Division by Zero and Bhāskara's Example

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April 3, 2019

Abstract: In this paper, we will introduce the division by zero calculus and Bhāskara's simple example which shows clearly both $1/0 = 0$ and $0/0 = 0$.

Key Words: Zero, division by zero, division by zero calculus, $0/0 = 1/0 = z/0 = 0$, Laurent expansion, Bhāskara's example.

2010 Mathematics Subject Classification: 30E20, 30C20, 00A05, 00A09.

1 Division by zero calculus

For the long history of division by zero, see [1, 19]. The division by zero with mysterious and long history was indeed trivial and clear as in the followings.

By the concept of the Moore-Penrose generalized solution of the fundamental equation $ax = b$, the division by zero was trivial and clear as $b/0 = 0$ in the **generalized fraction** that is defined by the generalized solution of the equation $ax = b$. Here, the generalized solution is always uniquely determined and the theory is very classical. See [5] for example.

Division by zero is trivial and clear also from the concept of repeated subtraction - H. Michiwaki.

Recall the uniqueness theorem by S. Takahasi on the division by zero. See [5, 29].

The simple field structure containing division by zero was established by M. Yamada ([8]). For a simple introduction, see H. Okumura [17].

Many applications of the division by zero to Wasan geometry were given by H. Okumura. See [11, 12, 13, 14, 15, 16] for example.

As the number system containing the division by zero, the Yamada field structure is perfect. However, for applications of the division by zero to **functions**, we need the concept of the division by zero calculus for the sake of uniquely determinations of the results and for other reasons.

For example, for the typical linear mapping

$$W = \frac{z - i}{z + i}, \tag{1.1}$$

it gives a conformal mapping on $\{\mathbf{C} \setminus \{-i\}\}$ onto $\{\mathbf{C} \setminus \{1\}\}$ in one to one and from

$$W = 1 + \frac{-2i}{z - (-i)}, \tag{1.2}$$

we see that $-i$ corresponds to 1 and so the function maps the whole $\{\mathbf{C}\}$ onto $\{\mathbf{C}\}$ in one to one.

Meanwhile, note that for

$$W = (z - i) \cdot \frac{1}{z + i}, \tag{1.3}$$

we should not enter $z = -i$ in the way

$$[(z - i)]_{z=-i} \cdot \left[\frac{1}{z + i}\right]_{z=-i} = (-2i) \cdot 0 = 0. \tag{1.4}$$

However, in many cases, the above two results will have practical meanings and so, we will need to consider many ways for the application of the division by zero and we will need to check the results obtained, in some practical viewpoints. We referred to this delicate problem with many examples in the references.

Therefore, we will introduce the division by zero calculus. For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n, \tag{1.5}$$

we **define** the identity, by the division by zero

$$f(a) = C_0. \tag{1.6}$$

Apart from the motivation, we define the division by zero calculus by (1.6). With this assumption, we can obtain many new results and new ideas. However, for this assumption we have to check the results obtained whether they are reasonable or not. By this idea, we can avoid any logical problems. – In this point, the division by zero calculus may be considered as a fundamental assumption like an axiom.

In addition, we will refer to the naturality of the division by zero calculus.

Recall the Cauchy integral formula for an analytic function $f(z)$; for an analytic function $f(z)$ around $z = a$ and for a smooth simple Jordan closed curve γ enclosing one time the point a, we have

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz.$$

Even when the function $f(z)$ has any singularity at the point a, we assume that this formula is valid as the division by zero calculus. We define the value of the function $f(z)$ at the singular point $z = a$ with the Cauchy integral.

The basic idea of the above may be considered that we can consider the value of a function by some mean value of the function.

The division by zero calculus opens a new world since Aristotele-Euclid. See, in particular, [2] and also the references for recent related results.

On February 16, 2019 Professor H. Okumura introduced the surprising news in Research Gate:

José Manuel Rodríguez Caballero
Added an answer
In the proof assistant Isabelle/HOL we have $x/0 = 0$ for each number x.

This is advantageous in order to simplify the proofs. You can download this proof assistant here: **https://isabelle.in.tum.de/**.

J.M.R. Caballero kindly showed surprisingly several examples by the system that

$$\tan\frac{\pi}{2} = 0,$$

$$\log 0 = 0,$$

$$\exp\frac{1}{x}(x = 0) = 1,$$

and others. Furthermore, for the presentation at the annual meeting of the Japanese Mathematical Society at the Tokyo Institute of Technology:

March 17, 2019; 9:45-10:00 in Complex Analysis Session, *Horn torus models for the Riemann sphere from the viewpoint of division by zero* with [2],

he kindly sent the message:

It is nice to know that you will present your result at the Tokyo Institute of Technology. Please remember to mention Isabelle/HOL, which is a software in which x/0 = 0. This software is the result of many years of research and a millions of dollars were invested in it. If x/0 = 0 was false, all these money was for nothing. Right now, there is a team of mathematicians formalizing all the mathematics in Isabelle/HOL, where x/0 = 0 for all x, so this mathematical relation is the future of mathematics. https://www.cl.cam.ac.uk/ lp15/Grants/Alexandria/

Surprisingly enough, he sent his e-mail at 2019.3.30.18:42 as follows:

Nevertheless, you can use that $x/0 = 0$, following the rules from Isabelle/HOL and you will obtain no contradiction. Indeed, you can check this fact just downloading Isabelle/HOL: https://isabelle.in.tum.de/

and copying the following code

theory DivByZeroSatoih imports Complex Main

begin

theorem T: ‹x/0 + 2000 = 2000› for x :: complex by simp

end

Meanwhile, on ZERO, S. K. Sen and R. P. Agarwal [26] published its long history and many important properties of zero. See also R. Kaplan [4] and E. Sondheimer and A. Rogerson [28] on the very interesting books on zero and infinity. In particular, for the fundamental relation of zero and infinity, we stated the simple and fundamental relation in [25] that

The point at infinity is represented by zero; and zero is the definite complex number and the point at infinity is considered by the limiting idea

and that is represented geometrically with the horn torus model [2].

S. K. Sen and R. P. Agarwal [26] referred to the paper [5] in connection with division by zero, however, their understandings on the paper seem to be not suitable (not right) and their ideas on the division by zero seem to be traditional, indeed, they stated as the conclusion of the introduction of the book that:

"Thou shalt not divide by zero" remains valid eternally.

However, in [23] we stated simply based on the division by zero calculus that

We Can Divide the Numbers and Analytic Functions by Zero with a Natural Sense.

They stated in the book many meanings of zero over mathematics, deeply.

We got the great book [3] of 608 pages and we can see some greatness of Bhāskara and it is quite famous that he considered as $1/0 = \infty$ and this result is considered even nowadays. However, we discovered that $1/0 = 0/0 = 0$ with the very natural sense, and many and many examples were given with their applications.

In this paper, from an example of Bhāskara, we will see the result $1/0 = 0/0 = 0$. **This very simple and clear example will be very attractive for general people over mathematicians** with the typical examples in [24].

2 Bhāskara's example

We will consider the circle such that its center is the origin and its radius R. We consider the point S (sun) on the circle such that $\angle SOI = \theta$; $O(0,0), I(R,0)$. For fixed $d > 0$, we consider the common point $(-L, -d)$ of two line OS and $y=-d$. Then we obtain the identity

$$L=\frac{R\cos\theta}{R\sin\theta}d,$$

([3], page 77.). That is the length of the shadow of the segment of $(0,0)-(0,-d)$ onto the line $y=-d$ of the sun S.

When we consider $\theta\to+0$ we see that, of course

$$L\to\infty.$$

Therefore, Bhāskara considered that

$$\frac{1}{0}=\infty. \tag{2.1}$$

Even nowaday, our mathematics and many people consider so.

However, for $\theta=0$, we have S=I and we can not consider any shadow on the line $y=-d$, so we should consider that $L=0$; that is

$$\frac{1}{0}=0. \tag{2.2}$$

Furthermore, for $R=0$; that is, for S=O, we see its shadow is the point $(0,-d)$ and so $L=0$ and

$$L=\frac{0\cos\theta}{0\sin\theta}d=0;$$

that is

$$\frac{0}{0}=0.$$

This example shows that the division by zero calculus is not almighty.

Note that both identities (2.1) and (2.2) are right in their senses. Depending on the interpretations of $1/0$, we obtain INFINITY and ZERO, respectively.

3 Another example

We consider a triangle ABC with $\overline{AB} = c$, $\overline{BC} = a$, $\overline{CA} = c$. Let x_i be the orthogonal projections of AB and AC to the line BC. Then we have

$$x_i = \frac{1}{2}\left\{a \mp \frac{(b+c)|b-c|}{a}\right\},$$

([3], pages 70-71.). If $b = c$, then, of course, $x_1 = x_2 = a/2$. For $a = 0$, by the division by zero, we have the reasonable value $x_1 = x_2 = 0$.

4 Remark

For the example ([3], pages 70-71.), we see that now there is no problem, because we have the relation

$$\frac{R}{j_c} = \frac{r}{R}.$$

Then, we have the right formula

$$y = r \sin \overline{\varphi}.$$

Acknowledgements

The author wishes to express his deep thanks to Professor Hiroshi Okumura and Mr. José Manuel Rodríguez Caballero for their very exciting informations.

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