Noname manuscript No.
(will be inserted by the editor)

A New Proof of The *ABC* Conjecture

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Received: date / Accepted: date

Abstract In this paper, using the recent result that $c < rad(abc)^2$, we will give the proof of the *abc* conjecture for $\epsilon \geq 1$, then for $\epsilon \in]0,1[$. We choose the constant $K(\epsilon)$ as $K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}$. Some numerical examples are presented.

Keywords Elementary number theory · real functions of one variable.

Mathematics Subject Classification (2010) 11AXX · 26AXX

To the memory of my Father who taught me arithmetic
To the memory of my colleague and friend Dr.Eng. Chedly Fezzani (1943-2019) for his important work in the field of Geodesy and the promotion of the Geographic Sciences in Africa

1 Introduction and notations

Let a positive integer $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as :

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{2}$$

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The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) ([1]). It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 1 (***abc*** **Conjecture**): Let a, b, c positive integers relatively prime with $c = a + b$, then for each $\epsilon > 0$, there exists a constant $K(\epsilon)$ such that :

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{3}$$

$K(\epsilon)$ depending only of ϵ.

The idea to try to write a paper about this conjecture was born after after the publication of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [2]. The difficulty to find a proof of the *abc* conjecture is due to the incomprehensibility how the prime factors are organized in c giving a, b with $c = a + b$. The *tour de force* of my proof is the use of recent result obtained by Constantin M. Petridi [3] that $c < rad(c)^2 \Longrightarrow c < rad(abc)^2$. So, I will give a simple proof that can be understood by undergraduate students.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ ([1]). A conjecture was proposed that $c < rad^2(abc)$ ([4]). It is the key to resolve the *abc* conjecture. The paper is organized as fellow: in the second section, we recall the result obtained recently by C.M. Petridi [3] , then the main proof of the *abc* conjecture is presented. The numerical examples are discussed in section three.

2 The Proof of the *abc* Conjecture

2.1 The new result for the *abc* conjecture

Let us recall the recently result concerning the *abc* conjecture. In last March, Prof. C.M. Petridi published in hal.archives-ouvertes.fr [3] a paper confirming that for $\forall c \geq 3$ an integer, $c < rad(c)^2$. It follows the important theorem that resolve the conjecture cited above:

Theorem 1 *Let a, b, c positive integers relatively coprime with $c = a + b$, $1 \leq b < a$, then:*

$$c < rad^2(abc) \Longrightarrow \frac{Logc}{Log(rad(abc))} < 2 \tag{4}$$

2.2 The Proof of the *abc* Conjecture (1), Case :$\epsilon \geq 1$

Using the result of the theorem $c < rad^2(abc)$, we have $\forall \epsilon \geq 1$:

$$c < R^2 \leq R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \; \epsilon \geq 1 \tag{5}$$

We verify easily that $K(\epsilon) > 1$ for $\epsilon \geq 1$. Then the abc conjecture is true.

2.3 The Proof of the abc Conjecture (1), Case :$\epsilon < 1$

2.4 Case: $\epsilon < 1$

2.4.1 Case: $c < R$

In this case, we can write :

$$c < R < R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = e^{\left(\dfrac{1}{\epsilon^2}\right)}, \ \epsilon < 1 \tag{6}$$

here also $K(\epsilon) > 1$ for $\epsilon < 1$ and the abc conjecture is true.

2.4.2 Case: $c > R$

In this case, we confirm that :

$$c < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = e^{\left(\dfrac{1}{\epsilon^2}\right)}, 0 < \epsilon < 1 \tag{7}$$

If not, then $\exists \epsilon_0 \in]0,1[$, so that the triplets (a,b,c) checking $c > R$ and:

$$c \geq R^{1+\epsilon_0}.K(\epsilon_0) \tag{8}$$

are in finite number. We have:

$$\begin{aligned} c \geq R^{1+\epsilon_0}.K(\epsilon_0) \Longrightarrow R^{1-\epsilon_0}.c \geq R^{1-\epsilon_0}.R^{1+\epsilon_0}.K(\epsilon_0) \Longrightarrow \\ R^{1-\epsilon_0}.c \geq R^2.K(\epsilon_0) > c.K(\epsilon_0) \Longrightarrow R^{1-\epsilon_0} > K(\epsilon_0) \end{aligned} \tag{9}$$

As $c > R$, we obtain:

$$c^{1-\epsilon_0} > R^{1-\epsilon_0} > K(\epsilon_0) \Longrightarrow$$

$$c^{1-\epsilon_0} > K(\epsilon_0) \Longrightarrow c > K(\epsilon_0)^{\left(\dfrac{1}{1-\epsilon_0}\right)} \tag{10}$$

We deduce that it exists an infinity of triples (a,b,c) verifying (8), hence the contradiction. Then the proof of the abc conjecture is finished. We obtain that $\forall \epsilon > 0$, $c = a + b$ with a, b, c relatively coprime, $a > b \geq 2$:

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \quad with \quad K(\epsilon) = e^{\left(\dfrac{1}{\epsilon^2}\right)} \tag{11}$$

Q.E.D

3 Examples

In this section, we are going to verify some numerical examples.

3.1 Example 1

The example is given by:

$$1 + 5 \times 127 \times (2 \times 3 \times 7)^3 = 19^6 \tag{12}$$

$a = 5 \times 127 \times (2 \times 3 \times 7)^3 = 47\,045\,880 \Rightarrow \mu_a = 2 \times 3 \times 7 = 42$ and $rad(a) = 2 \times 3 \times 5 \times 7 \times 127$,
$b = 1 \Rightarrow \mu_b = 1$ and $rad(b) = 1$,
$c = 19^6 = 47\,045\,880 \Rightarrow rad(c) = 19$. Then $rad(abc) = rad(ac) = 2 \times 3 \times 5 \times 7 \times 19 \times 127 = 506\,730$..

We have $c > rad(ac)$ but $rad^2(ac) = 506\,730^2 = 256\,775\,292\,900 > c = 47\,045\,880$.

3.1.1 Case $\epsilon = 0.01$

$c < K(\epsilon).rad(ac)^{1+\epsilon} \Longrightarrow 47\,045\,880 \overset{?}{<} e^{10000}.506\,730^{1.01}$. The expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{\frac{1}{0.0001}} = e^{10000} = 8,7477777149120053120152473488653e{+}4342 \tag{13}$$

We deduce that $c \ll K(0.01).506\,730^{1.01}$ and the equation (11) is verified.

3.1.2 Case $\epsilon = 0.1$

$K(0.1) = e^{\frac{1}{0.01}} = e^{100} = 2,6879363309671754205917012128876e + 43 \Longrightarrow c < K(0.1) \times 506\,730^{1.01}$. And the equation (11) is verified.

3.1.3 Case $\epsilon = 1$

$K(1) = e \Longrightarrow c = 47\,045\,880 < e.rad^2(ac) = 697\,987\,143\,184,212$. and the equation (11) is verified.

3.1.4 Case $\epsilon = 100$

$$K(100) = e^{0.0001} \Longrightarrow c = 47\,045\,880 \overset{?}{<} e^{0.0001}.506\,730^{101} = 1,5222350248607608781853142687284e + 576$$

and the equation (11) is verified.

3.2 Example 2

We give here the example of Eric Reyssat [1], it is given by:

$$3^{10} \times 109 + 2 = 23^5 = 6436343 \tag{14}$$

$a = 3^{10}.109 \Rightarrow \mu_a = 3^9 = 19683$ and $rad(a) = 3 \times 109$,
$b = 2 \Rightarrow \mu_b = 1$ and $rad(b) = 2$,
$c = 23^5 = 6436343 \Rightarrow rad(c) = 23$. Then $rad(abc) = 2 \times 3 \times 109 \times 23 = 15042$.
For example, we take $\epsilon = 0.01$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{9999.99} = 8,7477777149120053120152473488653e + 4342 \tag{15}$$

Let us verify (11):

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow c = 6436343 \overset{?}{<} K(0.01) \times (3 \times 109 \times 2 \times 23)^{1.01} \Longrightarrow$$
$$6436343 \ll K(0.01) \times 15042^{1.01} \tag{16}$$

Hence (11) is verified.

3.3 Example 3

The example of Nitaj about the ABC conjecture [1] is:

$$a = 11^{16}.13^2.79 = 613\,474\,843\,408\,551\,921\,511 \Rightarrow rad(a) = 11.13.79 \tag{17}$$
$$b = 7^2.41^2.311^3 = 2\,477\,678\,547\,239 \Rightarrow rad(b) = 7.41.311 \tag{18}$$
$$c = 2.3^3.5^{23}.953 = 613\,474\,845\,886\,230\,468\,750 \Rightarrow rad(c) = 2.3.5.953 \tag{19}$$
$$rad(abc) = 2.3.5.7.11.13.41.79.311.953 = 28\,828\,335\,646\,110 \tag{20}$$

3.3.1 Case 1

we take $\epsilon = 100$ we have:

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} e^{0.0001}.(2.3.5.7.11.13.41.79.311.953)^{101} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 2,7657949971494838920022381186039e + 1359$$

then (11) is verified.

3.3.2 Case 2

We take $\epsilon = 0.5$, then:

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow \tag{21}$$
$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} e^4.(2.3.5.7.11.13.41.79.311.953)^{1.5} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 8\,450\,961\,319\,227\,998\,887\,403,9993 \tag{22}$$

We obtain that (11) is verified.

3.3.3 Case 3

We take $\epsilon = 1$, then

$$
\begin{gathered}
c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow \\
613\,474\,845\,886\,230\,468\,750 \overset{?}{<} (2.3.5.7.11.13.41.79.311.953)^2 \Longrightarrow \\
613\,474\,845\,886\,230\,468\,750 < 831\,072\,936\,124\,776\,471\,158\,132\,100
\end{gathered} \tag{23}
$$

We obtain that (11) is verified.

3.4 Example 4

It is of Ralf Bonse about the ABC conjecture [4] :

$$2543^4.182587.2802983.85813163 + 2^{15}.3^{77}.11.173 = 5^{56}.245983 \tag{24}$$

$$
\begin{gathered}
a = 2543^4.182587.2802983.85813163 \\
b = 2^{15}.3^{77}.11.173 \\
c = 5^{56}.245983 \\
rad(abc) = 2.3.5.11.173.2543.182587.245983.2802983.85813163 \\
rad(abc) = 1.5683959920004546031461002610848e + 33
\end{gathered} \tag{25}
$$

3.4.1 Case 1

For example, we take $\epsilon = 10$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{0.01} = 1,0078157404282956743204617416779$$

Let us verify (11):

$$
\begin{gathered}
c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Rightarrow c = 5^{56}.245983 \overset{?}{<} \\
e^{0.01}.(2.3.5.11.173.2543.182587.245983.2802983.85813163)^{11} \\
\Longrightarrow 3.4136998783296235160378273576498e + 44 < \\
1,4236200596494908176008120925721e + 365
\end{gathered} \tag{26}
$$

The equation (11) is verified.

3.4.2 Case 2

We take $\epsilon = 0.4 \Longrightarrow K(\epsilon) = 12,182473474251512159126256696\,08$, then: The

$$
\begin{gathered}
c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Rightarrow c = 5^{56}.245983 \overset{?}{<} \\
e^{6.25}.(2.3.5.11.173.2543.182587.245983.2802983.85813163)^{1.4} \\
\Longrightarrow 3.4136998783296235160378273576498e + 44 < \\
3,6255465680011453642792720569685e + 47
\end{gathered} \tag{27}
$$

And the equation (11) is verified.

Ouf, end of the mystery!

4 Conclusion

We have given an elementary proof of the *abc* conjecture in the two cases $c = a' + 1$ and $c = a + b$, confirmed by some numerical examples. We can announce the important theorem:

Theorem 2 *(David Masser, Joseph Œsterlé & Abdelmajid Ben Hadj Salem; 2019) Let a, b, c positive integers relatively prime with $c = a + b$, then for each $\epsilon > 0$, there exists $K(\epsilon)$ such that :*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{28}$$

where $K(\epsilon)$ is a constant depending of ϵ proposed equal to $e^{\left(\frac{1}{\epsilon^2}\right)}$.

Acknowledgements The author is very grateful to Professors Mihăilescu Preda and Gérald Tenenbaum for their comments about errors found in previous manuscripts concerning proofs proposed of the *abc* conjecture. The author thanks very much Professor G. Tenenbaum for checking the paper before submitting it.

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