and AB, respectively for a point O on the segment AB (see Figure 1). We denote the arbelos and the radii of α and β by (α, β, γ) and a and b, respectively, and call the perpendicular to AB at O the axis. Circles of radius $r_{\rm A} = ab/(a+b)$ are said to be Archimedean, and the incircle of the curvilinear triangle made by α, γ and the axis is Archimedean, which is denoted by δ. Let σ be the reflection in the perpendicular to AB at the center of γ. We consider the following problem in [11] (see Figure 2).

Problem 1. Let ε be the circle touching α^σ externally γ internally and the axis from the side opposite to A. If ε and α have the same radius, find the radius of ε in terms of the difference of the radii of γ and δ.

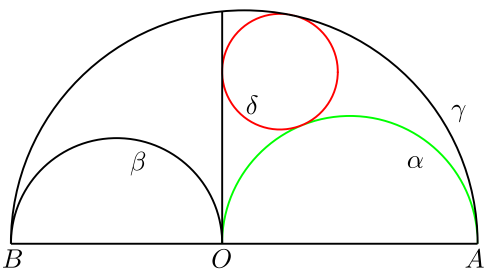

Figure 1: (α, β, γ) with δ.

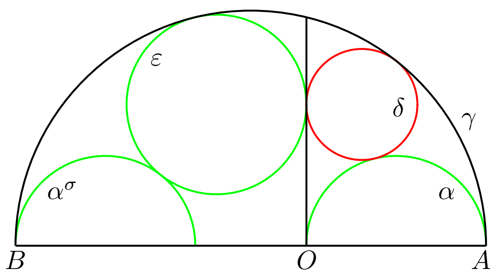

Figure 2.

The same sangaku problem proposed in 1891 [1]. If $a/b = \phi^{\pm 1}$, then (α, β, γ) is called a golden arbelos, where $\phi = (1+\sqrt{5})/2$. We will show that the figure of the problem forms a golden arbelos and the circles δ and ε touch. We will also give a condition in which the circles δ and ε touch in the case $a \neq b$.

2. Circles touching a perpendicular to AB at the same point

We use a rectangular coordinate system with origin O such that the farthest point on α from AB has coordinates (a, a). We use the next proposition.

Proposition 1. *It two externally touching circles of radii r_1 and r_2 touch a line at two points P and Q, then $|PQ| = 2\sqrt{r_1 r_2}$.*

Theorem 1. *Let ζ be the semicircle of diameter BO' constructed on the same side as γ for a point O' on the segment AB, and let ε be the circle touching γ internally, ζ externally and the axis from the side opposite to A. Then the following statements are equivalent.*
(i) *The circles δ and ε touch.*
(ii) *The circle ε has radius $b - r_{\mathrm{A}}$.*
(iii) *The semicircle ζ coincides with α^σ.*

Proof. Let e and z be the radii of ε and ζ, respectively, and let y be the y-coordinate of the center of ε (see Figure 3). Then we have $(a + b - e)^2 = (-e - (a - b))^2 + y^2$ and $(z + e)^2 = (-e - (-2b + z))^2 + y^2$. Solving the equations for e and z, respectively, we get

$$e = b - \frac{y^2}{4a} \tag{1}$$

and

$$z = b - e + \frac{y^2}{4b}. \tag{2}$$

While (i) is equivalent to $y = 2\sqrt{ar_{\mathrm{A}}}$ by Proposition 1. Therefore (1) implies that $y = 2\sqrt{ar_{\mathrm{A}}}$ if and only if $e = b - r_{\mathrm{A}}$, i.e., (i) and (ii) are equivalent. Substituting (1) in (2), we get

$$y^2 = 4zr_{\mathrm{A}}. \tag{3}$$

The equation gives that $y = 2\sqrt{ar_{\mathrm{A}}}$ if and only if $z = a$, i.e., (i) and (iii) are equivalent. □

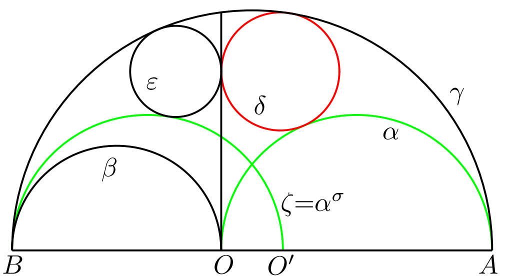

Figure 3.

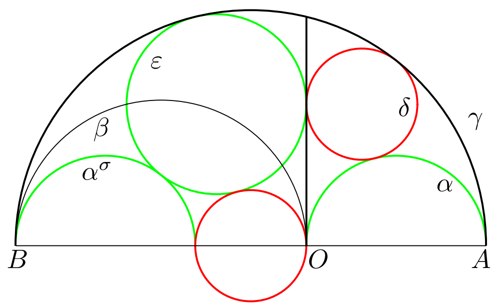

Figure 4.

We now consider the figure of Problem 1 and assume that the radius of the circle ε equals a (see Figure 4). Then by the equivalence of (ii) and (iii) in Theorem 1 we have

$$a = b - r_{\mathrm{A}}. \tag{4}$$

Let c be the radius of γ. Then $2a = a + b - r_{\mathrm{A}} = c - r_{\mathrm{A}}$, i.e., $a = (c - r_{\mathrm{A}})/2$, which is an answer of Problem 1. Solving (4) for b, we get $b = \phi a$. Therefore (α, β, γ) is a golden arbelos, where notice that r_{A}, a, b, c form a geometric progression with common ratio ϕ. Also (4) implies that there is an Archimedean circle concentric to γ touching the axis and the circles α, α^σ and ε externally.

The Archimedean circle touching ε externally and the axis at O can also be obtained in the case $b \neq \phi a$. Notice that the radius of the circle touching ε externally and the axis at O from the side opposite to A equals $y^2/(4e) = (z/e)r_{\mathrm{A}}$

by Proposition 1 and (3) in the proof of Theorem 1. Therefore we get (see Figure 5):

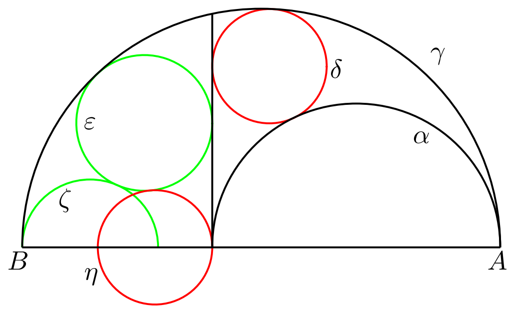

Figure 5.

Theorem 2. *Let ζ and ε be the semicircle and the circle in Theorem 1, and let η be the circle touching ε externally and the axis at O from the side opposite to A. Then η is Archimedean if and only if ζ and ε have the same radius. In this event, (α, β, γ) is a golden arbelos if and only if ζ and η touch.*

We have considered two circles touching a perpendicular to AB from the opposite side at the same point in a general way in [5]. Theorem 1 gives a special case in which such a pair of circles appears. Another condition using the reflection in the axis can also be found in [6].

3. Application of division by zero

We consider the relations (1), (2) with the recent definition of division by zero: $z/0 = 0$ for any real number z [3].

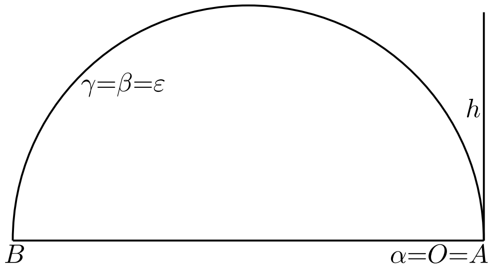

Figure 6: $a = 0$.

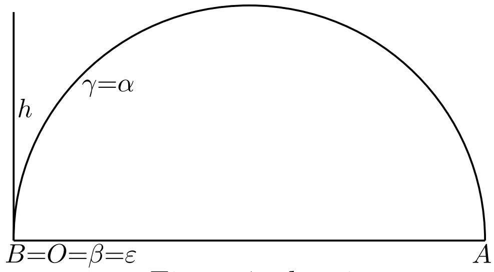

Figure 7: $b = 0$.

We consider (1). Notice that this relation is derived only from the assumption that the circle ε touches γ internally and the axis from the side opposite to A. If $a = 0$, then the semicircle α degenerates to the point A, β and γ coincide, and $y^2/(4a) = y^2/0 = 0$ by the definition of division by zero. Hence (1) implies $e = b$. Therefore the half part of the circle ε coincides with γ (see Figure 6).

If $b = 0$, then β and ε degenerate to the point B, i.e., $e = z = 0$, and $y^2/(4b) = 0$. Therefore (2) still holds (see Figure 6).

For more applications of division be zero to Wasan geometry see [2], [4], [7], [8], [9, 10].

4. A configuration arising from the golden arbelos

Let τ be the product of σ and the homothety of center A and ratio ϕ^{-1}. Let p be the x-coordinate of a point P on AB. Then we have $(p+p^{\sigma})/2 = a-b$ and $(p^{\sigma}-2a)/\phi = p^{\tau}-2a$, where p^{σ} and p^{τ} are the x-coordinates of the points P^{σ} and P^{τ}, respectively. Then $p^{\tau} = 2a+(p^{\sigma}-2a)/\phi = 2a+(-2b-p)/\phi = -p/\phi$. Therefore τ coincides with the homothety of center O with ratio $-1/\phi$. Hence $p^{\tau^n} = (-1)^n p/\phi^n$, i.e., P^{τ^n} has x-coordinate $(-1)^n p/\phi^n$, and the axis is fixed by τ. Notice that γ^{τ} passes through the point of tangency of δ and ε by Proposition 1, because $(2\sqrt{ar_{\mathrm{A}}})^2 = 2a\cdot 2\phi a = |A^{\tau}O||B^{\tau}O|$ (see Figure 8).

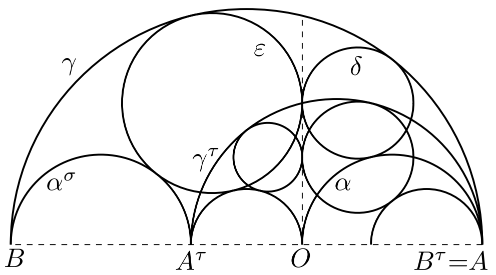

Figure 8: $\mathcal{K}\cup\mathcal{K}^{\tau} = \mathcal{K}_1\cup\mathcal{K}_2$.

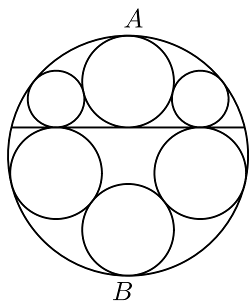

Figure 9.

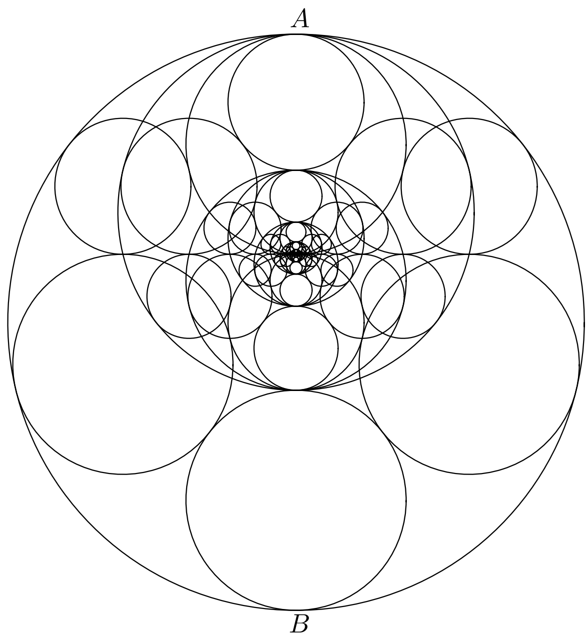

Figure 10: $\mathcal{K}_0$ with it reflection in AB.

Let $\mathcal{K}$ be the figure consisting of γ, α, α^{σ}, δ and ε in the case $b=\phi a$, which is obtained from Figure 2 by removing AB and the axis. Let $\mathcal{K}_i = \mathcal{K}^{\tau^{i-1}}$ for $i=1,2,3,\cdots$, and $\mathcal{K}_0 = \bigcup_{i\geq 1}\mathcal{K}_i$. It is a custom of Wasan geometry to describe the arbelos by three circles so that their centers lie on a vertical line. The original figure of Problem 1 is also described by $\mathcal{K}$ with the axis and its reflection in AB so that AB is a vertical segment as in Figure 9. Following to this custom, we also describe $\mathcal{K}_0$ so that AB is a vertical line with its reflection in AB (see Figure 10).

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