Newton's Second Law is Valid in Relativity for Proper Time

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Abstract

In Newtonian particle dynamics, time is invariant under inertial transformations, and speed has no upper bound. In special relativity, it is the observed particle's proper time, rather than the arbitrary observer's time, which is inertial-transformation invariant, and it is the particle's proper speed which has no upper bound. It is thus perhaps not surprising the Newton's Second Law is also valid in relativistic particle dynamics presented in terms of the particle's proper time. This follows from the special-relativistic dynamical principle that the time derivative of special-relativistic momentum is equal to the applied force, provided proper force is defined to have an additional factor of gamma. The four-vector fully Lorentz-covariant completion of proper force is obliged to have zero contraction with the particle's proper four-velocity. A special-relativistic particle in either a scalar or a four-vector (electromagnetic) potential is verified to adhere to the proper-time Newton's Second Law, and when it is consistently taken into account that a metric (gravitational) potential's observed effect on a particle is to alter the rate of change of its proper time with observer time, that adherence is verified to hold for metric potentials as well.

Proper versus perceived velocity in special relativity

An *implicit ingredient in the perceived speed* $|dx/dt|$ *of a relativistic moving object* in one spatial dimension is *its length contraction by the factor* γ^{-1} [1],

$$\gamma^{-1} \stackrel{\text{def}}{=} \left(1 - ((dx/dt)/c)^2\right)^{\frac{1}{2}};\;\; 0 < \gamma^{-1} \le 1. \tag{1a}$$

The *degree* of length contraction *is completely dependent on the inertial reference frame*, so perceived speed $|dx/dt|$ is *unsuitable* for some applications. The *effect* of length contraction on a moving object's perceived speed is *removed* by multiplying that speed by γ, *which produces the higher speed* $\gamma|dx/dt|$. Multiplication of the moving object's perceived speed $|dx/dt|$ by γ is *equivalent* to replacing $|dx/dt|$ by $|dx/d\tau|$, where *Lorentz-transformation invariant differential proper time* $d\tau$ *is defined as*,

$$d\tau \stackrel{\text{def}}{=} \left((dt)^2 - (dx/c)^2\right)^{\frac{1}{2}} = \left(1 - ((dx/dt)/c)^2\right)^{\frac{1}{2}} dt = \gamma^{-1}dt, \text{ so } (dt/d\tau) = \gamma \text{ and } |dx/d\tau| = \gamma|dx/dt|. \tag{1b}$$

Since it is Lorentz-transformation invariant, the proper time defined by Eq. (1b) *is somewhat analogous to the Galilean-transformation invariant time of Newtonian physics*. Also, despite the strict adherence of perceived speed to $|dx/dt| < c$, *proper speed* $|dx/d\tau| = \gamma|dx/dt|$ *is unbounded because* γ *is*, so proper speed *is somewhat analogous to the unbounded speed of Newtonian physics*.

In *three* spatial dimensions, differential proper time $d\tau$, γ and proper velocity $(d\mathbf{r}/d\tau)$ are given by,

$$d\tau \stackrel{\text{def}}{=} \left((dt)^2 - |d\mathbf{r}/c|^2\right)^{\frac{1}{2}} = \left(1 - |\dot{\mathbf{r}}/c|^2\right)^{\frac{1}{2}} dt, \text{ so } (dt/d\tau) = \gamma \stackrel{\text{def}}{=} \left(1 - |\dot{\mathbf{r}}/c|^2\right)^{-\frac{1}{2}}, \text{ and } (d\mathbf{r}/d\tau) = \gamma\dot{\mathbf{r}}. \tag{2}$$

The Lorentz-transformation invariant proper time defined by Eq. (2) *is somewhat analogous to the Galilean-transformation invariant time of Newtonian physics*. Also, despite $|\dot{\mathbf{r}}| < c$, *proper speed* $|d\mathbf{r}/d\tau| = \gamma|\dot{\mathbf{r}}|$ *is unbounded because* γ *is*, and thus *is somewhat analogous to the unbounded speed of Newtonian physics*.

The Lorentz-covariant proper-time extension of Newton's Second Law

The usual presentation of single-particle special-relativistic dynamics is,

$$(d\mathbf{p}/dt) = \mathbf{f}, \tag{3a}$$

where $\mathbf{f}$ is the force and the relativistic single-particle momentum $\mathbf{p}$ is given by,

$$\mathbf{p} = m\gamma\dot{\mathbf{r}}, \tag{3b}$$

where m is the particle's rest mass. From Eq. (2) we see that Eq. (3b) can be rewritten,

$$\mathbf{p} = m(d\mathbf{r}/d\tau), \tag{3c}$$

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so Eq. (3a) becomes,

$$m(d(d\mathbf{r}/d\tau)/dt) = \mathbf{f}. \tag{3d}$$

We now multiply the left side of Eq. (3d) by $(dt/d\tau)$ and its right side by γ, as per Eq. (2), which yields,

$$m(d(d\mathbf{r}/d\tau)/dt)(dt/d\tau) = \gamma\mathbf{f}. \tag{3e}$$

We simplify the left side of Eq. (3e) and denote $\gamma\mathbf{f}$ on its right side *as the proper force* $\mathbf{F}$ to obtain,

$$m\big(d^2\mathbf{r}/d\tau^2\big) = \mathbf{F}, \tag{3f}$$

the relativistic extension of Newton's Second Law via proper time. An *example* of Eq. (3f) is the proper force exerted by an electromagnetic field on a particle of charge e, namely,

$$\mathbf{F} = e\gamma(\mathbf{E} + ((\dot{\mathbf{r}}/c)\times\mathbf{B}))\,. \tag{3g}$$

The *fully Lorentz-covariant four-vector completion of* Eq. (3f) *must of course read*,

$$m\big(d^2x^\mu/d\tau^2\big) = F^\mu, \tag{3h}$$

but the nature of proper time *ensures that only three of the four components of the proper force* F^μ *can be mutually independent*. We begin the demonstration of this fact by using Eq. (2) to show that,

$$(dx^\mu/d\tau)\,(dx_\mu/d\tau) = (\dot{x}^\mu\dot{x}_\mu)(dt/d\tau)^2 = \big(c^2 - |\dot{\mathbf{r}}|^2\big)/\big(1 - |\dot{\mathbf{r}}/c|^2\big) = c^2, \tag{3i}$$

which furthermore implies that,

$$\big(d^2x^\mu/d\tau^2\big)(dx_\mu/d\tau) = \tfrac{1}{2}(d((dx^\mu/d\tau)(dx_\mu/d\tau))/d\tau) = \tfrac{1}{2}\big(d\big(c^2\big)/d\tau\big) = 0. \tag{3j}$$

Eq. (3h) together with Eq. (3j) implies that,

$$F^\mu(dx_\mu/d\tau) = m\big(d^2x^\mu/d\tau^2\big)(dx_\mu/d\tau) = 0. \tag{3k}$$

Therefore only three of the four components of the proper force F^μ can be mutually independent. In greater detail, Eq. (3k) together with Eq. (2) yields that,

$$0 = F^\mu(dx_\mu/d\tau) = (F^\mu\dot{x}_\mu)(dt/d\tau) = \big(F^0c - \mathbf{F}\cdot\dot{\mathbf{r}}\big)/\big(1 - |\dot{\mathbf{r}}/c|^2\big)^{\frac{1}{2}}, \text{ which implies that } F^0 = \mathbf{F}\cdot(\dot{\mathbf{r}}/c). \tag{3l}$$

We thus see that F^0 *vanishes altogether in the nonrelativistic limit* $|\dot{\mathbf{r}}/c| \to 0$, for which it is *also* true that $(dt/d\tau) = \gamma = \big(1 - |\dot{\mathbf{r}}/c|^2\big)^{-\frac{1}{2}} \to 1$, so in the nonrelativistic limit Eq. (3h) *reduces to Newton's* $m\ddot{\mathbf{r}} = \mathbf{f}$.

Eq. (3f) shows that the concept of inertial mass, which is *the same* as rest mass, *is just as relevant to relativistic physics as it is to Newtonian physics*. Indeed, the development of Higgs field physics [2] has put considerable flesh on the bones of the inertial mass concept. An interesting *relativistic issue* is the *existence* of particles, such as *photons*, which have *zero inertial mass* (these are asserted *to not couple at all to the Higgs field*). According to Eq. (3c), a zero-inertial-mass particle which has nonzero momentum $|\mathbf{p}| > 0$ *has infinite proper speed* $|d\mathbf{r}/d\tau| = \lim_{m\to 0}(|\mathbf{p}|/m) = \infty$ that corresponds *to perceived speed* c. To *demonstrate the last assertion*, we *invert* the Eq. (2) relation of proper velocity $(d\mathbf{r}/d\tau)$ to perceived velocity $\dot{\mathbf{r}}$, which is,

$$(d\mathbf{r}/d\tau) = \gamma\dot{\mathbf{r}} = \big(1 - |\dot{\mathbf{r}}/c|^2\big)^{-\frac{1}{2}}\,\dot{\mathbf{r}}.$$

This relation's *inverse* comes out to be,

$$\dot{\mathbf{r}} = (d\mathbf{r}/d\tau)\,\big(1 + |(d\mathbf{r}/d\tau)/c|^2\big)^{-\frac{1}{2}}\,, \tag{3m}$$

which has the asymptotic form,

$$\dot{\mathbf{r}} \sim c\,((d\mathbf{r}/d\tau)/|d\mathbf{r}/d\tau|) \text{ as } |(d\mathbf{r}/d\tau)/c| \to \infty. \tag{3n}$$

This result shows that zero-inertial-mass particles of nonzero momentum $|\mathbf{p}| > 0$, *which thus have infinite proper speed* $|d\mathbf{r}/d\tau| = \lim_{m\to 0}(|\mathbf{p}|/m) = \infty$, must therefore *have perceived speed* $|\dot{\mathbf{r}}|$ *equal to* c.

We shall now work out the explicit forms of the proper force F^μ exerted on a relativistic particle of mass m by (1) scalar potentials $\phi(x^\alpha)$, (2) four-vector (electromagnetic) potentials $A^\mu(x^\alpha)$ and (3) dimensionless second-rank symmetric-tensor metric (gravitational) potentials $g_{\mu\nu}(x^\alpha)$, which follow from the equations of motion generated by the corresponding relativistic-particle Lagrangians. Our first order of business is therefore the development of the relativistic-particle Lagrangian which corresponds to a given potential.

Guidelines for the development of special-relativistic particle Lagrangians

The *energy* of a rest-mass m special-relativistic particle with $\dot{\mathbf{r}} = \mathbf{0}$ is mc^2 *plus* its "at rest" *potential energy*,

$$H_{\text{rest}} = mc^2 + V_{\text{rest}}, \tag{4a}$$

so the special-relativistic particle's "at-rest" Lagrangian L_{rest} is,

$$L_{\text{rest}} = -H_{\text{rest}} = -\big(mc^2 + V_{\text{rest}}\big), \tag{4b}$$

since the usual additional term $\dot{\mathbf{r}}\cdot\mathbf{p}$ *vanishes entirely* for the $\dot{\mathbf{r}} = \mathbf{0}$ "at-rest" particle. Therefore the "at-rest" special-relativistic particle's action S_{rest} is given by,

$$S_{\text{rest}} = -\int \big(mc^2 + V_{\text{rest}}\big)dt. \tag{4c}$$

The *special-relativistic extension* of S_{rest} is required to be *Lorentz invariant*, and therefore is of the form,

$$S_{\text{inv}} = -\int \big(mc^2 + V_{\text{inv}}\big)d\tau = -\int \big(mc^2 + V_{\text{inv}}\big)(d\tau/dt)dt, \tag{4d}$$

where $d\tau$ is the particle's Lorentz-invariant differential proper time, and V_{inv} *is its Lorentz-invariant potential energy,* which *must reduce to* V_{rest} in the limit $\dot{\mathbf{r}} \to \mathbf{0}$. The *extension* of V_{rest} to *the Lorentz-invariant* V_{inv} is dealt with case-by-case. Eq. (4d) implies that the full special-relativistic Lagrangian L_{rel} is given by,

$$L_{\text{rel}} = -\big(mc^2 + V_{\text{inv}}\big)(d\tau/dt). \tag{4e}$$

The proper force exerted by a scalar potential

A relativistic particle of mass m which couples to a scalar potential $\phi(x^\alpha)$ with dimensionless coupling strength k has *both* V_{rest} *and* V_{inv} *equal to* $(k\phi)$, so from Eqs. (4e) and (2),

$$L_{\text{rel}} = -\big(mc^2 + k\phi\big)(d\tau/dt) = -\big(mc^2 + k\phi\big)\gamma^{-1} = -\big(mc^2 + k\phi\big)\big(1 - |\dot{\mathbf{r}}/c|^2\big)^{\frac{1}{2}}. \tag{5a}$$

Since the equation of motion implied by any single-particle Lagrangian L is,

$$d(\partial L/\partial \dot{x}^i)/dt = (\partial L/\partial x^i), \text{ where } i = 1, 2, 3, \tag{5b}$$

it is very useful in the case of the Lagrangian L_{rel} of Eq. (5a) to note that,

$$\big(\partial\left(\gamma^{-1}\right)/\partial \dot{x}^i\big) = -c^{-2}\gamma\dot{x}^i = -c^{-2}(dt/d\tau)\dot{x}^i = -c^{-2}(dx^i/d\tau). \tag{5c}$$

From Eqs. (5a)–(5c) we obtain that,

$$d\big(\big(m + \big(k\phi/c^2\big)\big)(dx^i/d\tau)\big)/dt = -k(\partial\phi/\partial x^i)(d\tau/dt). \tag{5d}$$

Upon multiplying both sides of Eq. (5d) by $(dt/d\tau)$ and noting that $x^i = -x_i$, it becomes,

$$d\big(\big(m + \big(k\phi/c^2\big)\big)(dx^i/d\tau)\big)/d\tau = k(\partial\phi/\partial x_i), \tag{5e}$$

whose fully Lorentz-covariant four-vector completion clearly is,

$$d\big(\big(m + \big(k\phi/c^2\big)\big)(dx^\mu/d\tau)\big)/d\tau = k(\partial\phi/\partial x_\mu). \tag{5f}$$

After fully carrying out the outer differentiation with respect to τ in Eq. (5f), *including noting that*,

$$(d\phi/d\tau) = (\partial\phi/\partial x_\nu)(dx_\nu/d\tau), \tag{5g}$$

followed by shifting all terms *except* $\big(m\big(d^2x^\mu/d\tau^2\big)\big)$ to the right side of Eq. (5f), we obtain,

$$m\big(d^2x^\mu/d\tau^2\big) = F^\mu, \tag{5h}$$

where *the proper force* F^μ exerted on the particle by the scalar potential $\phi(x^\alpha)$ is given by,

$$F^\mu = k\big[(\partial\phi/\partial x_\mu) - \big(1/c^2\big)\big[(dx^\mu/d\tau)(\partial\phi/\partial x_\nu)(dx_\nu/d\tau) + \phi\big(d^2x^\mu/d\tau^2\big)\big]\big]\,. \tag{5i}$$

By applying the results of Eqs. (3i) and (3j), namely that,

$$(dx^\mu/d\tau)(dx_\mu/d\tau) = c^2 \text{ and } \big(d^2x^\mu/d\tau^2\big)(dx_\mu/d\tau) = 0, \tag{5j}$$

it is readily verified that the proper force given by Eq. (5i) satisfies the requirement of Eq. (3k), i.e.,

$$F^\mu(dx_\mu/d\tau) = 0. \tag{5k}$$

We also note that if the scalar potential $\phi(x^\alpha)$ is *constant* in x^α, Eqs. (5h) and (5i) imply that,

$$\big(m + \big(k\phi/c^2\big)\big)\big(d^2x^\mu/d\tau^2\big) = 0, \tag{5l}$$

i.e., the particle's *mass* m is effectively *modified by the addition to it of the constant term* $\big(k\phi/c^2\big)$. The Higgs field is thought of as such a constant scalar potential which is able to give an effective mass to otherwise zero-mass particles if they have nonzero dimensionless coupling strength k with that scalar potential [2].

The proper force exerted by a four-vector (electromagnetic) potential

A particle of mass m and charge e at rest in a four-vector electromagnetic potential $A^\mu(x^\alpha)$ has potential energy $V_{\text{rest}} = eA^0$, with Lorentz-invariant extension $V_{\text{inv}} = (e/c)(dx_\nu/d\tau)A^\nu$. Thus from Eqs. (4e) and (2),

$$L_{\text{rel}} = -\big(mc^2 + (e/c)(dx_\nu/d\tau)A^\nu\big)(d\tau/dt) = -mc^2\gamma^{-1} - (e/c)\dot{x}_\nu A^\nu = -mc^2\gamma^{-1} - eA^0 + (e/c)\dot{\mathbf{r}}\cdot\mathbf{A}. \tag{6a}$$

Applying Eqs. (5b) and (5c) to this Lagrangian yields,

$$d\big(m(dx^i/d\tau) + (e/c)A^i\big)/dt = -(e/c)\dot{x}_\nu\big(\partial A^\nu/\partial x^i\big). \tag{6b}$$

Multiplying both sides of Eq. (6b) by $(dt/d\tau)$ and noting that $x^i = -x_i$ produces,

$$d\big(m(dx^i/d\tau) + (e/c)A^i\big)/d\tau = (e/c)(dx_\nu/d\tau)(\partial A^\nu/\partial x_i), \tag{6c}$$

which we reexpress as,

$$m\big(d^2x^i/d\tau^2\big) = (e/c)\big[(dx_\nu/d\tau)(\partial A^\nu/\partial x_i) - \big(dA^i/d\tau\big)\big]. \tag{6d}$$

Since,

$$\big(dA^i/d\tau\big) = \big(\partial A^i/\partial x_\nu\big)(dx_\nu/d\tau), \tag{6e}$$

we can rewrite Eq. (6d) as,

$$m\big(d^2x^i/d\tau^2\big) = (e/c)(dx_\nu/d\tau)\big[(\partial A^\nu/\partial x_i) - \big(\partial A^i/\partial x_\nu\big)\big], \tag{6f}$$

whose fully Lorentz-covariant four-vector completion clearly is,

$$m\big(d^2x^\mu/d\tau^2\big) = (e/c)(dx_\nu/d\tau)[(\partial A^\nu/\partial x_\mu) - (\partial A^\mu/\partial x_\nu)]. \tag{6g}$$

Therefore *the proper force* F^μ exerted on a particle of mass m and charge e by $A^\mu(x^\alpha)$ is,

$$F^\mu = (e/c)(dx_\nu/d\tau)[(\partial A^\nu/\partial x_\mu) - (\partial A^\mu/\partial x_\nu)], \tag{6h}$$

which satisfies the requirement $F^\mu(dx_\mu/d\tau) = 0$ of Eq. (3k) because $(dx_\nu/d\tau)(dx_\mu/d\tau)$ is *symmetric* under interchange of ν and μ, whereas $[(\partial A^\nu/\partial x_\mu) - (\partial A^\mu/\partial x_\nu)]$ is *antisymmetric* under that interchange. Eq. (6h) also implies Eq. (3g), since for $\mu = i = 1$, 2, or 3,

$$\begin{aligned} F^i = (e/c)(\gamma\dot{x}_\nu)\big[(\partial A^\nu/\partial x_i) - \big(\partial A^i/\partial x_\nu\big)\big] = \\ e\gamma\big[-\big(\partial A^0/\partial x^i\big) - (1/c)\dot{A}^i\big] + (e/c)\gamma\sum_{j=1}^{3}\big(\dot{x}^j\big)\big[\big(\partial A^j/\partial x^i\big) - \big(\partial A^i/\partial x^j\big)\big] = \\ e\gamma\big(-\big(\nabla_{\mathbf{r}}A^0\big) - (1/c)\dot{\mathbf{A}}\big)^i + (e/c)\gamma\left((\nabla_{\mathbf{r}}(\dot{\mathbf{r}}\cdot\mathbf{A})) - ((\dot{\mathbf{r}}\cdot\nabla_{\mathbf{r}})\mathbf{A})\right)^i = \\ e\gamma\left(\mathbf{E} + ((\dot{\mathbf{r}}/c)\times(\nabla_{\mathbf{r}}\times\mathbf{A}))\right)^i = e\gamma\left(\mathbf{E} + ((\dot{\mathbf{r}}/c)\times\mathbf{B})\right)^i. \end{aligned} \tag{6i}$$

The proper force exerted by a metric (gravitational) potential

A dimensionless second-rank symmetric-tensor metric potential $g_{\mu\nu}(x^\alpha)$ affects a particle's observed trajectory *via altering the dimensionless rate of change of the particle's proper time with the observer's time*, i.e., *via altering the factor* $(d\tau/dt)$ *of the* Eq. (4e) *basic special-relativistic Lagrangian.* Therefore we proceed *by using* $g_{\mu\nu}(x^\alpha)$ *to modify that Lagrangian's* $(d\tau/dt)$ *factor, in lieu of inserting a particle potential energy* V_{inv}.

Up to this point, we have always taken $(d\tau/dt)$ to have its basic special-relativistic value implied by Eq. (2), which depends *only on the particle's speed* $|\dot{\mathbf{r}}|$, namely,

$$(d\tau/dt) = \gamma^{-1} = \left(1 - |\dot{\mathbf{r}}/c|^2\right)^{\frac{1}{2}} = \left(c^2 - |\dot{\mathbf{r}}|^2\right)^{\frac{1}{2}}/c = (\dot{x}^\mu \dot{x}_\mu)^{\frac{1}{2}}/c = (\dot{x}^\mu \eta_{\mu\nu} \dot{x}^\nu)^{\frac{1}{2}}/c, \tag{7a}$$

where $\eta_{\mu\nu}$ is the special-relativistic dimensionless *constant diagonal Minkowski metric*,

$$\eta_{\mu\nu} \stackrel{\text{def}}{=} \begin{cases} +1 & \text{if } \mu = \nu = 0, \\ -1 & \text{if } \mu = \nu = 1,\ 2, \text{ or } 3, \\ 0 & \text{if } \mu \neq \nu. \end{cases} \tag{7b}$$

A metric potential $g_{\mu\nu}(x^\alpha)$ *alters* $(d\tau/dt)$ *from its* Eq. (7a) *basic special-relativistic value to*,

$$(d\tau/dt)_{g_{\mu\nu}} = (\dot{x}^\mu g_{\mu\nu} \dot{x}^\nu)^{\frac{1}{2}}/c = \left(\dot{x}^0 g_{00} \dot{x}^0 + 2\dot{x}^0 \sum_{j=1}^3 g_{0j} \dot{x}^j + \sum_{j=1}^3 \sum_{k=1}^3 \dot{x}^j g_{jk} \dot{x}^k\right)^{\frac{1}{2}}/c, \tag{7c}$$

where, of course, $\dot{x}^0 = c$. In general, $(d\tau/dt)_{g_{\mu\nu}}$ *will depend on the particle's space-time location* x^α *via* $g_{\mu\nu}(x^\alpha)$*, and it will also depend on the particle's direction of travel* $(\dot{\mathbf{r}}/|\dot{\mathbf{r}}|)$, instead of depending *only on the particle's speed* $|\dot{\mathbf{r}}|$, as it does in the basic special-relativistic case of Eq. (7a) where $g_{\mu\nu}(x^\alpha)$ reduces to $\eta_{\mu\nu}$.

A *key physical restriction on* $g_{\mu\nu}(x^\alpha)$ is that for all values of x^α, *its four matrix eigenvalues are required have the same signs as the matrix eigenvalues of* $\eta_{\mu\nu}$*, namely* $\{+, -, -, -\}$ [3]. Therefore for all values of x^α, $g_{\mu\nu}(x^\alpha)$ *has a matrix inverse*, which is conventionally denoted as $g^{\lambda\kappa}(x^\alpha)$. Thus, for example,

$$g^{\lambda\kappa}(x^\alpha) g_{\kappa\nu}(x^\alpha) = \delta^\lambda_\nu. \tag{7d}$$

For the metric potential $g_{\mu\nu}(x^\alpha)$, the particle Lagrangian $L_{g_{\mu\nu}}$ *has the same form* as that of the special-relativistic *free* particle, *but the* Eq. (7a) $(d\tau/dt) = (d\tau/dt)_{\eta_{\mu\nu}}$ *is replaced by the* Eq. (7c) $(d\tau/dt)_{g_{\mu\nu}}$, so,

$$L_{g_{\mu\nu}} = -mc^2 (d\tau/dt)_{g_{\mu\nu}}. \tag{7e}$$

Before we work out the particle equation of motion that follows from $L_{g_{\mu\nu}}$, we take note of the generalizations of the proper-velocity and proper-acceleration identities given by Eqs. (3i) and (3j) that ensue when $(d\tau/dt) = (d\tau/dt)_{\eta_{\mu\nu}}$ *is replaced by* $(d\tau/dt)_{g_{\mu\nu}}$. The Eq. (3i) identity's generalization is easily obtained, i.e.,

$$(dx^\mu/d\tau) g_{\mu\nu} (dx^\nu/d\tau) = (\dot{x}^\mu g_{\mu\nu} \dot{x}^\nu)/((d\tau/dt)_{g_{\mu\nu}})^2 = c^2, \tag{7f}$$

where the last equality follows from Eq. (7c). The Eq. (3j) identity's generalization is then obtained via differentiation with respect to τ of the Eq. (7f) result,

$$\begin{gathered} 0 = d\left(c^2\right)/d\tau = d((dx^\mu/d\tau) g_{\mu\nu} (dx^\nu/d\tau))/d\tau = \\ 2\left(d^2x^\mu/d\tau^2\right) g_{\mu\nu} (dx^\nu/d\tau) + (dx^\mu/d\tau)(\partial g_{\mu\nu}/\partial x^\kappa)(dx^\kappa/d\tau)(dx^\nu/d\tau), \end{gathered} \tag{7g}$$

which implies the following generalization of the Eq. (3j) identity,

$$\left(d^2x^\mu/d\tau^2\right) g_{\mu\nu} (dx^\nu/d\tau) = -\tfrac{1}{2}(dx^\mu/d\tau)(\partial g_{\mu\nu}/\partial x^\kappa)(dx^\kappa/d\tau)(dx^\nu/d\tau). \tag{7h}$$

In light of the Eq. (7h) identity, any purported proper force F^λ on a particle of mass m that is claimed to adhere to the equation,

$$m\left(d^2x^\mu/d\tau^2\right) = F^\mu, \tag{7i}$$

must be such that it satisfies the consistency requirement,

$$F^\mu g_{\mu\nu} (dx^\nu/d\tau) = -\tfrac{1}{2}m(dx^\mu/d\tau)(\partial g_{\mu\nu}/\partial x^\kappa)(dx^\kappa/d\tau)(dx^\nu/d\tau). \tag{7j}$$

We now work out the particle equation of motion that follows from the Eq. (7e) Lagrangian $L_{g_{\mu\nu}}$,

$$(\partial L_{g_{\mu\nu}}/\partial\dot{x}^i) = -\tfrac{1}{2}mc\Big(2g_{i0}\dot{x}^0 + 2\sum_{j=1}^{3} g_{ij}\dot{x}^j\Big)/(\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu)^{\frac{1}{2}} = \\ -mc(g_{i\nu}\dot{x}^\nu)/(c(d\tau/dt)_{g_{\mu\nu}}) = -mg_{i\nu}(dx^\nu/d\tau), \tag{7k}$$

and,

$$(\partial L_{g_{\mu\nu}}/\partial x^i) = -\tfrac{1}{2}mc(\dot{x}^\mu(\partial g_{\mu\nu}/\partial x^i)\dot{x}^\nu)/(\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu)^{\frac{1}{2}} = \\ -\tfrac{1}{2}mc(\dot{x}^\mu(\partial g_{\mu\nu}/\partial x^i)\dot{x}^\nu)/(c(d\tau/dt)_{g_{\mu\nu}}) = -\tfrac{1}{2}m(\dot{x}^\mu(\partial g_{\mu\nu}/\partial x^i)(dx^\nu/d\tau)). \tag{7l}$$

Using the Eq. (7k) and (7l) results, we obtain the following particle equation of motion,

$$-m(d(g_{i\nu}(dx^\nu/d\tau))/dt) = -\tfrac{1}{2}m(\dot{x}^\mu(\partial g_{\mu\nu}/\partial x^i)(dx^\nu/d\tau)). \tag{7m}$$

After dividing both sides of Eq. (7m) by $(d\tau/dt)_{g_{\mu\nu}}$, this equation of motion can be reexpressed as,

$$-m(d(g_{i\nu}(dx^\nu/d\tau))/d\tau) = -\tfrac{1}{2}m((dx^\mu/d\tau)(\partial g_{\mu\nu}/\partial x^i)(dx^\nu/d\tau)). \tag{7n}$$

The four-vector completion of Eq. (7n) clearly is,

$$-m(d(g_{\kappa\nu}(dx^\nu/d\tau))/d\tau) = -\tfrac{1}{2}m((dx^\mu/d\tau)(\partial g_{\mu\nu}/\partial x^\kappa)(dx^\nu/d\tau)). \tag{7o}$$

Fully carrying out the outer τ differentiation on the left side of Eq. (7o) yields two terms,

$$-mg_{\kappa\nu}\big(d^2x^\nu/d\tau^2\big) - m((dx^\mu/d\tau)(\partial g_{\kappa\nu}/\partial x^\mu)(dx^\nu/d\tau)) = -\tfrac{1}{2}m((dx^\mu/d\tau)(\partial g_{\mu\nu}/\partial x^\kappa)(dx^\nu/d\tau)). \tag{7p}$$

Because the entity $((dx^\mu/d\tau)(dx^\nu/d\tau))$ is *symmetric* under the interchange of μ and ν, Eq. (7p) can be rewritten as,

$$-mg_{\kappa\nu}\big(d^2x^\nu/d\tau^2\big) = \tfrac{1}{2}m((dx^\mu/d\tau)[(\partial g_{\kappa\nu}/\partial x^\mu) + (\partial g_{\kappa\mu}/\partial x^\nu) - (\partial g_{\mu\nu}/\partial x^\kappa)](dx^\nu/d\tau)). \tag{7q}$$

Making use of Eq. (7d), we multiply both sides of Eq. (7q) by $-g^{\lambda\kappa}$ and sum over the index κ to obtain,

$$m\big(d^2x^\lambda/d\tau^2\big) = -\tfrac{1}{2}mg^{\lambda\kappa}(dx^\mu/d\tau)[(\partial g_{\kappa\nu}/\partial x^\mu) + (\partial g_{\kappa\mu}/\partial x^\nu) - (\partial g_{\mu\nu}/\partial x^\kappa)](dx^\nu/d\tau) = F^\lambda, \tag{7r}$$

where F^λ *is the proper force exerted on a particle by a metric potential* $g_{\mu\nu}(x^\alpha)$. To check that F^λ satisfies the consistency requirement of Eq. (7j), we use the fact that $g^{\lambda\kappa} = g^{\kappa\lambda}$ to reexpress it as,

$$F^\lambda = -\tfrac{1}{2}m(dx^\mu/d\tau)[(\partial g_{\kappa\nu}/\partial x^\mu) + (\partial g_{\kappa\mu}/\partial x^\nu) - (\partial g_{\mu\nu}/\partial x^\kappa)](dx^\nu/d\tau)g^{\kappa\lambda}, \tag{7s}$$

which, since $g^{\kappa\lambda}g_{\lambda\gamma} = \delta^\kappa_\gamma$, yields that,

$$F^\lambda g_{\lambda\gamma}(dx^\gamma/d\tau) = -\tfrac{1}{2}m(dx^\mu/d\tau)[(\partial g_{\kappa\nu}/\partial x^\mu) + (\partial g_{\kappa\mu}/\partial x^\nu) - (\partial g_{\mu\nu}/\partial x^\kappa)](dx^\nu/d\tau)(dx^\kappa/d\tau) = \\ -\tfrac{1}{2}m(dx^\mu/d\tau)(\partial g_{\mu\nu}/\partial x^\kappa)(dx^\kappa/d\tau)(dx^\nu/d\tau), \tag{7t}$$

as required by Eq. (7j), where the last equality follows upon appropriately renaming contracted indices.

It is to be noted that Eq. (7r) is conventionally written using the Christoffel symbol $\Gamma^\lambda_{\mu\nu}$, i.e. [4],

$$\big(d^2x^\lambda/d\tau^2\big) + (dx^\mu/d\tau)\Gamma^\lambda_{\mu\nu}(dx^\nu/d\tau) = 0 \text{ where,} \\ \Gamma^\lambda_{\mu\nu} = \tfrac{1}{2}g^{\lambda\kappa}[(\partial g_{\kappa\nu}/\partial x^\mu) + (\partial g_{\kappa\mu}/\partial x^\nu) - (\partial g_{\mu\nu}/\partial x^\kappa)]. \tag{7u}$$

References

[1] Length contraction–Wikipedia, https://en.wikipedia.org/wiki/Length_contraction.

[2] Higgs field–Simple Wikipedia, https://simple.wikipedia.org/wiki/Higgs_field.

[3] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley & Sons, New York, 1972), Section 3.6, pp. 85–86.

[4] S. Weinberg, op. cit., Eq. (3.2.3), p. 71 and Eq. (3.3.7), p. 75.