A new tensor in differential geometry

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Abstract

We define here a 3-form for any differential manifold with connection.

1 The tensors in differential geometry

For a connection over a differential geometry, we can define tensors. Essentially they are obtained form the curvature tensor or the torsion if the fiber bundle is the tangent one. These two tensors are the following:

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

2 The new tensor

2.1 A lemma

The following operator is a tensor in X, Y:

$$X \nabla_Y Z - Y \nabla_X Z - \nabla_{[X,Y]} Z$$

2.2 An other lemma

The following operator is a tensor in X, Y:

$$\nabla_X YZ - \nabla_Y XZ - [X,Y]Z$$

But we can remark that the tensor is ill defined because YZ is not a vector, so that we are obliged to redefine the tensor:

$$\nabla_X [Y,Z] + \nabla_Y [Z,X] + \nabla_Z [X,Y] - [X,Y]Z - [Y,Z]X - [Z,X]Y$$

2.3 The tensor

The new tensor, which is a 3-form in X, Y, Z with values in the operators of 2 order over the functions, is the following:

$$X\nabla_Y Z - Y\nabla_X Z - \nabla_{[X,Y]}Z+$$

$$Y\nabla_Z X - Z\nabla_Y X - \nabla_{[Y,Z]}X+$$

$$Z\nabla_X Y - X\nabla_Z Y - \nabla_{[Z,X]}Y+$$

$$\nabla_X[Y,Z] + \nabla_Y[Z,X] + \nabla_Z[X,Y]-$$

$$-[X,Y]Z - [Y,Z]X - [Z,X]Y$$

References

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