Thermal Relativity, Corrections to Black-Hole Entropy, Born's Reciprocal Relativity Theory and Quantum Gravity *

Carlos Castro Perelman
Center for Theoretical Studies of Physical Systems
Clark Atlanta University, Atlanta, Georgia. 30314.
Ronin Institute, 127 Haddon Place, Montclair, N.J. 07043.
perelmanc@hotmail.com

January 2019

Abstract

Starting with a brief description of Born's Reciprocal Relativity Theory (BRRT), based on a maximal proper-force, maximal speed of light, inertial and non-inertial observers, we derive the *exact* Thermal Relativistic *corrections* to the Schwarzschild, Reissner-Nordstrom, Kerr-Newman black hole entropies, and provide a detailed analysis of the many *novel* applications and consequences to the physics of black holes, quantum gravity, minimal area, minimal mass, Yang-Mills mass gap, information paradox, arrow of time, dark matter, and dark energy. We finalize by outlining our proposal towards a Space-Time-Matter Unification program where matter can be converted into spacetime quanta, and vice versa.

Keywords : Thermal Relativity; Gravity; Black Holes; Entropy; Born Reciprocity; Phase Spaces; Maximal Acceleration; Dark Matter; Finsler Geometry. PACS : 04.50.Kd; 11.10.Kk; 14.80.-j

1 Introduction

The deep origins of the connection between Black Holes and Thermodynamics is still a mystery (to our knowledge). We shall argue that the principle of Thermal

*Dedicated to the loving memory of Juan Manuel Pombo, a Colombian gentleman and scholar who loved books more than Love itself

Relativity holds some clues. Not Long time ago, Tolman [1] argued that in accordance with the special theory of relativity all forms of energy, including heat, have inertia and hence in accordance with the equivalence principle also have weight. He found that a temperature gradient is a necessary accompaniment of thermal equilibrium in a gravitational field, in order to prevent the flow of heat from regions of higher to those of lower gravitational potential. The general result for the relation between the gravitational potential and the equilibrium temperature T_{obs} as measured by a local observer in proper coordinates was be given by the equation

$$\frac{dlnT_{obs}}{dr} = -\frac{1}{2}\frac{dlng_{tt}}{dr} \Rightarrow T_{obs}(r) \;=\; \frac{T}{\sqrt{|g_{tt}(r)|}} \tag{1.1}$$

where T is the temperature at infinity.

Whether a moving body with uniform velocity appears cooler, the same, or warmer has been the subject of debate. Planck/Einstein, Landsberg, and Ott argued, respectively, that the temperature of a moving body should be given in terms of the Lorentz dilation factor $\gamma(v) = (1 - \frac{v^2}{c^2})^{-\frac{1}{2}}$ by $T = \gamma^\alpha T'$, where the respective powers $\alpha = -1, 0, 1$ correspond to the views by Planck/Einstein, Landsberg and Ott. According to the authors [3] at the present time no temperature transformation has been agreed upon. An attempt was made by [3] based on the kinetic theory of ideal gases to find a consistent and logical form for the relativistic temperature transformation. To reach consensus, it seems necessary that firm experimental evidence needs to be obtained.

The authors [4] went further by studying the effects of the acceleration. They modified the Rindler space to include the existence of a maximal acceleration. The consequence was a change in the Unruh relation between acceleration and temperature given by

$$T_U \;=\; \frac{\hbar a}{2\pi k_B c}\;(1 - \frac{a^2}{a^2_{max}})^{-\frac{1}{2}} \tag{1.2}$$

Another interesting critical value of the temperature is the so called Hagedorn temperature $T_{Hagedorn} \sim \alpha'^{-1/2}$ given in terms of the string slope α' (inverse string tension) that arises in string's thermodynamics, above which the string partition function diverges [5].

As pointed out by [2], the idea of describing classical thermodynamics using geometric approaches has a long history [6]. Among various treatments, Weinhold [7] used the Hessian of internal energy to define a metric for thermodynamic fluctuations, Ruppeiner [8] used the Hessian of entropy for the same purpose. More recently, Quevedo [9] introduced a formalism called Geometrothermodynamics (GTD) which also introduces metric structures on the configuration space $\mathcal{E}$ of the thermodynamic equilibrium states spanned by all the extensive variables. The Quevedo metric is obtained via the pullback of the metric from the $2n + 1$-dimensional thermodynamic phase space $\mathcal{T}$ (comprised of n extensive variables, n intensive variables, and the thermodynamic potential) to the

n-dim configuration space $\mathcal{E}$. Geometrothermodynamics differs from earlier approaches in that it implements an invariance under Legendre transformations at the fundamental level. Unfortunately, one of the essence of Riemannian geometry, i.e. invariance under continuous coordinates transformations was not discussed in this picture.

Another fact that was missing is that the above authors (to my knowledge) did not realize that their constructions are particular examples of the many important applications of Finsler geometry [10], to the field of Thermodynamics, contact geometry and a vast number of many other topics [11] . Zhao [2] was able to outline the essential principles of Thermal Relativity; i.e. invariance under the group $\mathcal{G}$ of general coordinate transformations on the thermodynamic configuration space, and *introduced* a metric with a *Lorentzian* signature on the space. The line element was identified as the square of the *proper entropy*. Thus the first and second law of thermodynamics admitted an invariant formulation under general coordinate transformations, which justified the foundations for the principle of Thermal Relativity.

Jacobson [12] in his seminal work showed that Einstein's gravitational field equations can be obtained by demanding that the first law of Thermodynamics $d\mathcal{Q} = TdS$ holds at the Rindler horizon. $d\mathcal{Q}$ is the heat flux crossing the horizon (is associated with the matter stress energy tensor). The temperature T is the Unruh temperature seen by an accelerated observer, and S is the entropy of the horizon (associated with a conserved Noether charge and related to the curvature as Wald showed [13]).

The purpose of the present work is to include Born's Reciprocal Relativity into the picture of Thermal Relativity, Gravity and Black Hole Thermodynamics, unifying inertial and accelerated observers. We will derive the exact Thermal Relativistic corrections to the Schwarzschild, Reissner-Nordstrom and Kerr-Newman black hole entropies, and provide a list of the many *novel* applications and consequences to the physics of black holes, quantum gravity, minimal area, minimal mass, Yang-Mills mass gap, information paradox, arrow of time, dark matter, and dark energy. We finalize by outlining our proposal towards a Space-Time-Matter Unification program where matter can be converted into spacetime quanta, and vice versa.

2 Born's Reciprocal Relativity in Phase Space and Maximal Proper Force

The first indication that phase space should play a role in Quantum Gravity was raised by [14]. The principle of Born's reciprocal relativity [14] was proposed long ago based on the idea that coordinates and momenta should be unified on the same footing, and consequently, if there is a limiting speed (temporal derivative of the position coordinates) in Nature there should be a maximal force [15] as well, since force is the temporal derivative of the momentum. A *maximal* speed limit (speed of light) must be accompanied with a *maximal* proper force (which is also compatible with a *maximal* and *minimal* length duality).

The generalized velocity and acceleration boosts (rotations) transformations of the *flat* $8D$ Phase space, where $x^i, t, E, P^i; i = 1,2,3$ are *all* boosted (rotated) into each-other, were given by [17] based on the group $U(1,3)$ and which is the Born version of the Lorentz group $SO(1,3)$. The $U(1,3) = SU(1,3) \times U(1)$ group transformations leave invariant the symplectic 2-form $\Omega = - \, dt \wedge dp_0 + \delta_{ij} dx^i \wedge dp^j; i,j = 1,2,3$ and also the following Born-Green line interval in the *flat* $8D$ phase-space

$$(d\omega)^2 \; = \; (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 + \frac{1}{b^2} \left((dE)^2 \; - \; (dP_x)^2 \; - \; (dP_y)^2 \; - \; (dP_z)^2 \right) \tag{2.1}$$

the rotations, velocity and force (acceleration) boosts leaving invariant the symplectic 2-form and the line interval in the $8D$ phase-space are rather elaborate, see [17] for details. Born's reciprocity within the context of the conformal group $SU(2,2) \subset U(2,2)$ in $4D$, the coherent states of accelerated relativistic quantum particles, vacuum radiation, and the spontaneous breakdown of conformal symmetry was studied in detail by [20].

These transformations can be simplified drastically when the velocity and force (acceleration) boosts are both parallel to the x-direction and leave the transverse directions y, z, P_y, P_z intact. There is now a subgroup $U(1,1) = SU(1,1) \times U(1) \subset U(1,3)$ which leaves invariant the following line interval

$$(d\omega)^2 \; = \; (dt)^2 \; - \; (dx)^2 \; + \; \frac{(dE)^2 \; - \; (dP)^2}{b^2} =$$
$$(d\tau)^2 \left(1 \; + \; \frac{(dE/d\tau)^2 \; - \; (dP/d\tau)^2}{b^2} \right) = (d\tau)^2 \left(1 \; - \; \frac{F^2}{F^2_{max}} \right) \tag{2.2}$$

where one has factored out the proper time infinitesimal $(d\tau)^2 = dt^2 - dx^2$ in (2.2). The proper force interval $(dE/d\tau)^2 - (dP/d\tau)^2 = -F^2 < 0$ is "spacelike" when the proper velocity interval $(dt/d\tau)^2 - (dx/d\tau)^2 > 0$ is timelike. The analog of the Lorentz relativistic factor in eq-(2.2) involves the ratios of two proper *forces*.

If (in natural units $\hbar = c = 1$) one sets the maximal proper-force to be given by $b \equiv m_P A_{max}$, where $m_P = (1/L_P)$ is the Planck mass and $A_{max} = (1/L_p)$, L_P is the Planck length. Then $b = (1/L_P)^2$ may also be interpreted as the maximal string tension. The units of b would be of $(mass)^2$. In the most general case there are four scales of time, energy, momentum and length that can be constructed from the three constants $b, c, \hbar$ as follows

$$\lambda_t \; = \; \sqrt{\frac{\hbar}{bc}}; \;\; \lambda_l \; = \; \sqrt{\frac{\hbar \; c}{b}}; \;\; \lambda_p \; = \; \sqrt{\frac{\hbar \; b}{c}}; \;\; \lambda_e \; = \; \sqrt{\hbar \; b \; c} \tag{2.3}$$

The gravitational constant can be written as $G = \alpha_G \; c^4/b$ where α_G is a dimensionless parameter to be determined experimentally. If $\alpha_G = 1$, then the four scales in eq-(2.3) coincide with the *Planck* time, length, momentum and energy, respectively.

The $U(1,1)$ group transformation laws of the phase-space coordinates x, t, P, E which leave the interval (2.2) invariant are [17]

$$t' \;=\; t\; cosh\xi \;+\; (\frac{\xi_v\; x}{c^2} \;+\; \frac{\xi_a\; P}{b^2})\; \frac{sinh\xi}{\xi} \qquad (2.4a)$$

$$E' \;=\; E\; cosh\xi \;+\; (-\xi_a\; x \;+\; \xi_v P)\; \frac{sinh\xi}{\xi} \qquad (2.4b)$$

$$x' \;=\; x\; cosh\xi \;+\; (\xi_v\; t \;-\; \frac{\xi_a\; E}{b^2})\; \frac{sinh\xi}{\xi} \qquad (2.4c)$$

$$p' \;=\; p\; cosh\xi \;+\; (\frac{\xi_v\; E}{c^2} \;+\; \xi_a\; t)\; \frac{sinh\xi}{\xi} \qquad (2.4d)$$

ξ_v is the velocity-boost rapidity parameter and the ξ_a is the force (acceleration) boost rapidity parameter of the primed-reference frame. These parameters ξ_a, ξ_v, ξ are defined respectively in terms of the velocity $v = dx/dt$ and force $f = dP/dT$ (related to acceleration) as

$$tanh(\frac{\xi_v}{c}) = \frac{v}{c}; \quad tanh(\frac{\xi_a}{b}) \;=\; \frac{F}{F_{max}}, \;\; \xi \;=\; \sqrt{(\frac{\xi_v}{c})^2 + (\frac{\xi_a}{b})^2} \qquad (2.5)$$

One of the most salient features of the transformations in eqs-(2.4) is that under pure acceleration boosts, obtained by setting $\xi_v = 0, \xi_a \neq 0$ in (2.5), the spacetime coordinates in the new accelerated frame are now *mixed* with the energy-momentum variables.

It is straightforward to verify that the transformations (2.4) leave invariant the phase space interval $c^2(dt)^2 - (dx)^2 + ((dE)^2 - c^2(dP)^2)/b^2$ but *do not* leave separately invariant the proper time interval $(d\tau)^2 = dt^2 - dx^2$, nor the interval in energy-momentum space $\frac{1}{b^2}[(dE)^2 - c^2(dP)^2]$, like it occurs under ordinary Lorentz transformations (rotations and velocity boosts) . Only the *combination*

$$(d\omega)^2 \;=\; (d\tau)^2 \; \left(1 \;-\; \frac{F^2}{F^2_{max}} \right) \qquad (2.6)$$

is truly left invariant under force (acceleration) boosts (2.4). They also leave invariant the symplectic 2-form (phase space areas) $\Omega = -\, dt \wedge dE + dx \wedge dP$. One can verify that the transformations eqs-(2.4) are invariant under the discrete transformations

$$(t, x) \rightarrow (E, P); \quad (E, P) \rightarrow (-t, -x), \quad b \rightarrow \frac{1}{b} \qquad (2.7)$$

we argued [19] that the latter transformation $b \rightarrow \frac{1}{b}$ is a manifestation of the large/small tension T-duality symmetry in string theory. In natural units of $\hbar = c = 1$, the maximal proper force b has the same dimensions as a string tension (energy per unit length) $(mass)^2$.

More recently we have shown in [18] that relativity of locality and chronology are natural consequences of this theory, even in flat phase space. The advantage of Born's reciprocal relativity theory (BRRT) is that Lorentz invariance is preserved and there is no need to introduce Hopf algebraic deformations of the Poincare algebra, de Sitter algebra, nor noncommutative spacetimes. After a detailed study of the notion of *generalized* force, momentum and mass in phase space, we found that what one may interpret as "dark matter" in galaxies, for example, is just an effect of observing ordinary galactic matter in $different$ *accelerating* frames of reference than ours. Explicit calculations were provided that explained these novel relativistic effects due to the *accelerated* expansion of the Universe, and which may generate the present-day density parameter value $\Omega_{DM} \sim 0.25$ of dark matter. The physical origins behind the numerical coincidences in Black-Hole Cosmology were also explored. We finalized with a rigorous study of the curved geometry of (co) tangent bundles (phase space) within the formalism of Finsler geometry, and provided a short discussion on Hamilton spaces.

Because the quadratic Casimir of the Poincare algebra $P_\mu P^\mu = m^2$ is **not** the same as the quadratic Casimir of the pseudo-unitary algebra $U(1,3)$ [17], in the case of a four-dim phase space, one has then the following $U(1,1)$ quadratic Casimir

$$\mathcal{C}_2 \ = \ (\frac{t}{\lambda_t})^2 \ - \ (\frac{x}{\lambda_l})^2 \ + \ (\frac{E}{\lambda_e})^2 \ - \ (\frac{P}{\lambda_p})^2 \qquad (2.8)$$

where we explicitly re-inserted the four scales of time, energy, momentum and length of eq-(2.3) to make $\mathcal{C}_2$ dimensionless.

If the temporal and spatial displacements are represented by the energy and momentum operators $E \rightarrow \frac{\partial}{\partial t}, P \rightarrow \frac{\partial}{\partial x}$, (in units of $\hbar = c = 1$), the Born reciprocity principle dictates that the energy and momentum displacements should be represented by the time and position operators $t \rightarrow \frac{\partial}{\partial E}, x \rightarrow \frac{\partial}{\partial P}$. Therefore we shall choose to define our $U(1,1)$ quadratic Casimir to be the following

$$\mathcal{C}_2 \ = \ \mathbf{M}^2 \ = \ b^2 \ (t^2 \ - \ x^2) \ + \ E^2 \ - \ P^2, \ \ \hbar = c = 1 \qquad (2.9a)$$

and expressed in terms of the quantity $\mathbf{M} \neq m$, which has the same physical units of mass. In ordinary relativity m is the Lorentz invariant quantity given by $E^2 - P^2 = m^2$. However m is not invariant under *acceleration* boosts transformations. The true invariant, under both Lorentz *and* acceleration boosts is $\mathbf{M}$.

If one examines carefully the quadratic Casimir expression (2.9) for a point mass particle at rest at the origin $x = P = 0$, it leads to

$$\mathbf{M}^2 \ = \ b^2 \ t^2 \ + \ E^2 \qquad (2.9b)$$

and one may encounter something which seems to be *paradoxical*. If $\mathbf{M}$ is an invariant (Casimir), and E^2 coincides with the rest mass squared m^2 of the particle sitting fixed at $x = 0$, as the time t flows and flows, this means that m must decrease in time. We know that a point mass has infinite mass density and will generate a black hole whose horizon's radius is $2Gm$. Thus it will radiate away its mass according to Hawking as the time flows. But when the final mass has reached the value of zero, isn't the time supposed to continue to flow, or does it *stop* from flowing ?

One immediately would argue that Quantum Mechanics should come to our rescue by invoking the uncertainty principle. One cannot have a point mass to have a precise value of the position $x = 0$, and momentum $P = 0$ simultaneously. The point mass should be smeared, fuzzy, and/or its spacetime location should be fuzzy, consistent with the ideas of Noncommutative geometry. We shall return to these issues below and explain why due to the unification/equivalence of space-time and mass-energy-momentum, time in BRRT should cease to flow at the moment t_{final} when $m \rightarrow 0$ such that $\mathbf{M} \rightarrow bt_{final}$. To be more precise, we will show that Hawking evaporation *stops* when the final mass is of the order of the Planck mass M_P, so that $bt_{final} \sim \sqrt{\mathbf{M}^2 - M_P^2}$.

One may notice also that eq-(2.9a) does not yield the quadratic Casimir of the Poincare algebra $P_\mu P^\mu = m^2$ in the $b \rightarrow \infty$ limit unless $t^2 - x^2$ is constrained to zero. The Galilean algebra (plus rotations) is a Inonu-Wigner contraction of the Poincare algebra in the $c \rightarrow \infty$ limit. However, the Poincare algebra is *not* the contraction of the $U(1,3)$ algebra in the $b \rightarrow \infty$ limit. For this reason we should not expect that Special Relativity is recovered in the $b \rightarrow \infty$ limit.

Given, $d\omega = d\tau \sqrt{1 - \frac{F^2}{b^2}}$, and $\mathbf{M}$, the *generalized* momentum in flat phase space is defined as

$$\mathcal{P}^M \equiv \mathbf{M} \frac{dZ^M}{d\omega} = \mathbf{M} \left(\frac{dt}{d\omega}, \frac{dx}{d\omega}, \frac{dE}{d\omega}, \frac{dP}{d\omega} \right) \tag{2.10}$$

note that we have not explicitly inserted b^{-1} factors into the definition of $\mathcal{P}^M$, thus not all quantities in $\mathcal{P}^M$ have the same units. We shall re-insert these factors when we evaluate the norm

$$\mathcal{P}_M \mathcal{P}^M = \mathbf{M}^2 \left((\frac{dt}{d\omega})^2 - (\frac{dx}{d\omega})^2 + \frac{(\frac{dE}{d\omega})^2 - (\frac{dP}{d\omega})^2}{b^2} \right) = \mathbf{M}^2 \frac{(d\omega)^2}{(d\omega)^2} = \mathbf{M}^2 \tag{2.11}$$

recovering now the generalized dispersion relation in flat phase space and which is invariant under velocity and force/acceleration boosts transformations (2.4).

As stated earlier, what is an invariant is the phase space interval

$$b^2 (d\omega)^2 = b^2 \left((dT)^2 - (dX)^2 \right) + (dE)^2 - (dP)^2 = b^2 (d\tau)^2 - (d\mathcal{M})^2 \tag{2.12}$$

where the (spacelike) mass infinitesimal displacement is defined by

$$-(d\mathcal{M})^2 \equiv (dE)^2 - (dP)^2 \leq 0 \Rightarrow \int d\mathcal{M} = \mathcal{M} = \int \sqrt{(dP)^2 - (dE)^2} \tag{2.13}$$

$\mathcal{M}$ is given by the line integral (2.13) in energy-momentum space.

The components of the *generalized* proper force in (flat) phase space are now given by

$$\mathcal{F}^M \equiv \mathbf{M} \frac{d^2 Z^M}{d\omega^2} = \mathbf{M} \left(\frac{d^2 t}{d\omega^2}, \frac{d^2 x}{d\omega^2}, \frac{d^2 E}{d\omega^2}, \frac{d^2 P}{d\omega^2} \right) \tag{2.14}$$

In the particular case of a massive particle with a *uniform* linear acceleration (Rindler observer) in spacetime, the *generalized* proper force in (flat) phase space turned out to be be [18]

$$\mathbf{F} = \frac{\mathbf{M} g}{\sqrt{1 - \frac{F^2}{b^2}}} \leftrightarrow P = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2.16}$$

where $F = mg$ is the proper force experienced by the particle in spacetime, and g is its uniform proper acceleration in spacetime. On the right hand side of (2.16) we have the Special Relativistic correspondence with the momentum, and expressed in terms of the *rest* mass m and the velocity v. Furthermore, given the bound on the proper force by the maximal value $F = mg \leq b$, one could write $b = mA$, where $A = b/m$ is the maximal acceleration that a particle of mass m can sustain. The more massive the particle is the lower A is, and vice versa. Hence the left hand side eq-(2.16) can be also rewritten in terms of A as

$$\mathbf{F} = \frac{\mathbf{M} g}{\sqrt{1 - \frac{g^2}{A^2}}} \leftrightarrow P = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2.17}$$

such that the correspondence with the right hand side is more evident. In the case of a ($\mathbf{M} = 0$) massless particle (photon), the trajectory in phase-space defined by $t = x;\ E = P,\ d\omega = d\tau = 0,\ \omega = \tau = 0$, and described in terms of an affine parameter $\lambda = \omega/\mathbf{M}$, yields *identically zero* $\mathcal{F}_M \mathcal{F}^M = \Omega^2 = 0,\ F_\mu F^\mu = F^2 = 0$ values. Hence, a photon describes a *null* path, and it experiences a zero proper force magnitude, both in spacetime *and* in phase-space. Whereas a massive particle subjected to the maximal proper force b in *spacetime* will experience an *infinite* generalized proper force squared in phase-space $\Omega^2 = -\infty$, while having a *null* interval in phase-space $(d\omega)^2 = (d\tau)^2(1 - F^2/b^2) = 0$, but a *timelike* interval in spacetime $(d\tau)^2 > 0$.

In what follows we shall adopt the units $\hbar = c = k_B = 1$, and explain why the relativistic factor $(1 - \frac{a^2}{a^2_{max}})^{-\frac{1}{2}}$ should *not* appear in eq-(1.2) but $(1 - \frac{T^2}{T_P^2})^{-\frac{1}{2}}$ instead, where T_P is the Planck temperature. Having presented a very brief summary of BRRT, let us rewrite eq-(1.2) in the form

$$T'_U = \frac{1}{2\pi} \frac{g}{\sqrt{1 - \frac{g^2}{A^2}}} = \frac{T_U}{\sqrt{1 - (\frac{T_U}{T_U^{max}})^2}}, \quad T_U = \frac{g}{2\pi}, \quad T_U^{max} = \frac{A}{2\pi} \tag{2.18}$$

The problem is that the value of T_U^{max} is *not* universal, it *varies* from particle to particle accordingly to the values of their masses $m_1, m_2, m_3, \cdots$. The more massive the particle is, the lower is its corresponding maximal acceleration A, and the lower is the value of its corresponding maximal Unruh temperature T_U^{max}.

And vice versa, the less massive the particle is, the higher is its corresponding maximal acceleration A, and the higher is the value of its corresponding maximal Unruh temperature T_U^{max}.

Because we wish to assign a "universal" cutoff value for T_U^{max}, to all particles, independent of their masses, we shall set the maximal temperature equal to the Planck temperature T_P, consistent with the Thermal Relativity postulate of Zhao [2]. He presented a beautiful explanation of how the group $\mathcal{G}$ of general coordinate transformations on the thermodynamic configuration space is spanned by all the extensive variables and ensures the invariance of the first law of thermodynamics. He showed how can introduce a metric with a *Lorentzian* signature in the thermodynamic space, and whose corresponding line element is also invariant under the action of G. This line element was identified as the square of the *proper* entropy **s**, which must not be confused with the ordinary entropy S. Thus the second law of thermodynamics was also formulated in an invariant fashion as $(d\mathbf{s})^2 \geq 0$, and this laid down the foundation for the principle of Thermal Relativity. The additional ingredient we are adding in this work is the bridge between Zhao's Thermal Relativity and Born's Reciprocal Relativity Theory.

In our case above, one may implement Zhao's formulation [2] of Thermal Relativity in the flat analog of Minkowski space as

$$(d\mathbf{s})^2 \ = \ (T_P dS)^2 \ - \ (dM)^2 \ \leftrightarrow \ (d\tau)^2 \ = \ (cdt)^2 \ - \ (dx)^2 \qquad (2.19)$$

The maximal Planck temperature T_P plays the role of the speed of light, and **s** is the so-called *proper entropy* which is invariant under the thermodynamical version of Lorentz transformations [2]. Note the $\mathbf{s} \leftrightarrow \tau$ correspondence. Thus the flow of the proper entropy **s** is consistent with the arrow of time.

The left hand side of (2.19) yields, after recurring to the first law of Thermodynamics $TdS = dM \Rightarrow T = \frac{dM}{dS}$,

$$(d\mathbf{s})^2 \ = \ (T_P dS)^2 \left(1 \ - \ \frac{T^2}{T_P^2} \right) \ \Rightarrow \ (d\mathbf{s}) \ = \ (T_P \ dS) \sqrt{\left(1 \ - \ \frac{T^2}{T_P^2} \right)} \ =$$

$$T_P \ (\frac{dM}{T}) \sqrt{\left(1 \ - \ \frac{T^2}{T_P^2} \right)} \ \Rightarrow \ dM \ = \ \frac{T}{T_P} \ \frac{1}{\sqrt{1 - \frac{T^2}{T_P^2}}} \ d\mathbf{s} \qquad (2.20)$$

Eq-(2.20) is the one we shall use to *derive* the thermal relativistic corrections to the Black Hole Entropy.

However, we may proceed even further, by finding the BRRT *corrections* to the relation (2.20) once we incorporate the Thermodynamic **phase space** into the picture. The analog of the Thermodynamic Space/Spacetime correspondence given by eq-(2.19) in the Thermodynamic **phase space** case (thermodynamic phase space/cotangent spacetime correspondence) is

$$(d\mathbf{S})^2 \;=\; (d\mathbf{s})^2 \left(1 \;-\; \frac{\mathcal{F}^2}{\mathcal{F}^2_{max}} \right) \;\leftrightarrow\; (d\omega)^2 \;=\; (d\tau)^2 \left(1 \;-\; \frac{F^2}{F^2_{max}} \right) \quad (2.21)$$

where $\mathcal{F}, \mathcal{F}_{max}$ are the generalized "thermodynamic force" (maximal force) in the thermodynamic space. It is the quantity $\mathbf{S}$ appearing in the left hand side of eq-(2.21) that must be used in conjunction with $\mathbf{M}, \mathbf{T}$ in the following version of the first law

$$d\mathbf{M} \;=\; \mathbf{T}\; d\mathbf{S} \quad (2.22)$$

where $\mathbf{M}$, $\mathbf{S}$, $\mathbf{T}$ are the proper mass, entropy and temperature invariants in the Thermodynamic **phase space** within the context of Born's Reciprocal Relativity Theory ; i.e. invariant under the symmetry transformations (2.4). We shall leave the physical implications of eqs-(2.21, 2.22) behind and just *focus* for now on the key eq-(2.20).

If one were to write a *modified* Unruh temperature T'_U (in terms of the original T_U one) as

$$T'_U \;=\; \frac{T_U}{\sqrt{1 - \frac{T_U^2}{T^2_{cutoff}}}} \;=\; \frac{T_U}{\sqrt{1 - \frac{T_U^2}{T_P^2}}} \;\leq\; T_P \;\Rightarrow\; T_U < T_P \quad (2.23)$$

the above inequality relation would imply that T_U cannot ever attain the maximal value of T_P, and consequently, particles close to the Planck mass cannot ever attain their maximal acceleration $A \sim \frac{1}{L_P}$.

This is the fundamental reason why we propose instead eq-(2.20) so that the numerical factor $(1 - \frac{T^2}{T_P^2})^{-\frac{1}{2}}$ is the *thermal dilation* factor γ_T analog of the Lorentz dilation factor $(1 - \frac{v^2}{c^2})^{-\frac{1}{2}}$, and such that

$$1 \;\leq\; \gamma_T \;\equiv\; (1 - \frac{T_U^2}{T_P^2})^{-\frac{1}{2}} \;\leq\; \infty, \;\; 0 \;\leq\; T_U \;\leq\; T_P \quad (2.24)$$

Given the thermal dilation factor one can always define an "effective" temperature by

$$T_{eff} \;=\; \frac{T}{\sqrt{1 - \frac{T^2}{T_P^2}}} \quad (2.25)$$

such that eq-(2.20) $d\mathbf{M} = \gamma(T) T (d\mathbf{s}/T_P)$ becomes then the thermal relativistic analog of the Energy-Momentum relations $E = m_o c^2 (1 - \frac{v^2}{c^2})^{-\frac{1}{2}}$, $\vec{p} = m_o \vec{v} (1 - \frac{v^2}{c^2})^{-\frac{1}{2}}$ in Special Relativity, in terms of the rest mass m_o, velocity v, and maximal speed of light c. This line of reasoning behind eq-(2.20) is what leads to the notion of a "Thermal Relativity Theory", in agreement with [2], and which must not be confused with other notions of "thermal Relativity", "thermal gravitation" in the past.

In the rest of this work, because we will be referring to black holes, by the thermal-dilation factor γ_T one means

$$\gamma_T \equiv \frac{1}{\sqrt{1 - \frac{T_H^2}{T_P^2}}} \qquad (2.26)$$

where T_H is the Hawking-Unruh temperature for black holes. For a Schwarzschild black hole it is given by

$$T_H = \frac{1}{2\pi} a = \frac{1}{4\pi} | (\frac{\partial g_{tt}}{\partial r}) |_{r=r_h=2GM} = \frac{1}{8\pi GM} \qquad (2.27)$$

To sum up, by renaming $\tilde{S} \equiv (\mathbf{s}/T_P)$, in terms of the *proper* entropy $\mathbf{s}$, the first law of black hole thermal-relativity dynamics $dM = \gamma(T_H) T_H d\tilde{S}$ yields the *corrected* entropy

$$\int_{\tilde{S}_o}^{\tilde{S}} d\tilde{S} = \tilde{S} - \tilde{S}_o = \int_{M_o}^{M} \frac{dM}{\gamma(T_H) T_H} = \int_{M_o}^{M} dM \, \frac{\sqrt{1 - (T_H^2/T_P^2)}}{T_H} \qquad (2.28)$$

inserting $T_H(M) = (8\pi GM)^{-1}$ into eq-(2.28) gives, after setting $(T_P)^{-2} = (M_P)^{-2} = L_P^2 = G$, the following integral

$$\tilde{S} - \tilde{S}_o = \int_{M_o}^{M} dM \, (8\pi GM) \sqrt{1 - \frac{G}{(8\pi GM)^2}} = \int_{M_o}^{M} dM \, \sqrt{(8\pi GM)^2 - G} \qquad (2.29)$$

The indefinite integral

$$\int dx \, \sqrt{a^2 x^2 - b} = \frac{ax \, \sqrt{a^2 x^2 - b}}{2a} - \frac{b}{2a} \, ln \Big(a \, [\sqrt{a^2 x^2 - b} + ax] \Big) \qquad (2.30)$$

permits to evaluate the definite integral in the right hand side of (2.28) between the *upper* limit M, and a *lower* limit M_o defined by $(8\pi GM_o)^2 - G = 0$, giving

$$\tilde{S} - \tilde{S}_o = \frac{A}{4G} \sqrt{1 - \frac{1}{16\pi} (\frac{A}{4G})^{-1}} - \frac{1}{16\pi} \, ln \left(4\sqrt{\pi} \, (\frac{A}{4G})^{\frac{1}{2}} \, [\, 1 + \sqrt{1 - \frac{1}{16\pi} (\frac{A}{4G})^{-1}} \,] \right) \qquad (2.31)$$

after using the relations for the ordinary entropy in the Schwarzschild black hole

$$S = \frac{A}{4G} = 4\pi GM^2 \Rightarrow M = (\frac{A}{16\pi G^2})^{\frac{1}{2}} \qquad (2.32)$$

and $(8\pi GM_o)^2 = G \Rightarrow 8\pi GM_o = \sqrt{G}$. The lower limit M_o of integration is required in eq-(2.28) to ensure the terms inside the square root are positive definite and the integral is real-valued.

Due to the relations

$$A_o \;=\; 4\pi(2GM_o)^2 \;=\; 16\pi G^2 M_o^2, \;\; \frac{A_o}{4G} \;=\; \frac{1}{16\pi}, \;\; M_o^2 \;=\; \frac{1}{64\pi^2 G} \Rightarrow M_o = \frac{M_P}{8\pi} \tag{2.33}$$

the *minimum* value of $M_o = (M_P/8\pi)$ corresponds to the *minimum* value of the horizon area A_o given by $\frac{A_o}{4G} = \frac{1}{16\pi} \Rightarrow A_o = \frac{G}{4\pi} = \frac{L_P^2}{4\pi}$. The constant $\tilde{S}_o$ is the corrected entropy associated with the minimum area (mass) and is found to be zero

$$\tilde{S}_o \;\equiv\; \tilde{S}(A = A_o) \;=\; ln[\; 4\sqrt{\pi}\; (\frac{A_o}{4G})^{\frac{1}{2}} \;] \;=\; ln[1] = 0 \tag{2.34}$$

Because the corrected entropy corresponding to a *zero* area leads to an imaginary expression in the right hand side of eq-(2.31)

$$\tilde{S}(A = 0) \;-\; \tilde{S}_o \;=\; -\,\frac{ln(i)}{16\pi} \;=\; -\;\frac{ln(e^{i\pi/2})}{16\pi} \;=\; -\;\frac{i}{32\pi} \tag{2.35}$$

this is the reason why one must have a *minimal* non-zero horizon area A_o, and whose corresponding corrected entropy is $\tilde{S}_o = 0$. As a reminder, by *corrected* entropy $\tilde{S}$ we mean the *proper* entropy **s** divided by T_P. It is known that the third law of thermodynamics is not obeyed for black holes. For instance, the *extremal* massive Reissner-Nordstrom black hole, obeying $GM = Q$, and whose inner and outer horizons coincide $r_+ = r_- = GM$, has *zero* Hawking temperature but *nonzero* entropy πGM^2.

The most salient feature of the corrected black hole entropy expression in eq-(2.31) is that it is an *exact analytical* expression. Having $\tilde{S}_o = 0$, one may expand the first term of eq-(2.31) in a Taylor series giving

$$\tilde{S} \;=\; \frac{A}{4G} \;-\; \frac{1}{32\pi} \;-\; \frac{1}{8}(\frac{1}{16\pi})^2(\frac{A}{4G})^{-1} \;-\; \frac{1}{16}(\frac{1}{16\pi})^3(\frac{A}{4G})^{-2} \;+\; \mathcal{O}((\frac{A}{4G})^{-n})$$

$$-\frac{1}{16\pi} ln(4\sqrt{\pi}) \;-\; \frac{1}{32\pi}\; ln(\frac{A}{4G}) \;-\; \frac{1}{16\pi}\; ln \left(\; 1 \;+\; \sqrt{1 - \frac{1}{16\pi}(\frac{A}{4G})^{-1}} \;\right) \tag{2.36}$$

the *logarithmic* and power series corrections to the black hole entropy. In the *very large* A limit (compared to the Planck area) one has the following leading and sub-leading terms

$$\tilde{S} \sim \frac{A}{4G} \;-\; \frac{1}{32\pi}\; ln(\frac{A}{4G}) \;-\; \left(\; \frac{1}{32\pi} \;+\; \frac{1}{16\pi} ln(4\sqrt{\pi}) \;+\; \frac{1}{16\pi}\; ln(2) \;\right) \tag{2.37}$$

Logarithmic corrections to the black hole entropy have appeared before but they were obtained by completely different methods than the ones based on Thermal Relativity.

The above findings can be generalized to other black holes like the Reissner-Nordstrom and Kerr-Newman. In the latter case, following the same arguments as before pertaining the thermodynamic space/spacetime correspondence, the proper entropy infinitesimal displacement is

$$(d\mathbf{s})^2 \ = \ (T_P dS)^2 \ - \ (dM)^2 \ - \ (\Omega_{max} dJ)^2 \ - \ (\Theta_{Max} dQ)^2 \tag{2.38}$$

where in addition to the maximal temperature T_P, one must introduce the maximal angular velocity Ω_{max}, and maximal electrostatic potential Θ_{max} in order to match physical units. The latter two quantities are defined in units of $\hbar = c = k_B = 1$, and electric charge $|e| = 1$, by $\Omega_{max} = \Theta_{max} = M_P$.

The first law is in this case $dM = TdS + \Omega dJ + \Theta dQ$. Holding J, Q fixed $\Rightarrow dJ = dQ = 0$, one will end up with a similar expression as before

$$dM \ = \ \gamma(T) \ T \ \frac{d\mathbf{s}}{T_P} \ = \ \gamma(T) \ T \ d\tilde{S} \tag{2.39}$$

where $\gamma(T)$ is the corresponding thermal dilation factor associated with the Hawking temperature $T_H(M, Q, J)$ for the Kerr-Newman black hole written in terms of the inner r_- and outer horizon radius r_+ as

$$T_H(M, Q, \frac{J}{M}) \ = \ \frac{1}{2\pi} \ \frac{\sqrt{(GM)^2 - Q^2 - (\frac{J}{M})^2}}{2(GM)^2 \ - \ Q^2 \ + \ 2GM \ \sqrt{(GM)^2 - Q^2 - (\frac{J}{M})^2}} \ =$$

$$\frac{1}{4\pi} \frac{r_+ \ - \ r_-}{r_+^2 \ + \ (\frac{J}{M})^2}, \quad r_\pm \ = \ GM \pm \sqrt{(GM)^2 - Q^2 - (\frac{J}{M})^2} \tag{2.40}$$

Holding J, Q *fixed* in eq-(2.39) $dM = \gamma(T) T d\tilde{S}$, the Thermal Relativity corrections to the Kerr-Newman black hole entropy are obtained from the integral,

$$\tilde{S} - \tilde{S}_o \ = \ \int_{M_o}^{M} \ dM \ \frac{\sqrt{1 \ - \ \frac{T_H^2(M,Q,\frac{J}{M})}{T_P^2}}}{T_H(M, Q, \frac{J}{M})} \tag{2.41}$$

The above and much more complicated integral yields the sought-after *corrections* $\tilde{S}$ to the original expression for Kerr-Newman black hole entropy

$$S_{KN} \ = \ \frac{A}{4G} \ = \ \frac{4\pi[r_+^2 + (\frac{J}{M})^2]}{4G} \tag{2.42}$$

this value of entropy S_{KN} must *not* be confused with the statistical entropy of a Bose, Fermi field near the black hole horizon, see [23] and references therein. Setting $J = 0$ in the integral (2.41) furnishes the corrections to the Reissner-Nordstrom black hole entropy. These results provided by the integral are exact.

Other relevant physical findings pertaining Thermal Relativity are :

(i) Within the Thermal Relativity proposal advocated in this work we find that that the *corrected* $\tilde{S}_o$ entropy (proper entropy $\mathbf{s}_o$ divided by T_P) associated with the *nonzero* minimal area is *zero*.

(ii) The Thermal Relativity dilation factor $\gamma_T = (1 - (\frac{T_H}{T_P})^2)^{-1/2}$ yields an infinite value when $T_H \to T_P$ so that $M \to M_o$ reaches its *minimum* nonzero value of $M_o = (M_P/8\pi)$. At that stage the Hawking evaporation *stops* leaving a black hole *remnant* of the order of the Planck mass[1]

These results should be contrasted to the model of [22], inspired by the minimum length postulate, where the point-mass is smeared into a Gaussian mass density distribution leading to a de-Sitter core geometry very close to $r = 0$. As the black hole evaporates the temperature reaches a maximum point, after which it begins to decrease to a zero (and negative temperature) value. It was conjectured that Hawking evaporation stops at the zero temperature point. Infinite derivative gravity is also another approach to regularize point-mass distributions, turning delta functions into Gaussians, and eliminating singularities, see [25] for a vast list of references. Infinite derivatives also occur when one expands fractional/fractal derivates into an infinite power series of ordinary derivatives. This is the so-called fractional derivatives regularization method.

(iii) In thermal field theory the Euclidean temporal period is $t = \frac{1}{T}$. Therefore, having learned that the dilation-thermal factor becomes infinite when the black holes attains its minimum mass M_o, the evaporation time it takes from a black hole of mass $M > M_o$ to reach the minimal mass $M_o = (M_P/8\pi)$ is given by

$$\Delta t \;=\; t(M) - t(M_o) \;=\; \frac{1}{\gamma_T(M) T_H(M)} - \frac{1}{\gamma_T(M_o) T_H(M_o)} \;=\; \frac{\sqrt{1 - \frac{T_H^2(M)}{T_P^2}}}{T_H(M)} =$$

$$8\pi GM \sqrt{1 - \frac{L_P^2}{(8\pi GM)^2}}, \quad G \;=\; L_P^2 \tag{2.40}$$

Therefore, the black hole evaporation times, for masses closer to the minimal mass M_o, are much *faster* than in the ordinary case. If the cutoff temperature were $T_{cut} = \infty$, instead of the Planck temperature, in this thermal nonrelativistic limit we would recover the standard relation for the evaporation time for a black hole of mass M to reach the now zero minimal mass $M_o = 0$, and given by the Hawking time $t_H(M) = \frac{1}{T_H(M)} = 8\pi GM = 8\pi(M/M_p)t_P$.

It is at this stage when we can explore deeper the principle of relativity. Inserting back the physical constants one has $T_P = \frac{M_P c^2}{k_B}$. If the Planck scale were zero, this would mean that the Planck mass M_P would be ∞, consequently T_P would be also ∞, and the thermal relativistic effects would be absent, in the same vein that taking $c \to \infty$ leads to a Galilean non relativistic theory. There

[1] In ordinary Special Relativity it takes an infinite energy to accelerate a massive particle to the speed of light. The rest mass is finite but the mass in the moving frame is infinite. Similar arguments follow in Thermal Relativity

is another way in which T_P can be ∞ without M_P being ∞ (nor L_P being zero), and this would happen if the Boltzman constant k_B were set to zero. The fact that $k_B \neq 0$ is the reason why $T_P \neq \infty$ and one could postulate a Thermal Relativity Theory[2].

(iv) Concerning Dark Matter and Dark Energy, one can envision a universe populated by mini-black holes whose effective temperature is $T_{eff} = \gamma_T T_H$. The idea that primordial black holes might be a hypothetical source of dark matter/energy is *not* new. What is *novel* here is the very large enhancement effects resulting from the very large thermal relativistic dilation factors γ_T, for very small masses, close to the minimal mass M_o. Since their thermal dilation factors γ_T are very large, their contribution to the effective energy/mass of the universe will be very large. As they evaporate by shedding off their mass down to the minimal mass M_o, their *enhanced* radiation due to the very large effective temperature $T_{eff} = \gamma_T T_H$, and much $faster$ evaporation times, will yield a very large contribution to the energy of the universe.

(v) Given $\beta_{eff} = (\gamma(T)T)^{-1}$, the thermal relativistic corrections to the Bose-Einstein density distribution is $\tilde{\rho} = (e^{\beta_{eff}E} - 1)^{-1}$. As $T \rightarrow T_P, \gamma_T \rightarrow \infty, \beta_{eff} \rightarrow 0, \tilde{\rho} \rightarrow \infty$. The divergence of $\tilde{\rho}$ at the Planck temperature should signal a (spacetime) phase transition, like turning a smooth spacetime into a fractal one. The fact that $\tilde{\rho}$ *deviates* from a purely thermal distribution due to the thermal relativistic corrections will have an important impact to the resolution of the black hole information paradox.

(vi) Role in Quantum Gravity. The presence of a minimal area $A_o = L_P^2/4\pi$, zero minimal proper entropy, minimal mass $M_o = M_P/8\pi$, maximal temperature T_P, phase transition at Planck scales, arrow of time, within the context of Thermal Relativity Theory, should have profound consequences for Quantum Gravity[3]. Based on the gravity/gauge correspondence one should ask if the presence of the minimal mass is related to the mass gap problem in Yang-Mills.

The questions to be answered, among many, are what are the effects that Thermal Relativity has on Weinberg's Asymptotic Safety program in Quantum Gravity [26]; Nottale's Scale Relativity Theory [27] in fractal spacetimes; Loop Quantum Gravity, String Theory (M, F theory) ; Doubly Special Relativity; Noncommutative/Nonassociative Geometry; Finsler Geometry; Quantum Information Theory; Complexity; Enthropic gravity; Emergence; Holography; Gravity/Gauge (AdS/CFT) correspondence; Gravity/Fluid correspondence; Cosmology [28], $\cdots$

Due to the *mixing* of spacetime coordinates with energy-momenta in BRRT, under the transformations (2.4), we propose a Space-Time-Matter $unification$ program, where spacetime quanta (atoms of spacetime) comprised of closed

[2]The idea of Total Relativity, including Topological Relativity, has been advocated by some in the past. For instance, by David Finkelstein, private communication

[3]There is a growing consensus among some that gravity might not be needed to be quantized in order to have a consistent coupling of classical gravity with quantum field theory (back reaction of quantum matter on classical spacetime). This requires modifications, extensions of Quantum Mechanics, in what is called the post-quantum theory of classical gravity[24]

strings (loops) of minimal area (of the order of the Planck area) can be converted into matter (mass), and vice versa, where matter (mass) can be converted into spacetime quanta. Since there is no *local* notion of mass in general relativity, we may view the mass of the black hole as *delocalized* and spread across the horizon boundary; i.e. the whole conversion process of mass into spacetime quanta, and vice versa, is fully encoded in the boundary. Roughly speaking, we end by asking : is the delocalized black hole mass a condensate of spacetime quanta and the real source of black hole entropy via entanglement ?

In [29] we showed that the $4D$ *Euclideanized* Einstein-Hilbert gravitational action, associated to a point-mass delta function source and generating a Ricci scalar curvature $R = 4GM\delta(r)/r^2$, was exactly equal to the black hole entropy. The result is also valid in higher dimensions. This is a clear indication that mass is the source of the black hole entropy. The next project is to search for the $modified$ (Euclidean) gravitational action which corresponds to the corrected (proper) entropy found in eq-(2.31). Is it an infinite derivative nonlocal gravitational action ? The work by Vacaru [30] on Ricci flows based on the Perelman entropy is also a vey highly appealing project within the context of Thermal Relativity. We have only discussed systems in thermal equilibrium, it is essential to analyze Thermal Relativity within the context of non-equilibrium thermodynamics and what role does chaos and universality play, if any.

Acknowledgements

We thank M. Bowers for very kind assistance and to Sergiu Vacaru for many discussions on Finsler Geometry.

References

[1] R. C. Tolman, "On the Weight of Heat and Thermal Equilibrium in General Relativity", Phys. Rev. **35**, (1930) 904.

[2] L. Zhao Communications in Theoretical Physics, Vo **56**, Issue 6, (2011) 1052-1056.

[3] M. Khaleghy and F. Qassemi, "Relativistic Temperature Transformation Revisited, One hundred years after Relativity Theory" arXiv : physics/0506214.

[4] E. Benedetto and A. Feoli, "Unruh temperature with maximal acceleration" Mod. Phys. Lett **A 30** (2015) 1550075

A. Feoli, Maximal Acceleration or Maximal Accelerations ? arXiv : gr-qc/0210038.

[5] R. Parentani and R. Potting Phys. Rev. Lett. **63** (1989) 1945.

[6] J. Willard Gibbs, H.A. Bumstead, W.R. Longley, R.G. Van Name, *The Collected Works of J. Willard Gibbs*, (Longmans, Green and Co., Yew York (1928)) .

[7] F. Weinhold, "Metric Geometry of Equilibrium Thermodynamics" I, II, III, IV, V, J. Chem. Phys. **63** (1975) 2479; 2484; 2488; 2496; 65 (1976) 558.

[8] G. Ruppeiner, Phys. Rev. **A 20** (1979) 1608.

G. Ruppeiner, Rev. Mod. Phys. **67** (1995) 605; Erratum **68** (1996) 313.

[9] H. Quevedo, J. Math. Phys. **48** (2007) 013506

H. Quevedo, A. Sanchez, and A.Vazquez, "Invariant Geometry of the Ideal Gas" , arXiv: math-ph/0811.0222.

[10] S. Vacaru, "Stochastic processes and thermodynamics on curved spaces", Ann. Phys. (Leipzig) **9** (Special Issue), 175176 (2000).

[11] S. I. Vacaru, "On axiomatic formulation of gravity and matter field theories with MDRs and Finsler-Lagrange-Hamilton geometry on (co)tangent Lorentz bundles" arXiv : 1801.06444.

L. Bubuianu and Sergiu I. Vacaru, " Axiomatic formulations of modified gravity theories with nonlinear dispersion relations and Finsler-Lagrange-Hamilton geometry" Eur. Phys. J. **C 78** (2018) 969.

[12] T. Jacobson, Phys. Rev. Letts **75** (1995) 1260.

[13] R. Wald, "Black Hole Entropy is the Noether charge", Phys. Rev **D 48** (1993) R3427.

T. Padmanabhan, arXiv : 0912.3165.

[14] M. Born, "A suggestion for unifying quantum theory and relativity", *Proc. Royal Society* **A 165** (1938) 291.

M. Born, "Reciprocity Theory of Elementary Particles", *Rev. Mod. Physics* **21** (1949) 463.

[15] E.R. Cainiello, "Is there a Maximal Acceleration", Il. Nuovo Cim., **32** (1981) 65.

H. Brandt, "Finslerian Fields in the Spacetime Tangent Bundle" *Chaos, Solitons and Fractals* **10** (2-3) (1999) 267.

H. Brandt, "Maximal proper acceleration relative to the vacuum" Lett. Nuovo Cimento **38** (1983) 522.

M. Toller, "The geometry of maximal acceleration" Int. J. Theor. Phys **29** (1990) 963.

[16] C. Castro, "On Born's Deformed Reciprocal Complex Gravitational Theory and Noncommutative Gravity " Phys Letts **B 668** (2008) 442.

C. Castro, "Some consequences of Born's Reciprocal Relativity in Phase Spaces" *Foundations of Physics* **35**, no.6 (2005) 971.

[17] S. Low, " $U(3,1)$ Transformations with Invariant Symplectic and Orthogonal Metrics" , *Il Nuovo Cimento* **B 108** (1993) 841.

S. Low, "Canonically Relativistic Quantum Mechanics: Casimir Field Equations of the Quaplectic Group" arXiv:math-ph/0404077.

S. Low, "Representations of the canonical group, (the semi-direct product of the unitary and Weyl-Heisenberg groups), acting as a dynamical group on noncommutative extended phase space", J. Phys. **A** : Math. Gen., **35** (2002) 5711.

S. Low, "Canonically Relativistic Quantum Mechanics: Representations of the Unitary Semi-direct Heisenberg Group, $U(1,3) \times_s H(1,3)$" *J. Math. Phys* **38** (1997) 2197.

J. Govaerts, P. Jarvis, S. Morgan and G. Low, "World-line Quantisation of a Reciprocally Invariant System" J.Phys. **A40** (2007) 12095-12112.

[18] C. Castro, "Is Dark Matter and Black-Hole Cosmology an Effect of Born's Reciprocal Relativity Theory ?, " Canadian Journal of Physics, published online 29 May, 2018.

[19] C. Castro, "On Dual Phase Space Relativity, the Machian Principle and Modified Newtonian Mechanics" Progress in Physics **vol 1** (April 2005) 20-30.

[20] M. Calixto, E. Perez-Romero, "Conformal Spinning Quantum Particles in Complex Minkowski Space as Constrained Nonlinear Sigma Models in $U(2,2)$ and Born's Reciprocity" Int. J. Geom. Meth. Mod. Phys. **08** (2011) 587-619.

M Calixto, E. Perez-Romero and V. Aldaya, "Coherent states of accelerated relativistic quantum particles, vacuum radiation and the spontaneous breakdown of the conformal $SU(2,2)$ symmetry" J. Phys. **A**: Math. Theor. **45** (2012) 244010.

[21] C. Castro, "On Maximal Proper Force, Black Hole Horizons and Matter as Curvature in Momentum Space", Int. J. Geom. Meth. Mod. Phys. **15** (2017) no.03, 1850035.

[22] P. Nicolini, A. Smailagic, E. Spallucci, "Noncommutative geometry inspired Schwarzschild black hole" Phys. Letts **B 632**, issue 4 (2006) 547.

[23] Zhao Ren, Zhang Li-Chun, Li Huai-Fan and Wu Yue-Qin, "Entropy of Kerr-Newman black hole to all orders in the Planck length" arXiv : 0801.4118.

[24] J. Oppenheim, "A post-quantum theory of classical gravity ?" arXiv : 1811.03116

[25] A. Teimouri, "On Aspects of Infinite Derivatives Field Theories and Infinite Derivative Gravity" arXiv : 1811.09818.

[26] M. Reuter and F. Saueressig, *Quantum Gravity and the Functional Renormalization Group : The Road towards Asymptotic Safety"* (Cambridge University Press, 2019).

R. Percacci, *An Introduction to Covariant Quantum Gravity and Asymptotic Safety* (World Scientific, Singapore, 2017).

[27] L. Nottale, *Scale Relativity And Fractal Space-Time: A New Approach to Unifying Relativity and Quantum Mechanics* (World Scientific 2011)

L. Nottale, *Fractal Space-Time and Micro-physics* (World Scientific 1993).

[28] M. Lopez-Corredoira, "Tests and problems of the standard model in Cosmology", arXiv : 1701.08720, to appear in Foundations of Physics.

[29] C. Castro Perelman, "Novel Remarks on Point Mass sources, Firewalls, Null Singularities and Gravitational Entropy" Foundations of Physics **46** (2016) 46.

[30] S. Vacaru, "The entropy of Lagrange-Finsler spaces and Ricci flows", Rep. Math. Phys. **63**, 95110 (2009).

S. Vacaru, "Almost Kaehler Ricci flows and Einstein and Lagrange-Finsler structures on Lie algebroids" , Mediterr. J.Math. **12**, 13971427 (2015).

S. Vacaru, "Metric compatible or noncompatible Finsler-Ricci flows. Int. J. Geom. Methods Mod. Phys. **9**, 1250041 (2012).

S. Vacaru, "Fractional nonholonomic Ricci flows", Chaos Solitons Fractals **45**, 12661276 (2012).