A Final Tentative of The Proof of The *ABC* Conjecture - Case $c = a + 1$

Abdelmajid Ben Hadj Salem
6, Rue du Nil, Cité Soliman Er-Riadh,
8020 Soliman,
Tunisia
E-mail: abenhadjsalema@gmail.com

Abstract: In this paper, we consider the *abc* conjecture in the case $c = a + 1$. Firstly, we give the proof of the first conjecture that $c < rad^2(ac)$ using the polynomial functions. It is the key of the proof of the *abc* conjecture. Secondly, the proof of the *abc* conjecture is given for $\varepsilon \geq 1$, then for $\varepsilon \in]0,1[$ for the two cases: $c \leq rad(ac)$ and $c > rad(ac)$. We choose the constant $K(\varepsilon)$ as $K(\varepsilon) = e^{\left(\frac{1}{\varepsilon^2}\right)}$. A numerical example is presented.

A Final Tentative of The Proof of The *ABC* Conjecture - Case $c = a + 1$

To the memory of my Father who taught me arithmetic

To the memory of my colleague and friend Dr.Eng. Chedly Fezzani (1943-2019) for his important work in the field of Geodesy and the promotion of the Geographic Sciences in Africa

1. Introduction and notations

Let a a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1.1}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{1.2}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) ([1]). It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 1.3. *(**abc Conjecture**): Let a, b, c positive integers relatively prime with $c = a + b$, then for each $\varepsilon > 0$, there exists $K(\varepsilon)$ such that :*

$$c < K(\varepsilon).rad(abc)^{1+\varepsilon} \tag{1.4}$$

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.616751$ ([2]). A conjecture was proposed that $c < rad^2(abc)$ ([3]). Here we will give the proof of it, in the case $c = a + 1$, using a polynomial function.

Conjecture 1.5. *Let a, b, c positive integers relatively prime with $c = a + b$, then:*

$$c < rad^2(abc) \Longrightarrow \frac{Logc}{Log(rad(abc))} < 2 \tag{1.6}$$

This result, I think is the key to obtain a proof of the veracity of the *abc* conjecture.

2. A Proof of the conjecture (1.5) Case $c = a + 1$

Let a, c positive integers, relatively prime, with $c = a + 1$. If $c < rad(ac)$ then we obtain:

$$c < rad(ac) < rad^2(ac) \tag{2.1}$$

and the condition (1.6) is verified.

In the following, we suppose that $c \geq rad(ac)$.

2.1 Notations

We note:

$$a = \prod_i a_i^{\alpha_i} \Longrightarrow rad(a) = \prod_i a_i, \mu_a = \prod_i a_i^{\alpha_i - 1},\ i = 1, N_a \tag{2.2}$$

$$c = \prod_j c_j^{\beta_j} \Longrightarrow rad(c) = \prod_j c_j, \mu_c = \prod_j c_j^{\beta_j - 1},\ j = 1, N_c \tag{2.3}$$

with a_i, c_j prime integers and $N_a, N_c, \alpha, \beta \geq 1$ positive integers. Let:

$$R = rad(a).rad(c) = rad(ac) \tag{2.4}$$

$$\mathscr{R}(x) = \prod_i^{N_a}(x+a_i)^2.\prod_j^{N_c}(x+c_j) \Longrightarrow \mathscr{R}(x) > 0, \forall x \geq 0 \tag{2.5}$$

$$F(x) = \mathscr{R}(x) - \mu_c \tag{2.6}$$

From the last equations we obtain:

$$F(0) = \mathscr{R}(0) - \mu_c = rad^2(a).rad(c) - \mu_c \tag{2.7}$$

Then, our main task is to prove that $F(0) > 0 \Longrightarrow R^2 > c$.

2.1.1 The Proof of $c < rad^2(ac)$

From the definition of the polynomial $F(x)$, its degree is $2N_a + N_c$. We have :
1. $lim_{x \longrightarrow +\infty} F(x) = +\infty$,
2. $lim_{x \longrightarrow +\infty} \dfrac{F(x)}{x} = +\infty$, F is convex for x large,
3. if x_1 is the great real root of $F(x) = 0$, and from the points 1., 2. we deduce that $F"(x_1^+) > 0$,
4. if $x_1 < 0$, then $F(0) > 0$.
Let us study $F'(x)$ and $F"(x)$. We obtain:

$$F'(x) = \mathscr{R}'(x)$$

$$\mathscr{R}'(x) = \left[\prod_i^{N_a}(x+a_i)^2\right]'.\prod_j^{N_c}(x+c_j) + \prod_i^{N_a}(x+a_i)^2.\left[\prod_j^{N_c}(x+c_j)\right]' \Longrightarrow$$

$$\left[\prod_i^{N_a}(x+a_i)^2\right]' = 2\prod_i^{N_a}(x+a_i)^2.\left(\sum_i \frac{1}{x+a_i}\right)$$

$$\left[\prod_j^{N_c}(x+c_j)\right]' = \prod_j^{N_c}(x+c_j)\left(\sum_{j=1}^{j=N_b} \frac{1}{x+c_j}\right) \Longrightarrow$$

$$\mathscr{R}'(x) = \mathscr{R}(x).\left(\sum_i^{N_a} \frac{2}{x+a_i} + \sum_j^{N_c} \frac{1}{x+c_j}\right) > 0, \forall x \geq 0 \tag{2.8}$$

$$F'(x) = \mathscr{R}' = \mathscr{R}(x)\left(\sum_i^{N_a} \frac{2}{x+a_i} + \sum_j^{N_c} \frac{1}{x+c_j}\right) > 0, \forall x > 0 \Longrightarrow$$

$$F'(0) = \mathscr{R}(0).\left(\sum_i^{N_a} \frac{2}{a_i} + \sum_j^{N_c} \frac{1}{c_j}\right) = rad^2(a).rad(c).\left(\sum_i^{N_a} \frac{2}{a_i} + \sum_j^{N_c} \frac{1}{c_j}\right) > 0 \tag{2.9}$$

For $F"(x)$, we obtain:

$$F"(x) = \mathscr{R}" = \mathscr{R}'(x)\left(\sum_i^{N_a}\frac{2}{x+a_i}+\sum_j^{N_c}\frac{1}{x+c_j}\right)-\mathscr{R}(x)\left(\sum_i^{N_a}\frac{2}{(x+a_i)^2}+\sum_j^{N_c}\frac{1}{(x+c_j)^2}\right)\Longrightarrow \tag{2.10}$$

$$F"(x) = \mathscr{R}(x).\left[\left(\sum_i^{N_a}\frac{2}{x+a_i}+\sum_j^{N_c}\frac{1}{x+c_j}\right)^2-\sum_i^{N_a}\frac{2}{(x+a_i)^2}-\sum_j^{N_c}\frac{1}{(x+c_j)^2}\right]\Longrightarrow$$

$$F"(x) > 0, \forall x \geq 0 \tag{2.11}$$

We obtain also that $F"(0) > 0$.

Before we attack the proof, we take an example as: $1+8=9 \Longrightarrow c=9, a=8, b=1$. We obtain $rad(a)=2, rad(c)=3, \mu_c=3, R=rad(ac)=2\times 3=6<(c=9)$ and $c=9$ verifies $c<(R^2=6^2=36)$. We write the polynomial $F(x)=(x+2)^2(x+3)-3=x^3+7x^2+16x+9>0, \forall x>0$. Then $F'(x)=3x^2+14x+16$, we verifies that $F'(x)=0$ has not real roots and $F'(x)>0, \forall x\in\mathbb{R}$. We have also $F"(x)=6x+14$. $F"(x)=0 \Longrightarrow x=-7/3\approx-2.33 \Longrightarrow F(-7/3)=-79/27\approx-2.92$. The point $(-7/3,-79/27)$ is an inflexion point of the curve of $y=F(x)$. We deduce that the curve is convex for $x\geq-7/3$. Let us now find the roots of $F(x)=0$. As the degree of F is three, the number of the real roots are 1 or 3. As there is one inflexion point, we will find one real root.

2.2 The Resolution of $F(x)=0$

We want to resolve:

$$F(x)=x^3+7x^2+16x+9=0 \tag{2.12}$$

Let the change of variables $x=t-7/3$, the equation (2.12) becomes:

$$t^3-\frac{t}{3}-\frac{79}{27}=0 \tag{2.13}$$

For the resolution of (2.13), we introduce two unknowns:

$$t=u+v \Longrightarrow (u+v)(3uv-\tfrac{1}{3})+u^3+v^3-\tfrac{79}{27}=0 \Longrightarrow$$

$$\begin{cases} u^3+v^3=\dfrac{79}{3^3} \\ uv=\dfrac{1}{3^2} \end{cases} \tag{2.14}$$

Then u^3, v^3 are solutions of the equation:

$$X^2-\frac{79}{3^3}X+\frac{1}{3^6}=0 \tag{2.15}$$

and given below:

$$u^3=\frac{1}{2}.\frac{79+9\sqrt{77}}{3^3}\Longrightarrow\begin{cases} u_1=\sqrt[3]{\dfrac{1}{2}\left(\dfrac{79+9\sqrt{77}}{3^3}\right)}\approx 0.97515 \\ \\ u_2=j.u_1, \quad j=\frac{-1+i\sqrt{3}}{2}=e^{i\frac{2\pi}{3}} \\ u_3=j^2u_1=\bar{j}.u_1 \end{cases}$$

$$v^3 = \frac{1}{2}.\frac{79-9\sqrt{77}}{3^3} \Longrightarrow \begin{cases} v_1 = \sqrt[3]{\frac{1}{2}\left(\frac{79-9\sqrt{77}}{3^3}\right)} \approx 0.00016 \\ \\ v_2 = j^2.v_1 = \bar{j}.v_1 \\ v_3 = j.v_1 \end{cases} \tag{2.16}$$

Finally, taking into account the second condition of (2.14), we obtain the real root of (2.13):

$$t = u_1 + v_1 = \sqrt[3]{\frac{1}{2}\left(\frac{79+9\sqrt{77}}{3^3}\right)} + \sqrt[3]{\frac{1}{2}\left(\frac{79-9\sqrt{77}}{3^3}\right)} \approx 0.97531$$
$$x_1 = t - 7/3 \approx -1.35802 \tag{2.17}$$

Then the first root of $F(x) = 0$ is $x_1 \approx -1.358 < 0$, the correction to the first root of $\mathscr{R}(x) = (x+2)^2(x+3) = 0$ is $dx = x_1 - (-2) = -1.358 - (-2) = +0.642$. As in our example $F'(x) > 0$, the function $F(x)$ is an increasing function having a parabolic branch as $x \longrightarrow +\infty$, the curve $y = F(x)$ intersects the line $x = 0$ in the half-plane $y \geq 0 \Longrightarrow F(0) > 0 \Longrightarrow c < rad^2(ac)$ which is verified numerically.

2.3 The General Case

Let us return to the general case $c = a+1$. We denote $q = min(a_i, c_j)$. If we consider that $F(x) = \mathscr{R}(x)$, the equation $F(x) = 0 \Longrightarrow \mathscr{R}(x) = 0$ and the first real root is $x_1 = -q$, the product of all the roots is $P = \prod_i (x_i)^2.\prod_j (x_j) = (-1)^{2N_a+N_c}\prod_i (a_i)^2.\prod_j (c_j)$. But $F(x) = \mathscr{R}(x) - \mu_c$, the constant coefficient of $F(x)$ will be $\prod_i (a_i)^2.\prod_j (c_j) - \mu_c$. The new product of the roots is $P' = \prod_i (x'_i)^2.\prod_j (x'_j) = (-1)^{2N_a+N_c}(\prod_i (a_i)^2.\prod_j (c_j) - \mu_c)$. The first root $x_1 = -q$ becomes $x'_1 = -q + dx$. To estimate dx, we can write to the order two that:

$$F(-q+dx) = \mathscr{R}(-q+dx) - \mu_c = 0 \Longrightarrow \mathscr{R}(-q+dx) = \mu_c \Longrightarrow$$
$$\mathscr{R}(-q) + dx.\mathscr{R}'(-q) + \frac{dx^2}{2}\mathscr{R}''(-q) = \mu_c \tag{2.18}$$

Supposing that $a_1 = q = min(a_i, c_j)$, from the equations (2.5-2.8-2.10), we have :

$$\mathscr{R}(-a_1) = 0$$
$$\mathscr{R}'(-a_1) = 0$$
$$\mathscr{R}''(-a_1) = 2\prod_{i=2}^{N_a}(a_i - a_1)^2.\prod_{j=1}^{N_c}(c_j - a_1) > 0 \Longrightarrow dx^2 = \frac{\mu_c}{\prod_{i=2}^{N_a}(a_i - a_1)^2.\prod_{j=1}^{N_c}(c_j - a_1)} \tag{2.19}$$

We suppose that $c > rad^2(ac) \Longrightarrow \mu_c > rad^2(a).rad(c) \Longrightarrow \mu_c > \mathscr{R}(0)$. We deduce that $F(0) < 0$ and $x'_1 = -a_1 + dx > 0 \Longrightarrow dx > 0$. We take the positive value of dx, then we obtain:

$$dx = \frac{\sqrt{\mu_c}}{\prod_{i=2}^{N_a}(a_i - a_1).\sqrt{\prod_{j=1}^{N_c}(c_j - a_1)}} \tag{2.20}$$

But $\mu_c = \mathscr{R}(x'_1) = \prod_i^{N_a}(x'_1+a_i)^2.\prod_j^{N_c}(x'_1+c_j)$, we can write:

$$\mu_c = dx^2.\prod_{i=2}^{N_a}(dx+a_i-a_1)^2.\prod_j^{N_c}(dx+c_j-a_1) \Longrightarrow \mu_c > dx^2.\prod_{i=2}^{N_a}(a_i-a_1)^2.\prod_j^{N_c}(c_j-a_1) \quad (2.21)$$

because all the terms $a_i - a_1$ and $c_j - a_1$ are positive numbers. Using the last inequality and the expression of dx given by the equation (2.20), we obtain:

$$\mu_c > \frac{\mu_c}{\prod_{i=2}^{N_a}(a_i-a_1)^2.\prod_{j=1}^{N_c}(c_j-a_1)}.\prod_{i=2}^{N_a}(a_i-a_1)^2.\prod_j^{N_c}(c_j-a_1) \Longrightarrow$$
$$1 > 1 \Longrightarrow \text{ the contradiction} \Longrightarrow \mu_c < rad^2(a)rad(c) \quad (2.22)$$

So, our supposition that $c > rad^2(ac)$ is false and we obtain the important result that $c < rad^2(ac)$ and the conjecture (1.5) is verified.

2.3.1 Examples

In this section, we are going to verify the above remarks with a numerical example. The example is given by:

$$\begin{aligned} 1+5\times 127\times(2\times 3\times 7)^3 &= 19^6 \\ rad(a) = 2\times 3\times 5\times 7\times 127 &= 26670 \\ rad(c) &= 19 \\ c = 19^5 = 47\,045\,881, \quad \mu_c = 19^5 &= 2\,476\,099 \end{aligned} \quad (2.23)$$

Using the notations of the paper, we obtain:

$$\begin{aligned} \mathscr{R}(x) = (x+2)^2(x+3)^2(x+5)^2(x+7)^2(x+127)^2(x+19) \\ F(x) = \mathscr{R}(x) - \mu_c \end{aligned}$$

Let $X = x+2$, the expression of $\mathscr{R}(x)$ becomes:

$$\overline{\mathscr{R}}(X) = X^2(X+1)^2(X+3)^2(X+5)^2(X+125)^2(X+17)$$

The calculations gives:

$$\begin{aligned} \overline{\mathscr{R}}(X) = X^{11} + 285.X^{10} + 24\,808.X^9 + 657\,728.X^8 + 7\,424\,722.X^7 + 42\,772\,898.X^6 \\ +134\,002\,080.X^5 + 223\,508\,940.X^4 + 187\,753\,125.X^3 + 597\,656\,251.X^2 \end{aligned} \quad (2.24)$$

We want to estimate the first root of $F(x) = 0$, we write:

$$\begin{aligned} \overline{\mathscr{R}}(X) - \mu_c = 0 \Longrightarrow \\ X^{11} + 285.X^{10} + 24\,808.X^9 + 657\,728.X^8 + 7\,424\,722.X^7 + 42\,772\,898.X^6 \\ +134\,002\,080.X^5 + 223\,508\,940.X^4 + 187\,753\,125.X^3 + 597\,656\,251.X^2 - 2\,476\,099 = 0 \end{aligned} \quad (2.25)$$

If $x=-2 \Longrightarrow X=0 \Longrightarrow \overline{\mathscr{R}}(X)-\mu_c<0$. If we take $x_1=-1.936315 \Longrightarrow X_1=0.03685$, then we obtain that:

$$\mathscr{R}(x_1)-\mu_c=\overline{\mathscr{R}}(X_1)-\mu_c \approx 177.82>0 \tag{2.26}$$

Then, $\exists\, \xi$ with $-2<\xi<x_1$ so that $X'=2+\xi$ verifies $\overline{\mathscr{R}}(X')-\mu_c=0$ and ξ is the first root of $F(x)=0$ and $\xi<0 \Longrightarrow F(0)>0 \Longrightarrow rad^2(a)rad(c)-\mu_c>0 \Longrightarrow R^2>c$ that is true. We have also $\xi=-2+dx=a_1+dx$ and $0<dx<a_1$.

3. The Proof of The *ABC* Conjecture (1.3) Case: $c=a+1$

We denote $R=rad(ac)$.

3.1 Case: $\varepsilon \geq 1$

Using the result of the theorem above, we have $\forall \varepsilon \geq 1$:

$$c<R^2 \leq R^{1+\varepsilon}<K(\varepsilon).R^{1+\varepsilon}, \quad K(\varepsilon)=e^{\left(\frac{1}{\varepsilon^2}\right)},\ \varepsilon \geq 1 \tag{3.1}$$

We verify easily that $K(\varepsilon)>1$ for $\varepsilon \geq 1$ and it is a decreasing function from e the base of the neperian logarithm to 1.

3.2 Case: $\varepsilon<1$

3.2.1 Case: $c \leq R$

In this case, we can write :

$$c \leq R<R^{1+\varepsilon}<K(\varepsilon).R^{1+\varepsilon}, \quad K(\varepsilon)=e^{\left(\frac{1}{\varepsilon^2}\right)},\ \varepsilon<1 \tag{3.2}$$

here also $K(\varepsilon)>1$ for $\varepsilon<1$ and the *abc* conjecture is true.

3.2.2 Case: $c>R$

In this case, we confirm that :

$$c<K(\varepsilon).R^{1+\varepsilon}, \quad K(\varepsilon)=e^{\left(\frac{1}{\varepsilon^2}\right)}, 0<\varepsilon<1 \tag{3.3}$$

If not, then $\exists \varepsilon_0 \in]0,1[$, so that the triplets $(a,1,c)$ checking $c>R$ and:

$$c \geq R^{1+\varepsilon_0}.K(\varepsilon_0) \tag{3.4}$$

are in finite number. We have:

$$\begin{aligned} c \geq R^{1+\varepsilon_0}.K(\varepsilon_0) \Longrightarrow R^{1-\varepsilon_0}.c \geq R^{1-\varepsilon_0}.R^{1+\varepsilon_0}.K(\varepsilon_0) \Longrightarrow \\ R^{1-\varepsilon_0}.c \geq R^2.K(\varepsilon_0)>c.K(\varepsilon_0) \Longrightarrow R^{1-\varepsilon_0}>K(\varepsilon_0) \end{aligned} \tag{3.5}$$

As $c > R$, we obtain:

$$c^{1-\varepsilon_0} > R^{1-\varepsilon_0} > K(\varepsilon_0) \Longrightarrow$$

$$c^{1-\varepsilon_0} > K(\varepsilon_0) \Longrightarrow c > K(\varepsilon_0)^{\left(\dfrac{1}{1-\varepsilon_0}\right)} \tag{3.6}$$

We deduce that it exists an infinity of triples $(a,1,c)$ verifying (3.4), hence the contradiction. Then the proof of the *abc* conjecture in the case $c = a+1$ is finished. We obtain that $\forall \varepsilon > 0$, $c = a+1$ with a, c relatively coprime, $2 \leq a < c$:

$$c < K(\varepsilon).rad(ac)^{1+\varepsilon} \quad with \quad K(\varepsilon) = e^{\left(\dfrac{1}{\varepsilon^2}\right)} \tag{3.7}$$

Q.E.D

4. Examples

In this section, we are going to verify some cases of one numerical example. The example is given by:

$$1 + 5 \times 127 \times (2 \times 3 \times 7)^3 = 19^6 \tag{4.1}$$

$a = 5 \times 127 \times (2 \times 3 \times 7)^3 = 47\,045\,880 \Rightarrow \mu_a = 2 \times 3 \times 7 = 42$ and $rad(a) = 2 \times 3 \times 5 \times 7 \times 127$,
$b = 1 \Rightarrow \mu_b = 1$ and $rad(b) = 1$,
$c = 19^6 = 47\,045\,880 \Rightarrow rad(c) = 19$. Then $rad(abc) = rad(ac) = 2 \times 3 \times 5 \times 7 \times 19 \times 127 = 506\,730$..

We have $c > rad(ac)$ but $rad^2(ac) = 506\,730^2 = 256\,775\,292\,900 > c = 47\,045\,880$.

4.0.1 Case $\varepsilon = 0.01$

$c < K(\varepsilon).rad(ac)^{1+\varepsilon} \Longrightarrow 47\,045\,880 \overset{?}{<} e^{10000}.506\,730^{1.01}$. The expression of $K(\varepsilon)$ becomes:

$$K(\varepsilon) = e^{\frac{1}{0.0001}} = e^{10000} = 8,7477777149120053120152473488653e+4342 \tag{4.2}$$

We deduce that $c \ll K(0.01).506\,730^{1.01}$ and the equation (3.7) is verified.

4.0.2 Case $\varepsilon = 0.1$

$K(0.1) = e^{\frac{1}{0.01}} = e^{100} = 2,6879363309671754205917012128876e+43 \Longrightarrow c < K(0.1) \times 506\,730^{1.01}$. And the equation (3.7) is verified.

4.0.3 Case $\varepsilon = 1$

$K(1) = e \Longrightarrow c = 47\,045\,880 < e.rad^2(ac) = 697\,987\,143\,184,212.$ and the equation (3.7) is verified.

4.0.4 Case $\varepsilon = 100$

$$K(100) = e^{0.0001} \Longrightarrow c = 47\,045\,880 \overset{?}{<} e^{0.0001}.506\,730^{101} =$$
$$1,5222350248607608781853142687284e+576$$

and the equation (3.7) is verified.

5. Conclusion

This is an elementary proof of the *abc* conjecture in the case $c = a+1$. We can announce the important theorem:

Theorem 1. *Let a, c positive integers relatively prime with $c = a+1$, $a \geq 2$ then for each $\varepsilon > 0$, there exists $K(\varepsilon)$ such that :*

$$c < K(\varepsilon).rad(ac)^{1+\varepsilon} \tag{5.1}$$

where $K(\varepsilon)$ is a constant depending of ε equal to $e^{\left(\frac{1}{\varepsilon^2}\right)}$.

Acknowledgements: The author is very grateful to Professors Mihăilescu Preda and Gérald Tenenbaum for their comments about errors found in previous manuscripts concerning proofs proposed of the *abc* conjecture.

References

[1] Waldschmidt M.: On the abc Conjecture and some of its consequences presented at The 6th World Conference on 21st Century Mathematics, Abdus Salam School of Mathematical Sciences (ASSMS), Lahore (Pakistan), March 6-9, 2013. (2013)

[2] Robert O., Stewart C.L. and Tenenbaum G.: A refinement of the abc conjecture. Bull. London Math. Soc. **46**,6, 1156-1166 (2014).

[3] Mihăilescu P.: Around ABC. European Mathematical Society Newsletter **N° 93**, September 2014. 29-34, (2014)