Däumler's Horn Torus Model and Division by Zero - Absolute Function Theory - New World

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Abstract: In this paper, we will introduce a beautiful horn torus model by Puha and Däumler for the Riemann sphere in complex analysis attaching the zero point and the point at infinity. Surprisingly enough, we can introduce analytical structure of conformal to the model. Here, some basic opinions on the Däumler's horn torus model will be stated as the basic ones in mathematics.

Key Words: Infinity, discontinuous, point at infinity, stereographic projection, Riemann sphere, horn torus, Laurent expansion, conformal mapping, division by zero.

Mathematics Subject Classification (2010): 30C25, 30F10

Introduction

In this paper, first based on the paper [4] we introduce a horn torus model for the classical Riemann sphere from the viewpoint of the division by zero. In the model, the zero point and the point at infinity are attaching and surprisingly enough, we can introduce analytical structure of conformal in the horn torus. This model seems to be fundamental and important for us as a new world.

In Section 1, we will introduce the horn torus model by V.V. Puha and in Section 1.1, by modifying the Puha mapping, we introduce Däumler's horn torus model. In Section 1.2 we introduce division by zero and division by zero calculus with up-to-date information.

The main part of this paper, in Section 2, we would like to discuss the important viewpoints for the Däumler's horn torus model.

1 Horn torus models and division by zero calculus

We recall the essence of the paper [4] for horn torus models.

We will consider the three circles represented by

$$\xi^2 + \left(\zeta - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2,$$

$$\left(\xi - \frac{1}{4}\right)^2 + \left(\zeta - \frac{1}{2}\right)^2 = \left(\frac{1}{4}\right)^2, \tag{1.1}$$

and

$$\left(\xi + \frac{1}{4}\right)^2 + \left(\zeta - \frac{1}{2}\right)^2 = \left(\frac{1}{4}\right)^2.$$

By rotation on the space (ξ, η, ζ) on the (x, y) plane as in $\xi = x, \eta = y$ around ζ axis, we will consider the sphere with $1/2$ radius as the Riemann sphere and the horn torus made in the sphere.

The stereographic projection mapping from (x, y) plane to the Riemann sphere is given by

$$\xi = \frac{x}{x^2 + y^2 + 1},$$

$$\eta = \frac{y}{x^2 + y^2 + 1},$$

and

$$\zeta = \frac{x^2 + y^2}{x^2 + y^2 + 1}.$$

Of course,

$$\xi^2 + \eta^2 = \zeta(1-\zeta),$$

and

$$x = \frac{\xi}{1-\zeta}, y = \frac{\eta}{1-\zeta}, \tag{1.2}$$

([1]).

The mapping from (x, y) plane to the horn torus is given by

$$\xi = \frac{2x\sqrt{x^2+y^2}}{(x^2+y^2+1)^2},$$

$$\eta = \frac{2y\sqrt{x^2+y^2}}{(x^2+y^2+1)^2},$$

and

$$\zeta = \frac{(x^2+y^2-1)\sqrt{x^2+y^2}}{(x^2+y^2+1)^2} + \frac{1}{2}.$$

This Puha mapping has a simple and beautiful geometrical correspondence. At first for the plane we consider the stereographic mapping to the Riemann sphere and next, we consider the common point of the line connecting the point and the center (0,0,1/2) and the horn torus. This is the desired point on the horn torus for the plane point.

The inversion is given by

$$x = \xi\left(\xi^2 + \eta^2 + \left(\zeta - \frac{1}{2}\right)^2 - \zeta + \frac{1}{2}\right)^{(-1/2)} \tag{1.3}$$

and

$$y = \eta\left(\xi^2 + \eta^2 + \left(\zeta - \frac{1}{2}\right)^2 - \zeta + \frac{1}{2}\right)^{(-1/2)}. \tag{1.4}$$

For the properties of horn torus with physical applications, see [3].

1.1 Conformal mapping from the plane to the horn torus with a modified mapping

W. W. Däumler discovered a surprising conformal mapping from the extended complex plane to the horn torus model (2018.8.18):

https://www.horntorus.com/manifolds/conformal.html
and
https://www.horntorus.com/manifolds/solution.html

We can represent the direct Däumler mapping from the z plane onto the horn torus as follows (V. V. Puha: 2018.8.28.22:31):

With

$$\phi = 2\cot^{-1}(-\log|z|), \quad z = x + yi, \tag{1.5}$$

$$\xi = \frac{x \cdot (1/2)(\sin(\phi/2))^2}{\sqrt{x^2 + y^2}},$$

$$\eta = \frac{y \cdot (1/2)(\sin(\phi/2))^2}{\sqrt{x^2 + y^2}},$$

and

$$\zeta = -\frac{1}{4}\sin\phi + \frac{1}{2}.$$

We have the inversion formula from the horn torus to the x, y plane:

$$x = \frac{\xi}{\sqrt{\xi^2 + \eta^2}} \exp \pm \left\{ \frac{\sqrt{\zeta - (\xi^2 + \eta^2 + \zeta^2)}}{\sqrt{\xi^2 + \eta^2 + \left(\zeta - \frac{1}{2}\right)^2}} \right\} \tag{1.6}$$

and

$$y = \frac{\eta}{\sqrt{\xi^2 + \eta^2}} \exp \pm \left\{ \frac{\sqrt{\zeta - (\xi^2 + \eta^2 + \zeta^2)}}{\sqrt{\xi^2 + \eta^2 + \left(\zeta - \frac{1}{2}\right)^2}} \right\}. \tag{1.7}$$

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1.2 Division by zero and division by zero calculus

We will state the essence of division by zero and division by zero calculus with up-to-date information.

For the long history of division by zero, see [2, 20]. The division by zero with mysterious and long history was indeed trivial and clear as in the followings.

By the concept of the Moore-Penrose generalized solution of the fundamental equation $ax = b$, the division by zero was trivial and clear as $b/0 = 0$ in the **generalized fraction** that is defined by the generalized solution of the equation $ax = b$. Here, the generalized solution is always uniquely determined and the theory is very classical. See [6] for example.

Division by zero is trivial and clear also from the concept of repeated subtraction - H. Michiwaki.

Recall the uniqueness theorem by S. Takahasi on the division by zero. See [6, 28].

The simple field structure containing division by zero was established by M. Yamada ([9]). For a simple introduction, see H. Okumura [18].

Many applications of the division by zero to Wasan geometry were given by H. Okumura. See [12, 13, 14, 15, 16, 17] for example. Many applications to differential equations, see [19]. In particular, we may say that the theory of differential equations is still incomplete essentially, because we are missing the value of $\tan(\pi/2)$.

As the number system containing the division by zero, the Yamada field structure is perfect. However, for applications of the division by zero to **functions**, we need the concept of the division by zero calculus for the sake of uniquely determinations of the results and for other many reasons.

For example, for the typical linear mapping

$$W = \frac{z-i}{z+i}, \tag{1.8}$$

it gives a conformal mapping on $\{\mathbf{C} \setminus \{-i\}\}$ onto $\{\mathbf{C} \setminus \{1\}\}$ in one to one and from

$$W = 1 + \frac{-2i}{z-(-i)}, \tag{1.9}$$

we see that $-i$ corresponds to 1 and so the function maps the whole $\{\mathbf{C}\}$ onto $\{\mathbf{C}\}$ in one to one.

Meanwhile, note that for

$$W = (z-i) \cdot \frac{1}{z+i}, \tag{1.10}$$

we should not enter $z = -i$ in the way

$$[(z-i)]_{z=-i} \cdot \left[\frac{1}{z+i}\right]_{z=-i} = (-2i) \cdot 0 = 0. \tag{1.11}$$

However, in many cases, the above two results will have practical meanings and so, we will need to consider many ways for the application of the division by zero and we will need to check the results obtained, in some practical viewpoints. We referred to this delicate problem with many examples in the references.

Therefore, we will introduce the division by zero calculus. For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n (z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n (z-a)^n, \tag{1.12}$$

we **define** the identity

$$f(a) = C_0. \tag{1.13}$$

Apart from the motivation, we define the division by zero calculus by (1.13). With this assumption, we can obtain many new results and new ideas. However, for this assumption we have to check the results obtained whether they are reasonable or not. By this idea, we can avoid any logical problems. – In this point, the division by zero calculus may be considered as a fundamental assumption like an axiom.

We will refer to a basic meaning of the division by zero calculus.

Recall the Cauchy integral formula for an analytic function $f(z)$; for an analytic function $f(z)$ around $z = a$ and for a smooth simple Jordan closed curve γ enclosing one time the point a, we have

$$f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a} dz.$$

Even when the function $f(z)$ has any singularity at the point a, we assume that this formula is valid as the division by zero calculus. We define the value of the function $f(z)$ at the singular point $z = a$ with the Cauchy integral.

The basic idea of the above may be considered that we can consider the value of a function by some mean value of the function.

The division by zero calculus opens a new world since Aristotele-Euclid. See the references for recent related results.

On February 16, 2019 Professor H. Okumura introduced the surprising news in Research Gate:

José Manuel Rodríguez Caballero
Added an answer
In the proof assistant Isabelle/HOL we have $x/0 = 0$ for each number x. This is advantageous in order to simplify the proofs. You can download this proof assistant here: **https://isabelle.in.tum.de/**.

J.M.R. Caballero kindly showed surprisingly several examples by the system that

$$\tan \frac{\pi}{2} = 0,$$

$$\log 0 = 0,$$

$$\exp \frac{1}{x}(x = 0) = 1,$$

and others. Furthermore, for the presentation at the annual meeting of the Japanese Mathematical Society at the Tokyo Institute of Technology:

March 17, 2019; 9:45-10:00 in Complex Analysis Session, *Horn torus models for the Riemann sphere from the viewpoint of division by zero* with [4],

he kindly sent the message:

It is nice to know that you will present your result at the Tokyo Institute of Technology. Please remember to mention Isabelle/HOL, which is a software in which x/0 = 0. This software is the result of many years of research and a millions of dollars were invested in it. If x/0 = 0 was false, all these money was for nothing. Right now, there is a team of mathematicians formalizing all the mathematics in Isabelle/HOL, where x/0 = 0 for all x, so this mathematical relation is the future of mathematics. https://www.cl.cam.ac.uk/ lp15/Grants/Alexandria/

Surprisingly enough, he sent his e-mail at 2019.3.30.18:42 as follows:

Nevertheless, you can use that $x/0 = 0$, following the rules from Isabelle/HOL and you will obtain no contradiction. Indeed, you can check this fact just downloading Isabelle/HOL: https://isabelle.in.tum.de/

and copying the following code

theory DivByZeroSatoih imports Complex Main
begin
theorem T: ‹x/0 + 2000 = 2000› for x :: complex by simp
end

Meanwhile, on ZERO, S. K. Sen and R. P. Agarwal [25] published its long history and many important properties of zero. See also R. Kaplan [5] and E. Sondheimer and A. Rogerson [27] on the very interesting books on zero and infinity. In particular, for the fundamental relation of zero and infinity, we stated the simple and fundamental relation in [24] that

The point at infinity is represented by zero; and zero is the definite complex number and the point at infinity is considered by the limiting idea
and that is represented geometrically with the horn torus model [4].

S. K. Sen and R. P. Agarwal [25] referred to the paper [6] in connection with division by zero, however, their understandings on the paper seem to be not suitable (not right) and their ideas on the division by zero seem to be traditional, indeed, they stated as the conclusion of the introduction of the book that:

"Thou shalt not divide by zero" remains valid eternally.

However, in [23] we stated simply based on the division by zero calculus that

We Can Divide the Numbers and Analytic Functions by Zero with a Natural Sense.

They stated in the book many meanings of zero over mathematics, deeply.

In this paper we would like to state the author's opinions on Däumler's horn torus model from some fundamental viewpoints.

2 Opinions

We will discuss on Däumler's horn torus model from some fundamental viewpoints.

First of all, note that in the Puha mapping and the Däumler mapping, and even in the classical stereographic mapping, we find the division by zero $1/0 = 0/0 = 0$. See [4] for the details.

2.1 What is the number system?

What are the numbers? What is the number system? For these fundamental questions, we can say that the numbers are complex numbers **C** and the number system is given by the Yamada field with the simple structure **as a field containing the division by zero.**

Nowadays, we have still many opinions on these fundamental questions, however, this subsection excludes all those opinions as in the above.

2.2 What is the natural coordinates?

We represented the complex numbers **C** by the complex plane or by the points on the Riemann sphere. On the complex plane, the point at infinity is the ideal point and for the Riemann sphere representation, we have to accept the **strong discontinuity**. From these reasons, the numbers and the numbers system should be represented by the Däumler's horn torus model that is conformally equaivalent to the extended complex plane.

2.3 What is a function?, and what is the graph of a function?

A function may be considered as a mapping from a set of numbers into a set of numbers.

The numbers are represented by Däumler's horn torus model and so, we can consider that a function, in particular, an analytic function can be considered as a mapping from Däumler's horn torus model into Däumler's horn torus model.

2.4 Absolute function theory

Following the above considerings, for analytic functions when we consider them as the mappings from Däumler's horn torus model into Däumler's horn torus model we would like to say that it is an **absolute function theory.**

For the classical theory of analytic functions, discontinuity of functions at singular points will be the serious problems and the theory will be quite different from the new mathematics, when we consider the functions on the Däumler's horn torus model. Even for analytic function theory on bounded domains, when we consider their images on Däumler's horn torus model, the results will be very interesting.

2.5 New mathematics and future mathematicians

The structure of Däumler's horn torus model is very involved and so, we will need some computer systems like MATHEMATICA and Isabelle/HOL system for our research activity. Indeed, for the analytical proof of the conformal mapping of Däumler, we had to use MATHEMATICA, already. Here, we will be able see some future of mathematicans.

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