Archimedean incircle of a triangle

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Abstract. We generalize several Archimedean circles, which are the incircles of special triangles.

1. Introduction

We consider an arbelos configuration formed by three circles α, β and γ with diameters AO, BO and AB, respectively for a point O on the segment AB (see Figure 1). Let a and b be the radii of α and β, respectively. Circles of radius $ab/(a+b)$ are said to be Archimedean. In [3], a special Archimedean circle is considered, which is the incircle of a triangle formed by a point lying ourside of γ and the two points of tangency of γ from the point. Similar Archimedean circles are also considered in [2]. In this paper we generalize those circles.

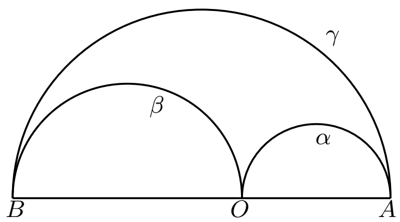

Figure 1

2. A circle generated by a point and a circle

In this section we generalize the Archimedean circles in [2, 3].

Theorem 1. *For a point P lying outside a circle δ of radius r, let Q and R be the points of tangency of the tangents of δ from P and $s = |PS|$, where S is the closest point on δ to P. Then the following statements hold.*
(i) *The point S coincides with the incenter of the triangle PQR.*
(ii) *The inradius of the triangle PQR equals $rs/(r+s)$.*

Proof. Assume that D is the center of δ, M is the midpoint of QS, and T is the midpoint of QR (see Figure 2). Since the triangles DMQ and QTS are similar, $2\angle SQT = \angle QDS$, while $2\angle SQP = \angle QDS$ by the inscribed angle theorem. Hence QS bisects $\angle PQR$. Therefore S is the incenter of PQR. This proves (i). If the inradius of PQR is x, then $x/s = |DQ|/|PD| = r/(r+s)$. This implies $x = rs/(r+s)$.

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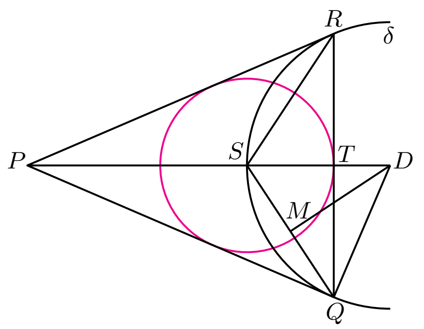

Figure 2

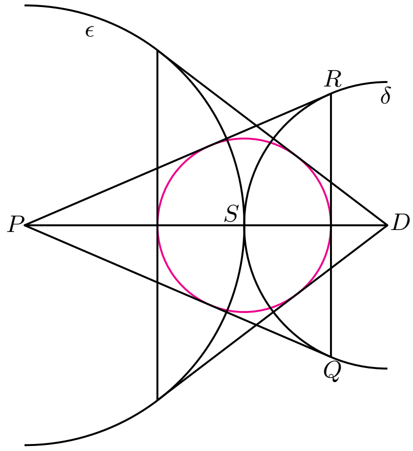

Figure 3

Let ϵ be the circle of center P passing through S in Theorem 1. As the part (ii) shows, the radius of the generated circle is symmetric in r and s. Hence the same circle is also generated by the point D and ϵ (see Figure 3).

Definition 1. If P is a point lying outside of a circle δ and the tangents of δ from P touches δ at points Q and R, then we call the incircle of PQR *the circle generated by* P *and* δ. *Also we say that the circle is generated by the circles* δ *and* ϵ, *where* ϵ *is the circle of center* P *touching* δ *externally.*

Now we consider the arbelos. Let O_a, O_b and O_c be the centers of the circles α, β and γ, respectively.

Corollary 1. *The following statement hold.*
(i) *The circle generated by a point* P *and the circle* α (*resp.* β) *is Archimedean if and only if* $|PO_a| = a + b$ (*resp.* $|PO_b| = a + b$).
(ii) *The circle generated by a point* P *and the circle* γ *is Archimedean if and only if* $|PO_c| = a + b + d$, *where* $d = ab(a + b)/(a^2 + ab + b^2)$.

Proof. The part (i) is obvious by Theorem 1(i). Solving the equation $(a + b)x/(a + b + x) = ab/(a + b)$ for x, we get $x = d$. This proves (ii).

Notice that d equals the inradius of the arbelos.

The two Archimedean circles generated by a point and one of the circles α and β given in [2] are obtained in a special case in the event of Corollary 1(i). Also the Archimedean circle generated by a point and the circle γ given in [3] is obtained in a special case in the event of Corollary 1(ii). Corollary 1(i) gives an interesting special case (see Figure 4).

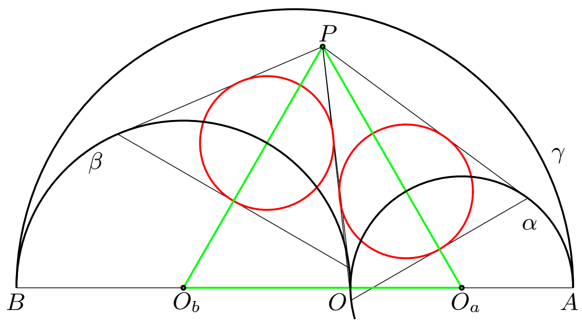

Figure 4.

Corollary 2. *If P is a point lying on the circle with center O_a (resp. O_b) congruent to γ, then the circle generated by P and α (resp. β) is Archimedean. Especially if PO_aO_b is an equilateral triangle, then the circles generated by P and each of α and β is Archimedean.*

The Archimedean circle of center O generated by α and β can be found in [1], which is denoted by W_8 .

References

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