The L/R symmetry and the categorization of natural numbers

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Abstract

"Every natural number, with the exception of 0 and 1, can be written in a unique way as a linear combination of consecutive powers of 2, with the coefficients of the linear combination being -1 or +1". According to this theorem we define the L/R symmetry of the natural numbers. The L/R symmetry gives the factors which determine the internal structure of natural numbers. As a consequence of this structure, an algorithm for the factorization of Fermat numbers is derived. Also, we determine a sequence of prime numbers, and we prove an essential corollary for the composite Mersenn numbers.

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1 Introduction

In this article, we start by proving the theorem: "Every natural number, with the exception of 0 and 1, can be written in a unique way as a linear combination of consecutive powers of 2, with the coefficients of the linear combination being -1 or +1". As a consequence of this theorem we have two fundamental symmetries of natural numbers: the symmetry L and the symmetry R. There exists a transformation which confesses the symmetries L and R. In fact, we have a single L/R symmetry instead of having two different symmetries.

The L/R symmetry categorizes the natural numbers and reveals to us the factors which determine their internal structure. Every natural number belongs to one of the following categories: it has symmetry L or it has symmetry R or it is not symmetric. In the categorization of natural numbers according to L/R symmetry there exist three numbers each of them is a distinct category contained of exactly one number. These numbers are 0, 1 and 3.

We prove an algorithm for the factorization of the asymmetric numbers. Fermat numbers are asymmetric, and therefore we can apply the algorithm for their factorization. Also, we prove that Fermat numbers F_0, F_1, F_2, F_3, F_4 cannot be composite numbers.

In the region of numbers in the form of $2^{2^S}, S \in \mathbb{N}$ we determine intervals in the form of [p, P] where p, P are prime numbers, in which the magnitude of the difference P-p is known. In that way we determine a sequence of pairs of prime numbers (p, P).

In the last chapter we prove a corollary for the composite Mersenn numbers.

2 Natural numbers as linear combination of consecutive powers of 2

We prove the following theorem:

Theorem 2.1. *Every natural number, with the exception of* 0, *and* 1, *can be uniquely written as a linear combination of consecutive powers of* 2, *with the coefficients of the linear combination being* -1 *or* +1.

Proof. Let the odd number Π as given from equation

$$\begin{aligned} &\Pi = \Pi(\nu, \beta_i) = 2^{\nu+1} + 2^{\nu} \pm 2^{\nu-1} \pm 2^{\nu-2} \pm .... \pm 2^1 \pm 2^0 = 2^{\nu+1} + 2^{\nu} + \sum_{i=0}^{\nu-1} \beta_i 2^i \\ &\beta_i = \pm 1, i = 0, 1, 2, ..., \nu - 1 \\ &\nu \in \mathbb{N} \end{aligned} \quad . \qquad (2.1)$$

From equation (2.1) for $\nu = 0$ we obtain

$$\Pi = 2^1 + 2^0 = 2 + 1 = 3.$$

We now examine the case where $\nu \in \mathbb{N}^* = \{1, 2, 3, ...\}$. The lowest value that the odd number Π of equation (2.1) can obtain is

$$\Pi_{\min} = \Pi(\nu) = 2^{\nu+1} + 2^{\nu} - 2^{\nu-1} - 2^{\nu-1} - .... 2^1 - 1$$

$$\Pi_{\min} = \Pi(\nu) = 2^{\nu+1} + 1. \qquad (2.2)$$

The largest value that the odd number Π of equation (2.1) can obtain is

$$\Pi_{\max} = \Pi(\nu) = 2^{\nu+1} + 2^{\nu} + 2^{\nu-1} + ... 2^1 + 1$$

$$\Pi_{\max} = \Pi(\nu) = 2^{\nu+2} - 1. \qquad (2.3)$$

Thus, for the odd numbers $\Pi = \Pi(\nu, \beta_i)$ of equation (2.1) the following inequality holds

$$\Pi_{\min} = 2^{\nu+1} + 1 \leq \Pi(\nu, \beta_i) \leq 2^{\nu+2} - 1 = \Pi_{\max} . \qquad (2.4)$$

The number $N(\Pi(\nu, \beta_i))$ of odd numbers in the closed interval $[2^{\nu+1} + 1, 2^{\nu+2} - 1]$ is

$$N(\Pi(\nu, \beta_i)) = \frac{\Pi_{\max} - \Pi_{\min}}{2} + 1 = \frac{(2^{\nu+2} - 1) - (2^{\nu+1} + 1)}{2} + 1$$

$$N(\Pi(\nu, \beta_i)) = 2^{\nu} . \qquad (2.5)$$

The integers $\beta_i, i=0,1,2,........,\nu-1$ in equation (2.1) can take only two values, $\beta_i=-1 \vee \beta_i=+1$, thus equation (2.1) gives exactly $2^\nu = N\left(\Pi\left(\nu,\beta_i\right)\right)$ odd numbers. Therefore, for every $\nu \in \mathbb{N}^*$ equation (2.1) gives all odd numbers in the interval $\left[2^{\nu+1}+1, 2^{\nu+2}-1\right]$.

We now prove the theorem for the even numbers. Every even number α which is a power of 2 can be uniquely written in the form of $\alpha = 2^\nu, \nu \in \mathbb{N}^*$. We now consider the case where the even number α is not a power of 2. In that case, the even number α is written in the form of

$$\alpha = 2^l \Pi, \Pi = \text{odd}, \Pi \neq 1, l \in \mathbb{N}^*. \tag{2.6}$$

We now prove that the even number α can be uniquely written in the form of equation (2.6). If we assume that the even number α can be written in the form of

$$\begin{aligned} &\alpha = 2^l \Pi = 2^{l'} \Pi' \\ &l \neq l' (l > l') \\ &\Pi \neq \Pi' \\ &l, l' \in \mathbb{N}^* \\ &\Pi, \Pi' = odd \end{aligned} \tag{2.7}$$

the we obtain

$$\begin{aligned} &2^l \Pi = 2^{l'} \Pi' \\ &2^{l-l'} \Pi = \Pi' \end{aligned}$$

which is impossible, since the first part of this equation is even and the second odd. Thus, it is $l = l'$ and we take that $\Pi = \Pi'$ from equation (2.7). Therefore, every even number α that is not a power of 2 can be uniquely written in the form of equation (2.6). The odd number Π of equation (2.6) can be uniquely written in the form of equation (2.1), thus from equation (2.6) it is derived that every even number α that is not a power of 2 can be uniquely written in the form of equation

$$\begin{aligned} &\alpha = \alpha\left(l,\nu,\beta_i\right) = 2^l \left(2^{\nu+1} + 2^\nu + \sum_{i=0}^{\nu-1} \beta_i 2^i\right) \\ &l \in \mathbb{N}^*, \nu \in \mathbb{N} \\ &\beta_i = \pm 1, i = 0,1,2,........,\nu-1 \end{aligned} \tag{2.8}$$

and equivalently

$$\begin{aligned} &\alpha = \alpha\left(l,\nu,\beta_i\right) = 2^{l+\nu+1} + 2^{l+\nu} + \sum_{i=0}^{\nu-1} \beta_i 2^{l+i} \\ &l \in \mathbb{N}^*, \nu \in \mathbb{N} \qquad . \\ &\beta_i = \pm 1, i = 0,1,2,........,\nu-1 \end{aligned} \tag{2.9}$$

For 1 we take

$1 = 2^0$

$1 = 2^1 - 2^0$

thus, it can be written in two ways in the form of equation (2.1). Both the odds of equation (2.1) and the evens of the equation (2.8) are positive. Thus, 0 cannot be written either in the form of equation (2.1) or in the form of equation (2.8). $\square$

In order to write an odd number $\Pi \neq 1, 3$ in the form of equation (2.1) we initially define the $\nu \in \mathbb{N}^*$ from inequality (2.4). Then, we calculate the sum

$2^{\nu+1} + 2^{\nu}$.

If it holds that $2^{\nu+1} + 2^{\nu} < \Pi$ we add the $2^{\nu-1}$, whereas if it holds that $2^{\nu+1} + 2^{\nu} > \Pi$ then we subtract it. By repeating the process exactly ν times we write the odd number Π in the form of equation (2.1). The number of ν steps needed in order to write the odd number Π in the form of equation (2.1) is extremely low compared to the magnitude of the odd number Π, as derived from inequality (2.4).

Example 2.1. For the odd number $\Pi = 23$ we obtain from inequality (2.4)

$2^{\nu+1} + 1 < 23 < 2^{\nu+2} - 1$

$2^{\nu+1} + 2 < 24 < 2^{\nu+2}$

$2^{\nu} < 12 < 2^{\nu+1}$

thus $\nu = 3$. Then, we have

$2^{\nu+1} + 2^{\nu} = 2^4 + 2^3 = 24 > 23$ (thus 2^2 is subtracted)

$2^4 + 2^3 - 2^2 = 20 < 23$ (thus 2^1 is added)

$2^4 + 2^3 - 2^2 + 2^1 = 22 < 23$ (thus $2^0 = 1$ is added)

$2^4 + 2^3 - 2^2 + 2^1 + 1 = 23$.

Fermat numbers F_s can be written directly in the form of equation (2.1), since they are of the form $\Pi_{\min}$,

$$\begin{gathered} F_s = 2^{2^s} + 1 = \Pi_{\min}\left(2^s - 1\right) = 2^{2^s} + 2^{2^s-1} - 2^{2^s-2} - 2^{2^s-3} - .... - 2^1 - 1 \\ s \in \mathbb{N} \end{gathered} \qquad (2.10)$$

Mersenne numbers M_p can be written directly in the form of equation (2.1), since they are of the form $\Pi_{\max}$,

$$\begin{gathered} M_p = 2^p - 1 = \Pi_{\max}\left(p-2\right) = 2^{p-1} + 2^{p-2} + 2^{p-3} + ... + 2^1 + 1 \\ p = prime \end{gathered} \qquad (2.11)$$

In order to write an even number α that is not a power of 2 in the form of equation (2.1), initially it is consecutively divided by 2 and it takes of the form of equation (2.6). Then, we write the odd number Π in the form of equation (2.1).

Example 2.2. By consecutively dividing the even number $\alpha = 368$ by 2 we obtain

$\alpha = 368 = 2^4 \cdot 23$.

Then, we write the odd number $\Pi = 23$ in the form of equation (2.1),

$23 = 2^4 + 2^3 - 2^2 + 2^1 + 1$,

and we get

$$368 = 2^4 \left(2^4 + 2^3 - 2^2 + 2^1 + 1\right)$$

$$368 = 2^8 + 2^7 - 2^6 + 2^5 + 2^4.$$

This equation gives the unique way in which the even number $\alpha = 368$ can be written in the form of equation (2.9).

From inequality (2.4) we obtain

$$2^{\nu+1} + 1 \le \Pi \le 2^{\nu+2} - 1$$

$$2^{\nu+1} < 2^{\nu+1} + 1 \le \Pi \le 2^{\nu+2} - 1 < 2^{\nu+2}$$

$$2^{\nu+1} < \Pi < 2^{\nu+2}$$

$$(\nu+1)\ln 2 < \ln \Pi < (\nu+2)\ln 2$$

from which we get

$$\frac{\ln \Pi}{\ln 2} - 1 < \nu + 1 < \frac{\ln \Pi}{\ln 2}$$

and finally

$$\nu + 1 = \left[\frac{\ln \Pi}{\ln 2}\right] \qquad (2.12)$$

Where $\left[\frac{\ln \Pi}{\ln 2}\right]$ the integer part of $\frac{\ln \Pi}{\ln 2} \in \mathbb{R}$.

We now give the following definition:

Definition 2.1. *We define as the conjugate of the odd*

$$\begin{aligned} &\Pi = \Pi(\nu, \beta_i) = 2^{\nu+1} + 2^\nu + \sum_{i=0}^{\nu-1} \beta_i 2^i \\ &\beta_i = \pm 1, i = 0,1,2,........,\nu - 1 \\ &\nu \in \mathbb{N}^* \end{aligned} \qquad (2.13)$$

the odd Π^*,

$$\begin{aligned} &\Pi^* = \Pi^*(\nu, \gamma_j) = 2^{\nu+1} + 2^\nu + \sum_{j=0}^{\nu-1} \gamma_j 2^j \\ &\gamma_i = \pm 1, j = 0,1,2,........,\nu - 1 \\ &\nu \in \mathbb{N}^* \end{aligned} \qquad (2.14)$$

for which it holds

$$\gamma_k = -\beta_k \forall k = 0,1,2,........,\nu - 1. \tag{2.15}$$

For conjugate odds, the following corollary holds:

Corollary 2.1. *For the conjugate odds* $\Pi = \Pi(\nu, \beta_i)$ *and* $\Pi^* = \Pi^*(\nu, \gamma_i)$ *the following hold:*

1. $(\Pi^*)^* = \Pi$. (2.16)

2. $\Pi^* = 3 \cdot 2^{\nu+1} - \Pi$. (2.17)

3. Π *is divisible by* 3 *if and only if* $\boldsymbol{\Pi^*}$ *is divisible by* 3.

4. *Two conjugate odd numbers cannot have common factor greater than 3.*

Proof. 1. The 1 of the corollary is an immediate consequence of definition 4.1.

2. From equations (2.13), (2.14) and (2.15) we get

$$\Pi + \Pi^* = \left(2^{\nu+1} + 2^{\nu}\right) + \left(2^{\nu+1} + 2^{\nu}\right)$$

and equivalently

$$\Pi + \Pi^* = 3 \cdot 2^{\nu+1}.$$

3. If the odd Π is divisible by 3 then it is written in the form $\Pi = 3x, x = odd$ and from equation (4.17) we get $3x + \Pi^* = 3 \cdot 2^{\nu+1}$ and equivalently $\Pi^* = 3\left(2^{\nu+1} - x\right)$. Similarly we can prove the inverse

4. If $\Pi = xy, \Pi^* = xz$, x, y, z odd numbers, from equation (2.17) we have $x(y+z) = 3 \cdot 2^{\nu+1}$ and consequently is $x = 3$. □

From corollary 2.1 we have that 3 the only odd number which is equal to its conjugate: $3^* = 3 \cdot 2^{0+1} - 3 = 3$.

3 The L/R symmetry

We now give the following definition:

Definition 3.1. *Define as "symmetry" every specific algorithm which determines the signs of* $\beta_i = \pm 1, i = 0,1,2,........,\nu - 1$ *in equation* (2.1):

$$\Pi = \Pi(\nu, \beta_i) = 2^{\nu+1} + 2^{\nu} \pm 2^{\nu-1} \pm 2^{\nu-2} \pm .... \pm 2^1 \pm 2^0 = 2^{\nu+1} + 2^{\nu} + \sum_{i=0}^{\nu-1} \beta_i 2^i$$
$$\beta_i = \pm 1, i = 0,1,2,........,\nu - 1 .$$
$$\nu \in \mathbb{N}$$

In this article we study the symmetries L and R, which are determined by the following definition:

Definition 3.2.1. *The odd number Π in the equation* (2.1) *has symmetry L when there exists an index L so that*

$$\begin{aligned}&\beta_L = +1\\&\beta_{L-1} = \beta_{L-2} = ... = \beta_1 = \beta_0 = -1.\\&L \in \{1, 2, 3, ..., \nu - 1\}\end{aligned} \qquad (3.1)$$

2. *The odd number Π in the equation* (2.1) *has symmetry R when there exists an index R so that*

$$\begin{aligned}&\beta_R = -1\\&\beta_{R-1} = \beta_{R-2} = ... = \beta_1 = \beta_0 = +1.\\&R \in \{1, 2, 3, ..., \nu - 1\}\end{aligned} \qquad (3.2)$$

3. *We will call asymmetric* the *odd numbers which have neither symmetry L nor symmetry R.*

4. *For each even number α,*

$$\alpha = 2^l \Pi, \Pi = \text{odd}, \Pi \neq 1, l \in \mathbb{N}^*$$

we define as the symmetry of α the symmetry of the odd Π.

We will note the symmetry of an odd Π by L=L(Π)=LΠ, or by R=R(Π)=RΠ. At first the L/R symmetry categorizes the odd numbers, and then the even numbers by 4 of definition 3.2. The odd number Π=1 cannot uniquely be written in the form of equation (2.1). So 1 and the powers of 2 are asymmetric numbers.

The odd numbers of the form

$$As = As(\nu) = 2^\nu + 1, \nu \in \mathbb{N}$$

have $\beta_i = -1 \forall i = 0, 1, 2, ..., \nu - 1$ in the equation (2.1), and so these are the only asymmetric odd numbers. From its definition we have that the Fermat numbers are asymmetric numbers. However, although 3 is a Fermat number it is asymmetric because of a different reason: It is the unique natural number which comes from equation (2.1) for ν=0,

$$3 = 2^1 + 2^0 = 2^1 + 1, (\nu = 0).$$

In the categorization of natural numbers according to L/R symmetry, 3 is a distinct category contained just one element, number 3. There are two other natural number with this property, 0 and 1.

The even numbers of the form

$$\begin{aligned}&\alpha = 2^l \cdot As\\&l \in \mathbb{N}^*\end{aligned}$$

where As is asymmetric number, as well as the powers of 2 are the asymmetric even numbers. The rest even numbers are symmetric (so the symmetric even numbers are more than the asymmetric ones).

The symmetry of an odd number can be found by writing it in the form of the equation (2.1). According to 4 of corollary 3.1, the factors, prime numbers or composites of Fermat numbers have symmetry L. Next, we have two examples:

Example 3.1. The prime number Q= 45592577 is a factor of $F_{10} = 2^{1024} + 1$. From the equation (2.12) we have ν+1=25, and then (see example 2.1) from the equation 2.1 we have

$$Q = 2^{25} + 2^{24} - 2^{23} + 2^{22} - 2^{21} + 2^{20} + 2^{19} - 2^{18} + 2^{17} + 2^{16} + 2^{15} + 2^{14} - 2^{13} + 2^{12} \\ + 2^{11} - 2^{10} - 2^{9} - 2^{8} - 2^{7} - 2^{6} - 2^{5} - 2^{4} - 2^{3} - 2^{2} - 2^{1} - 1 .$$

So the factor 45592577 of F_{10} has symmetry L 45592577=11.

Example 3.2. The prime number Q=568630647535356955169033410940867804839360742060818433 is a factor of $F_{12} = 2^{4096} + 1$. From the equation (2.12) we have ν+1=178, and then from equation 2.1 we have

$$\begin{aligned} Q = {} & 2^{178} + 2^{177} - 2^{176} + 2^{175} + 2^{174} + 2^{173} + 2^{172} - 2^{171} + 2^{170} + 2^{169} + 2^{168} + 2^{167} + 2^{166} \\ & + 2^{165} - 2^{164} + 2^{163} - 2^{162} - 2^{161} - 2^{160} - 2^{159} + 2^{158} + 2^{157} + 2^{156} - 2^{155} - 2^{154} - 2^{153} - 2^{152} \\ & - 2^{151} + 2^{150} - 2^{149} + 2^{148} - 2^{147} - 2^{146} + 2^{145} - 2^{144} + 2^{143} - 2^{142} - 2^{141} - 2^{140} + 2^{139} + 2^{138} \\ & - 2^{137} - 2^{136} + 2^{135} - 2^{134} - 2^{133} + 2^{132} - 2^{131} + 2^{130} - 2^{129} + 2^{128} - 2^{127} + 2^{126} - 2^{125} - 2^{124} \\ & - 2^{123} - 2^{122} - 2^{121} + 2^{120} - 2^{119} + 2^{118} - 2^{117} + 2^{116} - 2^{115} + 2^{114} - 2^{113} - 2^{112} - 2^{111} - 2^{110} \\ & - 2^{109} - 2^{108} + 2^{107} - 2^{106} + 2^{105} - 2^{104} + 2^{103} - 2^{102} + 2^{101} - 2^{100} + 2^{99} + 2^{98} - 2^{97} + 2^{96} - 2^{95} \\ & - 2^{94} + 2^{93} - 2^{92} + 2^{91} + 2^{90} - 2^{89} + 2^{88} - 2^{87} + 2^{86} + 2^{85} + 2^{84} - 2^{83} + 2^{82} - 2^{81} + 2^{80} + 2^{79} \\ & - 2^{78} - 2^{77} - 2^{76} - 2^{75} + 2^{74} + 2^{73} - 2^{72} - 2^{71} - 2^{70} + 2^{69} + 2^{68} + 2^{67} + 2^{66} + 2^{65} + 2^{64} - 2^{63} \\ & - 2^{62} + 2^{61} - 2^{60} - 2^{59} - 2^{58} - 2^{57} - 2^{56} + 2^{55} - 2^{54} - 2^{53} - 2^{52} - 2^{51} - 2^{50} - 2^{49} + 2^{48} + 2^{47} \\ & - 2^{46} + 2^{45} + 2^{44} + 2^{43} + 2^{42} - 2^{41} - 2^{40} + 2^{39} - 2^{38} - 2^{37} - 2^{36} + 2^{35} - 2^{34} - 2^{33} + 2^{32} + 2^{31} \\ & - 2^{30} + 2^{29} + 2^{28} + 2^{27} + 2^{26} + 2^{25} + 2^{24} - 2^{23} + 2^{22} + 2^{21} + 2^{20} - 2^{19} - 2^{18} - 2^{17} - 2^{16} + 2^{15} \\ & + 2^{14} - 2^{13} - 2^{12} - 2^{11} - 2^{10} - 2^{9} - 2^{8} - 2^{7} - 2^{6} - 2^{5} - 2^{4} - 2^{3} - 2^{2} - 2^{1} - 1 \end{aligned}$$

So the factor 568630647535356955169033410940867804839360742060818433 of F_{12} has symmetry L 568630647535356955169033410940867804839360742060818433=14.

From Lucas theorem for the Fermat numbers [1] the following corollary is derived:

Corollary 3.1. 1. *The prime numbers factors of Fermat numbers have symmetry L.*

2. *For the symmetry L of the prime numbers factors of a Fermat number*

$$F_S = 2^{2^S} + 1, S \in \mathbb{N} \tag{3.3}$$

holds

$$L \in \Phi_S = \{S+1, S+2, S+3, ...\}\,. \tag{3.4}$$

We have the following example.

Example 3.5. For the known factors, prime numbers and composites of $F_{12} = 2^{4096} + 1$ we have:

S=12

L114689=13

L26017793=15

L63766529=15

L190274191361=13

L1256132134125569=13

L568630647535356955169033410940867804839360742060818433=14

L(C1133)=13

where C1133 is a composite, non-factorized factor of F_{12} with 1133 digits. From the equations (3.3) we have

$$Q_1 = 114689 = 3\cdot 2^{15} + 2^{14}\cdot 1 + 1$$

$$Q_2 = 26017793 = 3\cdot 2^{23} + 2^{16}\cdot 13 + 1$$

$$Q_3 = 63766529 = 3\cdot 2^{24} + 2^{16}\cdot 205 + 1$$

$$Q_4 = 190274191361 = 3\cdot 2^{36} - 2^{14}\cdot 969497 + 1$$

$$Q_5 = 1256132134125569 = 3\cdot 2^{49} - 2^{14}\cdot 26410994027 + 1$$

$$Q_6 = 568630647535356955169033410940867804839360742060818433$$
$$= 3\cdot 2^{177} - 2^{15}\cdot 18478943754124043931111829347223324638874599 4813+1$$

$$C1133 = 3\cdot 2^{3761} + 2^{14}\cdot \Pi + 1$$

where Π is a negative number with 1128 digits.

4 The basic study of the L/R symmetry

In this chapter we prove the basic theorems for the L/R symmetry.

Theorem 4.1.1. *Every odd number Q with symmetry L can be written in the form*

$$\begin{aligned} Q &= 3\cdot 2^{\nu} + 2^{L+1}\cdot \sum_{i=1}^{\nu-L-1} \beta_{\nu-i}\cdot 2^{\nu-L-1-i} + 1 = 3\cdot 2^{\nu} + 2^{L+1}\cdot \Pi + 1 \\ &= 2^{L+1}\cdot\left(3\cdot 2^{\nu-L-1} + \Pi\right) + 1 = 2^{L+1}\cdot K + 1, \nu + 1 = \left[\frac{\ln Q}{\ln 2}\right] \end{aligned}\,. \tag{4.1}$$

The odd number $\Pi \in \mathbb{Z}^*$,

$$\Pi = \sum_{i=1}^{\nu-L-1} \beta_{\nu-i} \cdot 2^{\nu-L-i} \tag{4.2}$$

has the same sign as $\beta_{\nu-1} = \pm 1$, *and satisfies the inequality*

$$-2^{\nu-L-1} + 1 \le \Pi \le 2^{\nu-L-1} - 1. \tag{4.3}$$

2. *Every odd number D with symmetry R can be written in the form*

$$\begin{aligned} D &= 3\cdot 2^{\nu} + 2^{R+1} \cdot \sum_{i=1}^{\nu-R-1} \beta_{\nu-i} \cdot 2^{\nu-R-1-i} - 1 = 3\cdot 2^{\nu} + 2^{R+1} \cdot \Pi - 1 \\ &= 2^{R+1} \cdot \left(3\cdot 2^{\nu-R-1} + \Pi\right) - 1 = 2^{R+1} \cdot K - 1, \nu + 1 = \left[\frac{\ln D}{\ln 2}\right] \end{aligned} . \tag{4.4}$$

The odd number $\Pi \in \mathbb{Z}^*$,

$$\Pi = \sum_{i=1}^{\nu-R-1} \beta_{\nu-i} \cdot 2^{\nu-R-i} \tag{4.5}$$

has the same sign as $\beta_{\nu-1} = \pm 1$, *and satisfies the inequality*

$$-2^{\nu-R-1} + 1 \le \Pi \le 2^{\nu-R-1} - 1. \tag{4.6}$$

Proof. We prove the part 1 of the corollary. The proof of the part 2 is similar. If Q has symmetry L, from equation (2.1) we have

$$Q = 2^{\nu+1} + 2^{\nu} + \sum_{i=\nu-1}^{L+1} \beta_i \cdot 2^i + 2^L - 2^{L-1} - 2^{L-2} - ..... - 2^1 - 1$$

$$Q = 3\cdot 2^{\nu} + \sum_{i=\nu-1}^{L+1} \beta_i \cdot 2^i + 2^L - \left(2^{L-1} + 2^{L-2} + ... + 2^1 + 1\right)$$

$$Q = 3\cdot 2^{\nu} + \sum_{i=\nu-1}^{L+1} \beta_i \cdot 2^i + 2^L - \left(2^L - 1\right)$$

$$Q = 3\cdot 2^{\nu} + \sum_{i=\nu-1}^{L+1} \beta_i \cdot 2^i + 1$$

and taking into account that the highest power of 2 in the sum $\sum_{i=\nu-1}^{L+1} \beta_i \cdot 2^i$ is 2^{L+1} we take the equation (4.1). From equation (4.1) we have for the odd number Π,

$$\Pi = \sum_{i=1}^{\nu-L-1} \beta_{\nu-i} \cdot 2^{\nu-L-i}$$

which is the sum of successive powers of 2 with highest power $\beta_{\nu-1}\cdot 2^{\nu-L-1}$. So the odd number Π has the same sign as $\beta_{\nu-1}=\pm 1$. Moreover, the minimum value of Π is

$$\Pi_{\min}=\sum_{i=1}^{\nu-L-1}-2^{\nu-L-1-i}=-2^{\nu-L-1}+1$$

and the maximum

$$\Pi_{\max}=\sum_{i=1}^{\nu-L-1}2^{\nu-L-1-i}=2^{\nu-L-1}-1. \ \square$$

The following theorem concerns the symmetry of conjugate odd numbers.

Theorem 4.2.1. *For the odd number Q, with symmetry L, holds*

$$\begin{aligned}&Q=3\cdot 2^{\nu}+2^{L+1}\cdot\Pi+1\Leftrightarrow Q^{*}=3\cdot 2^{\nu}-2^{R+1}\cdot\Pi-1\\&R=L\end{aligned}. \tag{4.7}$$

2. *For the odd number D, with symmetry R, holds*

$$\begin{aligned}&D=3\cdot 2^{\nu}+2^{R+1}\cdot\Pi-1\Leftrightarrow D^{*}=3\cdot 2^{\nu}-2^{L+1}\cdot\Pi+1\\&L=R\end{aligned}. \tag{4.8}$$

Proof. Theorem is an immediate consequence of definitions 3.2, 2.1 and transformation (2.17). $\square$

From equations (4.7) and (4.8) we have

$$Q\cdot Q^{*}+\left(2^{L+1}\cdot\Pi+1\right)^{2}=9\cdot 2^{2\nu} \tag{4.9}$$

$$D\cdot D^{*}+\left(2^{R+1}\cdot\Pi+1\right)^{2}=9\cdot 2^{2\nu}. \tag{4.10}$$

These equations are independent from the transformation of the conjugation, which is the transformation (2.17).

Now, we prove the following theorem:

Theorem 4.3.1. *For the odd numbers Q with symmetry L the equation*

$$\Pi=\Pi_{L}=\frac{Q-3\cdot 2^{\nu}-1}{2^{L+1}} \tag{4.11}$$

gives the value of L, and the equation

$$\Pi=\Pi_{R}=\frac{D-3\cdot 2^{\nu}+1}{2^{R+1}} \tag{4.12}$$

gives R=0, and

$$\Pi_{L}=\frac{\Pi_{R}-1}{2^{L}}. \tag{4.13}$$

2. *For the odd numbers D with symmetry R the equation (4.12) gives the value of R, the equation (4.11) gives L=0, and*

$$\Pi_R = \frac{\Pi_L - 1}{2^R}. \tag{4.14}$$

Proof. We prove the part 1 of the theorem. The proof of part 2 is similar. Trying to calculate the value of R, in case of an odd number Q with symmetry L in the form of equation (4.4), we get $Q = 3\cdot 2^{\nu} + 2^{R+1}\cdot \Pi_R - 1$. Combining this equation with the equation (4.1) we have

$$Q = 3\cdot 2^{\nu} + 2^{L+1}\cdot \Pi_L + 1 = 3\cdot 2^{\nu} + 2^{R+1}\cdot \Pi_R - 1$$
$$2 = 2^{R+1}\cdot \Pi_R - 2^{L+1}\cdot \Pi_L$$
$$1 = 2^R \cdot \Pi_R - 2^L \cdot \Pi_L$$

and finally

$$\left(1 = 2^R \cdot \left(\Pi_R - 2^{L-R}\cdot \Pi_L\right)\right) \vee \left(1 = 2^L \cdot \left(2^{R-L}\cdot \Pi_R - \Pi_L\right)\right).$$

These equations hold if and only if R=0 or L=0. Number Q has symmetry L, so R=0. Moreover we have

$$1 = \Pi_R - 2^{L-R}\cdot \Pi$$

and because R=0 we take the equation (4.13).□

As an example, we calculate again the L and Π for the number Q of example 3.2 by using the equations (4.11) and (4.12):

Example 4.1. For the odd number A=568630647535356955169033410940867804839360742060818433 we have ν=177 from equation (2.5). Then, the equation (4.12) gives R=0. So number A has symmetry L. Then we observe that the equation (4.11) is verified for L=1, L=2, L=3, ..., L=14. For the maximum value of L=14 the equation (4.11) gives Π=184789 437541 240439 311118 293472 233246 388745 994813.

From theorem 4.2 we conclude that symmetries L and R commute from transformation (2.17). So we have L/R symmetry. Theorem 4.3 gives one of the pairs $\left(L \geq 1 \wedge R = 0\right) \vee \left(L = 0 \wedge R \geq 1\right)$ for every odd number, independently of its symmetry. So, it gives a pair for the Fermat numbers:

$$\begin{aligned} &F_S = 2^{2^S} + 1, S \in \mathbb{N} \\ &L\left(F_S\right) = 2^S - 1 \\ &R\left(F_S\right) = 0 \end{aligned} \quad . \tag{4.15}$$

Now we prove the following corollary:

Corollary 4.1.1. *For every odd number D with symmetry R the next odd number D+2=Q has symmetry L, and holds*

$$\nu\left(D+2\right) = \nu\left(D\right) \Rightarrow L\left(D+2\right) = R \wedge \Pi_L\left(D+2\right) = \Pi_R\left(D\right). \tag{4.16}$$

2. *For every odd number Q with symmetry L the previous odd number Q-2=D has symmetry R, and holds*

$$\nu(Q-2)=\nu(Q)\Rightarrow R(Q-2)=L\wedge\Pi_R(Q-2)=\Pi_L(Q). \qquad (4.17)$$

Poof. This corollary is an immediate consequence of theorem 4.1:

$$D+2=\left(3\cdot 2^{\nu}+2^{R+1}\cdot\Pi_R-1\right)+2=3\cdot 2^{\nu}+2^{R+1}\cdot\Pi_R+1=3\cdot 2^{\nu}+2^{L+1}\cdot\Pi_L+1=Q,$$

$$Q-2=\left(3\cdot 2^{\nu}+2^{L+1}\cdot\Pi_L+1\right)-2=3\cdot 2^{\nu}+2^{L+1}\cdot\Pi_L-1=3\cdot 2^{\nu}+2^{R+1}\cdot\Pi_R-1=D.$$

Theorem 2.1 makes a partition to the set of natural numbers contained of intervals of the form $\left[2^{\nu+1}+1,2^{\nu+2}-1\right],\nu\in\mathbb{N}^*$. From corollary 4.1 we have that the L/R symmetry makes a partition of the odd numbers of these intervals in $2^{\nu-1},\nu\geq 1$ pairs. We prove the following corollary:

Corollary 4.2. *There are four numbers in the interval*

$$\Omega(\nu)=\left[2^{\nu+1}+1,2^{\nu+2}-1\right]=\left[2^{\nu+1}+1,3\cdot 2^{\nu}-1\right]\cup\left[3\cdot 2^{\nu}+1,2^{\nu+2}-1\right] \qquad (4.18)$$
$$\nu\in\mathbb{N}^*$$

with symmetry L/R=ν-1:

1.

$$\Phi_1(\nu)=A_S(\nu+1)=2^{\nu+1}+1$$
$$L\left(\Phi_1(\nu)\right)=L\left(A_S(\nu+1)\right)=L\left(2^{\nu+1}+1\right)=\nu-1. \qquad (4.19)$$

2.

$$\Phi_2(\nu)=3\cdot 2^{\nu}-1$$
$$R\left(\Phi_2(\nu)\right)=R\left(3\cdot 2^{\nu}-1\right)=\nu-1. \qquad (4.20)$$
$$\Pi_R=0$$

3.

$$\Phi_3(\nu)=3\cdot 2^{\nu}+1$$
$$L\left(\Phi_3(\nu)\right)=L\left(3\cdot 2^{\nu}+1\right)=\nu-1. \qquad (4.21)$$
$$\Pi_L=0$$

4.

$$\Phi_4(\nu)=3\cdot 2^{\nu+2}-1$$
$$R\left(\Phi_4(\nu)\right)=R\left(3\cdot 2^{\nu+2}-1\right)=\nu-1. \qquad (4.22)$$

Proof. Corollary 4.2 is an immediate consequence of equations (4.11), (4.12). □

Using the last parts of equations (4.1), (4.4) Newton's binomial theorem we can calculate the symmetry of the powers of the odd numbers.

Next, we list two examples.

Example 4.2. The powers of 3=2+1 with even exponent have symmetry L. For the powers of the form 3^{2^S} the following equation holds

$$\begin{aligned} &L\left(3^{2^S}\right)=S \\ &S\in\mathbb{N} \end{aligned}.$$

For the rest of the powers of 3 with even exponent the following equation holds

$$\begin{aligned} &L\left(3^{2^S\cdot\Pi}\right)=S+1 \\ &S,\Pi\in\mathbb{N},\Pi\neq 1,\Pi=odd \end{aligned}.$$

The powers of 3 with odd exponent have constant symmetry R=1.

Example 4.3. The powers of $61=2^2\cdot 15+1, L=1$ have symmetry L. For powers of 61 with exponent being a power of 2 the following equation holds

$$\begin{aligned} &L\left(61^{2^S}\right)=S \\ &S\in\mathbb{N} \end{aligned}.$$

For the rest of the powers of 61 with even exponent the following equation holds

$$\begin{aligned} &L\left(61^{2^S\cdot\Pi}\right)=S+1 \\ &S,\Pi\in\mathbb{N},\Pi\neq 1,\Pi=odd \end{aligned}.$$

The odd powers of 61 have constant symmetry L=1.

Now, we prove the following corollary:

Corollary 4.3. *For the symmetric prime numbers A and B with symmetry L or R we have the following:*

1. L(A)<L(B)=>L(AB)=L(A).

2. L(A)<R(B)=>R(AB)=L(A)

3. R(A)<L(B)=>R(AB)=R(A).

4. R(A)<R(B)=>L(AB)=R(A).

5. Symmetry(A)= Symmetry(B)=> Symmetry(AB)> Symmetry(A)= Symmetry(B).

6. *The powers of odd numbers Π with even exponent,* $\Pi^{2l},\Pi=\text{odd},l\in\mathbb{N}^*$ *have symmetry L.*

Proof. The corollary is derived from the last parts of equations (4.1) and (4.4),

$$\begin{aligned} &Q=2^{L+1}\cdot K+1 \\ &K=odd \end{aligned} \qquad (4.23)$$

$$\begin{aligned} &D = 2^{R+1} \cdot K - 1 \\ &K = odd \end{aligned} \tag{4.24}$$

and equation

$$A_S = A_S(\nu + 1) = 2^{\nu+1} + 1, \nu \in \mathbb{N} \tag{4.25}$$

for asymmetric numbers As. □

We give two examples.

Example 4.4. L(641)=6<L(114689)=13 =>L(641×114689)=6.

Example 4.5. R(607)= 4<R(16633)=6 =>L(607×16633)=4.

From corollary 4.4 we can determine the L/R symmetry of at least of one composite odd number whose factors are unknown. Next, we list two examples.

Example 4.6. From equation (2.12), for the number C1133 which is composite factor of F_{12} with 1133 digits, we get $\nu(C1133) = 3761$. Then, from equations (4.11), (4.12) we get L(C1133)=13. The factors of Fermat numbers have symmetry L, so from part 1 of corollary 4.3 we have that at least one of the factors of C1133 has symmetry L=13.

Example 4.7. For RSA-232 =
1009881397871923546909564894309468582818233821955573955141120516205831021338528545374366109757154363664913380084917065169921701524733294389270280234380960909804976440540711201965410747553824948672771374075011577182305398340606162079, from equation (2.12) we get that $\nu(RSA-232) = 766$. Then, from equations (4.11), (4.12) we have R(RSA-232)=4. The only acceptable combination which is compatible with corollary 4.3 is the following: The one factor of the RSA-232 has symmetry L and the other has symmetry R, where $L = R < 4$ or $L = 4 \wedge R > 4$ or $L > 4 \wedge R = 4$.

Equations (4.23) and (4.24) provide the simplest way for the determination of the symmetry of a symmetric number. We give one example.

Example 4.8. For number 18303 we have

$$\begin{aligned} &18303 - 1 = 2^1 \times 9151 \\ &18303 + 1 = 2^7 \times 143 \end{aligned}.$$

Therefore, is $R(18303) = 7 - 1 = 6$.

We now prove the following corollary:

Corollary 4.4. 1. *Every composite asymmetric number has at least two factors the symmetries of which have equal values.*

2. *Every composite Fermat number has at least two prime numbers factors* $Q_1 \neq Q_2$ *with* $L(Q_1) = L(Q_2)$.

Proof. 1. Part 1 of corollary comes from the 1 and 5 of the corollary 4.3.

2. Part 2 of corollary comes from the 1 and 5 of the corollary 4.3, and additionally taking into account that Fermat numbers are asymmetric. □

5 One essential corollary for the asymmetric numbers

In this chapter we prove one essential corollary for the asymmetric numbers:

Corollary 5.1. *For every composite asymmetric number*

$$A_S = A_S(\nu) = 2^{\nu} + 1, \nu \in \mathbb{N} \tag{5.1}$$

one of the following holds:

1. *There is a natural number L and odd numbers* K_1, K_2, K_3 *so that*

$$A_S(\nu) = 2^{\nu} + 1 = \left(2^{L+1} \cdot K_1 + 1\right)\left(2^{L+1} \cdot K_2 + 1\right) \tag{5.2}$$

$$2^{\nu-L-1} = 2^{L+1} \cdot K_1 K_2 + K_1 + K_2 \tag{5.3}$$

$$K_1 + K_2 = 2^{L+1} \cdot K_3 \tag{5.4}$$

$$2^{\nu-2L-2} = K_1 K_2 + K_3 \tag{5.5}$$

$$K_1 = 2^L \cdot K_3 - \sqrt{2^{2L} \cdot K_3^2 + K_3 - 2^{\nu-2L-2}} \tag{5.6}$$

$$K_2 = 2^L \cdot K_3 + \sqrt{2^{2L} \cdot K_3^2 + K_3 - 2^{\nu-2L-2}}\,. \tag{5.7}$$

2. *There is a natural number R and odd numbers* K_1, K_2, K_3 *so that*

$$A_S(\nu) = 2^{\nu} + 1 = \left(2^{R+1} \cdot K_1 - 1\right)\left(2^{R+1} \cdot K_2 - 1\right) \tag{5.8}$$

$$2^{\nu-R-1} = 2^{R+1} \cdot K_1 K_2 - K_1 - K_2 \tag{5.9}$$

$$K_1 + K_2 = 2^{R+1} \cdot K_3 \tag{5.10}$$

$$2^{\nu-2R-2} = K_1 K_2 - K_3 \tag{5.11}$$

$$K_1 = 2^R \cdot K_3 - \sqrt{2^{2R} \cdot K_3^2 - K_3 - 2^{\nu-2R-2}} \tag{5.12}$$

$$K_2 = 2^R \cdot K_3 + \sqrt{2^{2R} \cdot K_3^2 - K_3 - 2^{\nu-2R-2}}\,. \tag{5.13}$$

Proof. From equation (4.19) and corollary 4.4 we have that every asymmetric number A_S can be written in the form $A_S(\nu) = 2^{\nu} + 1 = Q_1 Q_2, L(Q_1) = L(Q_2) = L$ or in the form $A_S(\nu) = 2^{\nu} + 1 = D_1 D_2, R(D_1) = R(D_2) = R$. In the first case part 1 of the corollary holds and for the second case part 2 holds. We prove part 1 of the corollary. The proof of the part 2 is similar.

In case $A_S(\nu)=2^\nu+1=Q_1Q_2, L(Q_1)=L(Q_2)=L$, from the last part of equation (4.1) we have

$$\begin{aligned}&A_S(\nu)=2^\nu+1=\left(2^{L+1}\cdot K_1+1\right)\left(2^{L+1}\cdot K_2+1\right)\\&\nu,L,K_1,K_2\in\mathbb{N},\nu>2(L+1),K_1,K_2=odd\end{aligned}\qquad(5.14)$$

and after some calculations we have

$$2^{\nu-L-1}=2^{L+1}\cdot K_1K_2+K_1+K_2.\qquad(5.15)$$

The sum K_1+K_2 is an even number so there is a natural number x and an odd number K_3 so that

$$K_1+K_2=2^x\cdot K_3.\qquad(5.16)$$

For $x\neq L+1$ the one side of the equation (5.15) is an even number and the other is an odd number, so we have that $x=L+1$, and the equations (5.16), (5.15) take the form

$$K_1+K_2=2^{L+1}\cdot K_3$$

$$2^{\nu-2L-2}=K_1K_2+K_3.$$

Solving the system of equations (5.4), (5.5) we obtain equations (5.6) and (5.7).

It is easy to prove that for the odd numbers K_1 and K_2 it holds that $L(K_1)=R(K_2)$ or $R(K_1)=L(K_2)$; in any other case, by simplifying using a proper power of 2 the one part of equations (5.3), (5.9) will be even and the other odd. □

Next, we have two examples.

Example 5.1. For $A_S(36)$ we have

$$A_S(36)=2^{36}+1=\left(2^2\right)^9+1=17\times241\times433\times38737=1774001\times38737$$

$$L(1774001)=L(38737)=3$$

$$\begin{aligned}&1774001=2^4\times110875+1\\&38737=2^4\times2421+1\end{aligned}.$$

Therefore is $\nu=36, L=3, K_1=110875, K_2=2421$, and from corollary 5.1 we obtain

$$\begin{aligned}&110875+2421=2^4\times7081\\&K_3=7081\\&2^{28}=110875\times2421+7081\end{aligned}.$$

From equalities

$$
\begin{aligned}
&A_S(36) = 17 \times 241 \times 433 \times 38737 \\
&= (17 \times 241 \times 433) \times 38737 \\
&= (17 \times 433) \times (241 \times 38737) \\
&= (17 \times 241) \times (433 \times 38737) \\
&= (17 \times 38737) \times (241 \times 433) \\
&= 17 \times (241 \times 433 \times 38737) \\
&= (241 \times 17) \times (433 \times 38737) \\
&= 433 \times (17 \times 241 \times 38737)
\end{aligned}
$$

seven cases are derived by applying corollary 5.1 for $A_S(36)$.

Example 5.2.

$$
\begin{aligned}
&A_s(11) = 2^{11} + 1 = 3 \times 683 = (1 \times 2^1 + 1)(171 \times 2^2 - 1) = \left(\mathbf{1} \times \mathbf{2^2} - \mathbf{1}\right)\left(\mathbf{171} \times \mathbf{2^2} - \mathbf{1}\right), \\
&K_1 = 1, K_2 = 171, R = 1, K_3 = 43, \\
&2^7 = 1 \times 171 - 43
\end{aligned}
$$

Equivalently, from equations (5.6), (5.7) and (5.12), (5.13) we get

$$
\begin{aligned}
&2^{2L} \cdot K_3^2 + K_3 - 2^{\nu-2L-2} = N^2 \\
&N \in \mathbb{N}^*, N = odd
\end{aligned}
\tag{5.17}
$$

$$
\begin{aligned}
&2^{2L} \cdot K_3^2 - K_3 - 2^{\nu-2L-2} = N^2 \\
&N \in \mathbb{N}^*, N = odd
\end{aligned}
\tag{5.18}
$$

The odd number K_3 can be written in the form of $K_3 = 2^{x_4} \cdot K_4 + 1, x_4, K_4 \in \mathbb{N}, K_4 = odd$ or in the form of $K_3 = 2^{x_4} \cdot K_4 - 1, x_4, K_4 \in \mathbb{N}, K_4 = odd$. For known ν and L we can determine a term $K_j, j \in \mathbb{N}, j > 3$ of the sequence

$$
\begin{aligned}
&K_i = 2^{x_{i+1}} \cdot K_{i+1} \pm 1 \\
&x_{i+1}, K_{i+1} \in \mathbb{N}^*, K_{i+1} = odd \\
&i = 3, 4, 5, \ldots
\end{aligned}
\tag{5.19}
$$

and in turn we can determine K_3 of equations (5.17), (5.18).

For Fermat numbers $F_S = 2^{2^S} + 1, S \in \mathbb{N}$ the part 1 of corollary 5.1 holds, while $\nu = 2^S$ and $L \in \{S+1, S+2, S+3, \ldots\}$. Therefore, the factorization of Fermat numbers can be achieved by determining the terms of the sequence of odd numbers $K_i, i = 3, 4, 5, \ldots$. From Corollary 4.3 it is derived that for the Fermat numbers with more than two factors there exist more than one sets of three K_1, K_2, K_3.

For $K_1, K_2 \neq 1$ is $K_1 + K_2 < K_1 K_2$ and with equation (5.4) we obtain

$$2^{L+1} \cdot K_3 < K_1 K_2 . \tag{5.20}$$

From equation (5.3) we get $K_1 K_2 < 2^{\nu-2L-2}$ and with inequality (5.20) we obtain

$$2^{L+1} \cdot K_3 < K_1 K_2 < 2^{\nu-2L-2}$$
$$2^{L+1} \cdot K_3 < 2^{\nu-2L-2}$$

and finally we obtain

$$K_3 < 2^{\nu-3L-3} . \tag{5.21}$$

From inequality (5.21) it is derived that

$$\nu \geq 3L+3 . \tag{5.22}$$

From inequality (5.22), for $F_4 = 2^{16}+1$ we get

$$16-1 > 3L+3$$
$$L \in \Phi_4 = \{5, 6, 7, ...\}$$

which is impossible. Similarly it can be proven that F_0, F_1, F_2, F_3 cannot be composite numbers. Inequality (5.22) holds only for Fermat numbers F_S with $S \geq 5$.

6 A sequence of prime numbers

The following corollary provides a sequence of prime numbers:

Corollary 6.1 (Conjecture) *For every asymmetric number of the form*

$$\Theta(2, S) = 2^{2^S}, S \in \mathbb{N} \tag{6.1}$$

exists an interval around this number, whose length is of order

$$\varepsilon = 2^S \tag{6.2}$$

and this interval does not contain any prime numbers.

Because of the accumulation of small prime numbers close to 0 the part 1 of the corollary holds for these values of S which satisfy $S \geq 5$.

In equation (6.2) we know the length (6.2). This allows us to determine [2-6] prime numbers by using the equations

$$\begin{aligned} &P = 2^{2^S} \mp 1 - 2x \\ &P = 2^{2^S} \mp 1 + 2x \\ &\varepsilon = 2^{S+l}, l \in \mathbb{R} \\ &S, x \in \mathbb{N}, S \geq 5 \end{aligned} . \tag{6.3}$$

From equation (6.3) for S=5, 6, 7, 8, 9 we get the first 10 prime numbers:

$S = 5$

$P = 2^{32} - 1 - 2 \cdot 2 = 2^{32} + 1 - 2 \cdot 3 = 4294\ 967291$

$P = 2^{32} - 1 + 2 \cdot 8 = 2^{32} + 1 + 2 \cdot 7 = 4294\ 967311$

$\varepsilon = 2 \cdot 8 - (-2 \cdot 2) = 20$

$S = 6$

$P = 2^{64} - 1 - 2 \cdot 29 = 2^{64} + 1 - 2 \cdot 30 = 18\ 446744\ 073709\ 551557$

$P = 2^{64} - 1 + 2 \cdot 7 = 2^{64} + 1 + 2 \cdot 6 = 18\ 446744\ 073709\ 551629$

$\varepsilon = 2 \cdot 7 - (-2 \cdot 29) = 72$

$S = 7$

$P = 2^{128} - 1 - 2 \cdot 79 = 2^{128} - 1 - 2 \cdot 79 = 2^{128} + 1 - 2 \cdot 80$

$= 340\ 282366\ 920938\ 463463\ 374607\ 431768\ 211297$

$P = 2^{128} - 1 + 2 \cdot 26 = 2^{128} + 1 + 2 \cdot 25$

$= 340\ 282366\ 920938\ 463463\ 374607\ 431768\ 211507$

$\varepsilon = 2 \cdot 26 - (-2 \cdot 79) = 210$

$S = 8$

$P = 2^{256} - 1 - 2 \cdot 217 = 2^{256} + 1 - 2 \cdot 218$

$= 115792\ 089237\ 316195\ 423570\ 985008\ 687907$

$853269\ 984665\ 640564\ 039457\ 584007\ 913129\ 639501$

$P = 2^{256} - 1 + 2 \cdot 149 = 2^{256} + 1 + 2 \cdot 148$

$= 115792\ 089237\ 316195\ 423570\ 985008\ 687907$

$853269\ 984665\ 640564\ 039457\ 584007\ 913129\ 640233$

$\varepsilon = 2 \cdot 149 - (-2 \cdot 217) = 732$

$S = 9$

$P = 2^{512} - 1 - 2 \cdot 284 = 2^{512} + 1 - 2 \cdot 285$

$= 13407\ 807929\ 942597\ 099574\ 024998\ 205846\ 127479\ 365820\ 592393\ 377723$

$561443\ 721764\ 030073\ 546976\ 801874\ 298166\ 903427\ 690031\ 858186\ 486050$

$853753\ 882811\ 946569\ 946433\ 649006\ 083527$

$P = 2^{512} - 1 + 2 \cdot 38 = 2^{512} + 1 + 2 \cdot 37$

$= 13407\ 807929\ 942597\ 099574\ 024998\ 205846\ 127479\ 365820\ 592393\ 377723$

$561443\ 721764\ 030073\ 546976\ 801874\ 298166\ 903427\ 690031\ 858186\ 486050$

$853753\ 882811\ 946569\ 946433\ 649006\ 084171$

$\varepsilon = 2 \cdot 38 - (-2 \cdot 285) = 646$

For $S \to +\infty$ we obtain large prime numbers.

An initial statistical investigation showed that for $S = 3l, l \in \mathbb{N}^*$ and $S = 3l+2, l \in \mathbb{N}^*$ the range ε tends to 2^S, taking larger values. For $S = 3l+1, l \in \mathbb{N}^*, l \geq 3$ the range ε tends to 2^S, taking smaller values. A further investigation will allow us to determine with greater precision the primer number found in the limits of the intervals of corollary 6.1.

7 One essential corollary for the Mersenn numbers

From equation (2.11) we get for the Mersenn numbers

$$\begin{aligned} &M_p = 2^p - 1 \\ &p \in \mathbb{N}, p = prime \end{aligned} \tag{7.1}$$

that have symmetry R,

$$R = R\left(M_p\right) = R\left(2^p - 1\right) = p - 3. \tag{7.2}$$

Therefore, from corollary 4.3 it is derived that for the composite Mersenn numbers it holds

$$\begin{aligned} &M_p = 2^p - 1 = \left(2^{R+1} \cdot K_1 - 1\right)\left(2^{L+1} \cdot K_2 + 1\right) = \left(2^{p-2} \cdot K_1 - 1\right)\left(2^{L+1} \cdot K_2 + 1\right) \\ &L, K_1, K_2 \in \mathbb{N}, K_1, K = odd \\ &R < L \Leftrightarrow p < L + 3 \end{aligned} \tag{7.3}$$

or

$$\begin{aligned} &M_p = 2^p - 1 = \left(2^{x+1} \cdot K_1 - 1\right)\left(2^{x+1} \cdot K_2 + 1\right). \\ &x, K_1, K_2 \in \mathbb{N}, K_1, K_2 = odd \end{aligned} \tag{7.4}$$

Equation (7.3) is impossible: By conducting the calculations an equation is derived in which the one part is even number and the other is odd, due to inequality p<L+3. Therefore, for the composite Mersenn numbers equation (7.4) holds, from which the following corollary is derived:

Corollary 7.1. *For every composite Mersenn number*

$$\begin{aligned} &M_p = 2^p - 1 \\ &p \in \mathbb{N}, p = prime \end{aligned}$$

there is a natural number x and odd numbers K_1, K_2, K_3, $K_1, K_2 \in \mathbb{N}, K_3 \in \mathbb{Z}$ *so that*

$$2^{p-x-1} = 2^{x+1} \cdot K_1 K_2 + K_1 - K_2 \tag{7.5}$$

$$2^{p-2x-2} = K_1 K_2 + K_3, K_3 \in \mathbb{Z} \tag{7.6}$$

$$K_1 - K_2 = 2^{x+1} \cdot K_3, K_3 \in \mathbb{Z} \tag{7.7}$$

$$K_1 = 2^x \cdot K_3 + \sqrt{2^{2x} \cdot K_3^2 - K_3 + 2^{p-2x-2}} \tag{7.8}$$

$$K_2 = 2^x \cdot K_3 - \sqrt{2^{2x} \cdot K_3^2 - K_3 + 2^{p-2x-2}} \tag{7.9}$$

$$2^{2x} \cdot K_3^2 - K_3 + 2^{p-2x-2} = N^2, N \in \mathbb{N}, N = odd\ . \tag{7.10}$$

Proof. The proof is similar to the one of corollary 5.1. □

Next, we have one example.

Example 7.1.

$M_{29} = 2^{29} - 1 = 1103 \times 486737$

$1103 = 2^4 \cdot 69 - 1$

$486737 = 2^4 \cdot 30421 + 1$

$x = 3, K_1 = 69, K_2 = 30421$.

$K_3 = \dfrac{69 - 30421}{2^4} = -1897$

$2^{29-6-2} = 2^{21} = 69 \cdot 30421 + (-1897)$

$2^6 \cdot (1897)^2 - (-1897) + 2^{21} = 15245^2$

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