# Values of the Riemann Zeta Function by Means of Division by Zero Calculus

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**Abstract:** In this paper, we will give the values of the Riemann zeta function for any positive integers by means of the division by zero calculus.

**Key Words:** Zero, division by zero, division by zero calculus, $0/0 = 1/0 = z/0 = \tan(\pi/2) = \log 0 = 0$, Laurent expansion, Riemann zeta function, Gamma function, Psi function, Digamma function.

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## 1 Division by zero and division by zero calculus

For the long history of division by zero, see [2, 21]. The division by zero with mysterious and long history was indeed trivial and clear as in the followings.

By the concept of the Moore-Penrose generalized solution of the fundamental equation $ax = b$, the division by zero was trivial and clear as $b/0 = 0$ in the **generalized fraction** that is defined by the generalized solution of the equation $ax = b$. Here, the generalized solution is always uniquely determined and the theory is very classical. See [7] for example.

Division by zero is trivial and clear also from the concept of repeated subtraction - H. Michiwaki.

Recall the uniqueness theorem by S. Takahasi on the division by zero. See [7, 30].

The simple field structure containing division by zero was established by M. Yamada ([10]). For a simple introduction, see H. Okumura [19].

Many applications of the division by zero to Wasan geometry were given by H. Okumura. See [13, 14, 15, 16, 17, 18] for example.

As the number system containing the division by zero, the Yamada field structure is perfect. However, for applications of the division by zero to **functions**, we need the concept of the division by zero calculus for the sake of uniquely determinations of the results and for other reasons.

For example, for the typical linear mapping

$$W = \frac{z-i}{z+i}, \tag{1.1}$$

it gives a conformal mapping on $\{\mathbf{C} \setminus \{-i\}\}$ onto $\{\mathbf{C} \setminus \{1\}\}$ in one to one and from

$$W = 1 + \frac{-2i}{z-(-i)}, \tag{1.2}$$

we see that $-i$ corresponds to 1 and so the function maps the whole $\{\mathbf{C}\}$ onto $\{\mathbf{C}\}$ in one to one.

Meanwhile, note that for

$$W = (z-i) \cdot \frac{1}{z+i}, \tag{1.3}$$

we should not enter $z = -i$ in the way

$$[(z-i)]_{z=-i} \cdot \left[\frac{1}{z+i}\right]_{z=-i} = (-2i) \cdot 0 = 0. \tag{1.4}$$

However, in many cases, the above two results will have practical meanings and so, we will need to consider many ways for the application of the division by zero and we will need to check the results obtained, in some practical viewpoints. We referred to this delicate problem with many examples in the references.

Therefore, we will introduce the division by zero calculus. For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n, \tag{1.5}$$

we **define** the identity, (by the division by zero)

$$f(a) = C_0. \tag{1.6}$$

(Note that here, there is no problem on any convergence of the expansion (1.5) at the point $z = a$, because all the terms $(z-a)^n$ are zero at $z = a$ for $n \neq 0$.)

**Apart from the motivation, we define the division by zero calculus by (1.6).** With this assumption, we can obtain many new results and new ideas. However, for this assumption we have to check the results obtained whether they are reasonable or not. By this idea, we can avoid any logical problems. – In this point, the division by zero calculus may be considered as a fundamental assumption like an axiom.

The division by zero calculus opens a new world since Aristotele-Euclid. See, in particular, [4] and also the references for recent related results.

On February 16, 2019 we (H. Okumura) found the surprising news in Research Gate:

José Manuel Rodríguez Caballero
Added an answer
In the proof assistant Isabelle/HOL we have $x/0 = 0$ for each number $x$. This is advantageous in order to simplify the proofs. You can download this proof assistant here: **https://isabelle.in.tum.de/**.

J.M.R. Caballero kindly showed surprisingly several examples by the system for our questions that

$$\tan\frac{\pi}{2} = 0,$$

$$\log 0 = 0,$$

$$\exp\frac{1}{x}(x=0) = 1,$$

and others. Furthermore, for the presentation at the annual meeting of the Japanese Mathematical Society at the Tokyo Institute of Technology:

March 17, 2019 : 9: 45-10: 00 in Complex Analysis Session, *Horn torus models for the Riemann sphere from the viewpoint of division by zero* with [4],

he kindly sent the kind message:

It is nice to know that you will present your result at the Tokyo Institute of Technology. Please remember to mention Isabelle/HOL, which is a software in which x/0 = 0. This software is the result of many years of research and a millions of dollars were invested in it. If x/0 = 0 was false, all these money was for nothing. Right now, there is a team of mathematicians formalizing all the mathematics in Isabelle/HOL, where x/0 = 0 for all x, so this mathematical relation is the future of mathematics. https://www.cl.cam.ac.uk/ lp15/Grants/Alexandria/

Meanwhile, on ZERO, the authors S. K. Sen and R. P. Agarwal [28] published its long history and many important properties of zero. See also R. Kaplan [6] and E. Sondheimer and A. Rogerson [29] on the very interesting books on zero and infinity. In particular, for the fundamental relation of zero and infinity, we stated the simple and fundamental relation in [26] that

**The point at infinity is represented by zero; and zero is the definite complex number and the point at infinity is considered by the limiting idea** as an ideal point of one point compactification and that is represented geometrically with the horn torus model [4].

S. K. Sen and R. P. Agarwal [28] referred to the paper [7] in connection with division by zero, however, their understandings on the paper seem to be not suitable (not right) and their ideas on the division by zero seem to be traditional, indeed, they stated as the conclusion of the introduction of the book that:

**"Thou shalt not divide by zero" remains valid eternally.**

However, in [25] we stated simply based on the division by zero calculus that

**We Can Divide the Numbers and Analytic Functions by Zero with a Natural Sense.**

They stated in the book many meanings of zero over mathematics, deeply.

In this paper, we will examine the values of the Riemann zeta function for positive integers $2 \leq n$ by using the division by zero calculus. For the values of the Riemann zeta functions at positive integers, see [5]. In particular, note that for odd integers, their values were mysterious. We can give the values in the both senses of analytical and numerical.

# 2 Simple applications of the division by zero calculus

As the first try, we will see some simple applications of the division by zero calculus to some typical formulas in order to look for the values of the Riemann zeta function.

## 2.1 Method 1

First, we recall the basic identity

$$\frac{1}{\sin^2 z} = \sum_{k=-\infty}^{\infty} \frac{1}{(z - k\pi)^2}$$

([1], page 75: 4.3.92), because the right hand side becomes, by the division by zero calculus, at $z = 0$, by taking $n$ times derivative

$$\frac{2(n-1)!}{\pi^n}\zeta(n).$$

However, note that this formula is valid for an even $n$.

Meanwhile, we will use the expansion

$$\frac{1}{\sin z} = \frac{1}{z} + \frac{z}{6} + \frac{7}{360}z^3 + \frac{31}{15120}z^5 + \cdots$$
$$+\frac{(-1)^{n-1}2(2^{2n-1}-1)B_{2n}}{(2n)!}z^{2n-1} + \cdots \quad (|z| < \pi) \tag{2.1}$$

([1], page 75: 4.3.68). We will calculate the square of this expansion and by taking $n$ order derivative, we can calculate the value at $z = 0$, by the division by zero calculas.

We can obtain simply the following results, by this method

$$\zeta(2) = \frac{\pi^2}{6},$$

$$\eta(2) = \frac{\pi^2}{12}$$

and

$$\zeta(4) = \frac{\pi^4}{90}.$$

For the values of the Riemann zeta function for even integers, we know good results, and so we do not examine any further details here.

## 2.2 Method 2

We will use the identities:

$$\cot z = \frac{1}{z} + 2z \sum_{k=1}^{\infty} \frac{1}{z^2 - k^2\pi^2} \tag{2.2}$$

([1], page 75, 4.3.93) and

$$\cot z = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \frac{2z^5}{945} - \cdots \tag{2.3}$$

$$-\frac{(-1)^{n-1}2^{2n}B_{2n}}{(2n)!} z^{2n-1} - \cdots \quad (|z| < 1)$$

([1], page 75, 4.3.70).

From (2.2), we have

$$\left(\frac{\cot z}{z}\right)' = -\frac{2}{z^3} - 4z \sum_{k=1}^{\infty} \frac{1}{(z^2 - k^2\pi^2)^2}.$$

Therefore, we have

$$\left(\frac{\cot z}{z}\right)' \frac{1}{z} = -\frac{2}{z^4} - 4 \sum_{k=1}^{\infty} \frac{1}{(z^2 - k^2\pi^2)^2}$$

$$= -\frac{2}{z^4} - \frac{2}{45} -$$

Hence, we have

$$\zeta(4) = \frac{\pi^4}{90}.$$

By induction, we can obtain $\zeta(2n)$.

## 2.3 Method 3

We will use the identities:

$$\frac{1}{\sin z} = \frac{1}{z} + 2z \sum_{k=1}^{\infty} \frac{(-1)^k}{z^2 - k^2\pi^2} \tag{2.4}$$

([1], page 75, 4.3.93) and (2.1).

From these identities, we obtain

$$\frac{1}{z^2} + 2 \sum_{k=1}^{\infty} \frac{(-1)^k}{z^2 - k^2\pi^2}$$

$$= \frac{1}{z^2} + \frac{1}{6} + \frac{7}{360} + \cdots$$

Therefore, we obtain the result

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{5^2} + \cdots = \frac{\pi^2}{12}.$$

## 2.4 Method 4

Recall the expansion

$$\cot z = \frac{1}{z} + \sum_{k=-\infty, k\neq 0}^{\infty} \left( \frac{1}{z - k\pi} + \frac{1}{k\pi} \right). \tag{2.5}$$

By taking $n$ order derivatives that are very simple with (2.2) we obtain the values of the Riemann zeta function $\zeta(n)$, easily. However, note that $n$ has to be even integers.

# 3 Some general definite result

Recall the expansion

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} - \sum_{k=1}^{\infty} \left( \frac{1}{z+k} - \frac{1}{k} \right) \tag{3.1}$$

([3], page 53). We obtain, taking $n-1; (n > 2)$ order derivative, by the division by zero calculus

$$\zeta(n) = \frac{(-1)^n}{(n-1)!} \psi^{(n-1)}(z)|_{z=0}. \tag{3.2}$$

Recall the expansion

$$\psi(z+1) = -\gamma + \sum_{k=2}^{\infty} (-1)^k \zeta(k) z^{k-1} \quad (|z| < 1) \tag{3.3}$$

([1], page 259, 6.3.14). Then we obtain

$$\frac{\psi(z+1)}{z^{n-1}} = \frac{-\gamma}{z^{n-1}} + \sum_{k=2}^{\infty} (-1)^k \zeta(k) \frac{z^{k-1}}{z^{n-1}} \quad (|z| < 1). \tag{3.4}$$

Hence, by the division by zero calculus, we obtain, for $n > 2$

$$\zeta(n) = (-1)^n \frac{\psi(z+1)}{z^{n-1}}|_{z=0}. \tag{3.5}$$

Then, by using (3.3), we obtain for $n = 2$, by MATHEMATICA

$$\zeta(3) = 1 - \frac{\psi^{(2)}(2)}{2} \sim 1.20206.$$

Note that with MATHEMATICA, we can derive the Laurent expansion for many analytic functions and so we can obtain the division by zero calculus for many analytic functions.

In general, we have

**Theorem:**

$$\zeta(n) = 1 - \frac{\psi^{(n-1)}(2)}{n-1}. \tag{3.6}$$

These values may be calculated easily as follows:

$$\zeta(5) = 1 - \frac{1}{24}\psi^{(4)}(2) \sim 1.03693,$$

$$\zeta(6) = \frac{\pi^6}{945} \sim 1.01734$$

$$\zeta(7) = 1 - \frac{1}{720}\psi^{(6)}(2) \sim 1.00835,$$

$$\zeta(8) = \frac{\pi^8}{9450} \sim 1.00408,$$

$$\zeta(9) = 1 - \frac{1}{40320}\psi^{(8)}(2) \sim 1.00201.$$

Note that the value of the function $\psi(z)$ may be calculated easily by MATHEMATICA.

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