A Simple Proof That Finite Mathematics Is More Fundamental Than Classical One

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Abstract

Classical mathematics (involving such notions as infinitely small/large and continuity) is usually treated as fundamental while finite mathematics is treated as inferior which is used only in special applications. We give a simple proof that the situation is the opposite: classical mathematics is only a degenerate special case of finite one. Motivation and implications are discussed.

Keywords: classical mathematics, finite mathematics, quantum theory

1 Motivation

A belief of the overwhelming majority of scientists is that classical mathematics (involving the notions of infinitely small/large and continuity) is fundamental while finite mathematics is something inferior what is used only in special applications. This belief is based on the fact that the history of mankind undoubtedly shows that classical mathematics has demonstrated its power in many areas of science.

The notions of of infinitely small/large, continuity etc. were proposed by Newton and Leibniz more than 300 years ago. At that time people did not know about existence of atoms and elementary particles and believed that any body can be divided by an arbitrarily large number of arbitrarily small parts. However, now it is clear that standard division has only a limited applicability because when we reach the level of atoms and elementary particle the division operation looses its meaning. In nature there are no infinitely small objects and no continuity because on the very fundamental level nature is discrete. So, as far as applications of mathematics to physics is concerned, classical mathematics is only an approximation which in many cases works with very high accuracy but the ultimate quantum theory cannot be based on classical mathematics.

A typical situation in physics can be described by the following

Definition: *Let theory A contain a finite parameter and theory B be obtained from theory A in the formal limit when this parameter goes to zero or infinity. Suppose that with any desired accuracy theory A can reproduce any result of theory B by choosing a value of the parameter. On the contrary, when the limit is already taken then one cannot return back to theory A and theory B cannot reproduce all results of* *theory A. Then theory A is more general than theory B and theory B is a special degenerate case of theory A.*

Probably the most known example is that nonrelativistic theory (NT) can be obtained from relativistic theory (RT) in the formal limit $c \to \infty$ where c is the velocity of light. RT can reproduce any result of NT with any desired accuracy if c is chosen to be sufficiently large. On the contrary, when the limit is already taken then one cannot return back from NT to RT, and NT can reproduce results of RT only in relatively small amount of cases when velocities are much less than c. Therefore RT is more general than NT and NT is the special degenerate case of RT. Other known examples are that classical theory is a special degenerate case of quantum one in the formal limit $\hbar \to 0$ where $\hbar$ is the Planck constant, and Poincare invariant theory is a special degenerate case of de Sitter invariant theories in the formal limit $R \to \infty$ where R is the parameter defining contraction from the de Sitter Lie algebras to the Poincare Lie algebra.

In our publications (see e.g. Refs. [1, 2]) we discussed an approach called Finite Quantum Theory (FQT) where quantum theory is based not on classical but on finite mathematics. It has been shown that FQT is more general than standard quantum theory and the latter is a special degenerate case of the former in the formal limit when the characteristic of the field or ring in FQT goes to infinity. In Sec. 2 we prove the main statement of the present work that *even classical mathematics itself is a special degenerate case of finite mathematics in the formal limit when the characteristic of the field or ring in the latter goes to infinity.* Section 3 is discussion.

2 Proof of the main statement

Classical mathematics starts from natural numbers but here only addition and multiplication are always possible. In order to make addition invertible we introduce negative integers and get the ring of integers Z. However, if instead of all natural numbers we consider only a set R_p of p numbers 0, 1, 2, ... $p-1$ where addition and multiplication are defined as usual but modulo p then we get a ring without adding new elements. In our opinion the notation Z/p for R_p is not quite adequate because it may give a wrong impression that finite mathematics starts from the infinite set Z and that Z is more general than R_p. However, although the number of elements in Z is greater than in R_p, Z cannot be more general than R_p because Z does not contain operations modulo p.

We assume for definiteness that p is odd; the case of even p can be considered analogously. Since operations in R_p are modulo p, one can represent R_p as a set of elements $\{0, \pm i\}$ $(i = 1, ...(p-1)/2)$. Let f be a function from R_p to Z such that $f(a)$ has the same notation in Z as a in R_p.

If elements of Z are depicted as integer points on the x axis of the xy plane then the elements of R_p can be depicted as points of the circumference in Fig. 1. This picture is natural since R_p has a property that if we take any element $a \in R_p$ and successively add 1 to this element then after p steps we will exhaust the whole

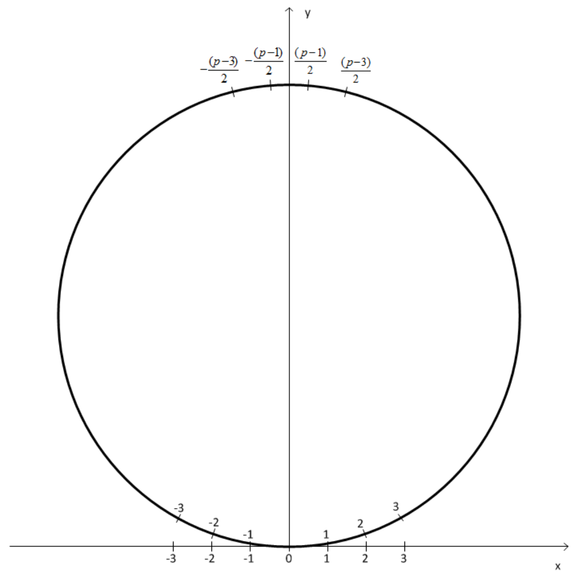

Figure 1: Relation between R_p and Z

set R_p by analogy with the property that if we move along a circumference in the same direction then sooner or later we will arrive at the initial point.

Let m be a natural number and $U(m)$ be a set of elements $a \in R_p$ such that $|f(a)| \leq [(p-1)/2]^{1/m}$. Then for any $n \leq m$ the result of any n operations of addition, subtraction or multiplication of elements $a \in U(m)$ is the same as for the corresponding elements $f(a)$ in Z, i.e. in this case operations modulo p are not explicitly manifested.

Let us now choose m as $m = g(p) = int(\{ln[(p-1)/2]\}^{1/2})$ where $int(x)$ is the integer part of x. Since $[(p-1)/2]^{1/g(p)}$ becomes infinitely large when $p \to \infty$ then in the formal limit $p \to \infty$ the set $U(g(p))$ becomes Z and for the infinite number of additions, subtractions and multiplications of the elements of $U(g(p))$ the result is the same as in Z. In other words, Z can be treated as a formal limit of $U(g(p))$ when $p \to \infty$.

We have proved the following

Statement: *The result of any finite combination of additions, subtractions and multiplications in Z can be reproduced in R_p if p is chosen to be sufficiently large.* On the contrary, when the limit $p \to \infty$ is already taken then one cannot return back from Z to R_p, and in Z it is not possible to reproduce all results in R_p because in Z there are no operations modulo p. According to the **Definition** in Sec. 1 this means that *the ring R_p is more general than Z, and Z is the special degenerate case of R_p.*

When p is very large then $U(g(p))$ is a relatively small part of R_p, and the results in Z and R_p are the same only in $U(g(p))$. This is analogous to the fact mentioned in Sec. 1 that the results of NT and RT are the same only in relatively small cases when velocities are much less than c. However, when the radius of the circumference in Fig. 1 becomes infinitely large then a relatively small vicinity of zero in R_p becomes the infinite set Z when $p \to \infty$.

A question arises whether the fact that R_p is more general than Z implies that finite mathematics is more general than classical one. In classical mathematics the ring Z is the starting point for introducing the notions of rational and real numbers. For example, if p is prime then R_p becomes the Galois field F_p, and the results in F_p considerably differ from those in the set Q of rational numbers. For example, $1/2$ in F_p is a very large number $(p+1)/2$. However, for describing nature those notions play only auxiliary role because, as noted in Sec. 1, standard division has a limited applicability and, as explained in Ref. [1], since states in standard quantum theory are projective then with any desirable accuracy they can be described by using only integers. Therefore on the basis of the above properties of the transition from R_p to Z we conclude that, at least in applications to physics, *classical mathematics is a degenerate case of finite one when formally* $p \to \infty$.

3 Discussion

The above construction has a well-known historical analogy. For many years people believed that the Earth was flat and infinite, and only after a long period of time they realized that it was finite and curved. It is difficult to notice the curvature when we deal only with distances much less than the radius of the curvature. Analogously one might think that the set of numbers describing physics in our Universe has a "curvature" defined by a very large number p but we do not notice it when we deal only with numbers much less than p.

In the preceding sections we argue that classical mathematics is a degenerate case of finite one in the formal limit $p \to \infty$ and that ultimate quantum theory will be based on finite mathematics. The fact that at the present stage of the Universe p is a huge number explains why in many cases classical mathematics describes natural phenomena with a very high accuracy. At the same time, as shown in our publications in Ref. [2], the explanation of several phenomena can be given only in the theory where p is finite.

Although classical mathematics is a degenerate case of finite one, a problem arises whether classical mathematics can be substantiated as an abstract science. It is well-known that, in spite of great efforts of many great mathematicians, the problem of foundation of classical mathematics has not been solved. For example, Gödel's incompleteness theorems state that no system of axioms can ensure that all facts about natural numbers can be proven and the system of axioms in classical mathematics cannot demonstrate its own consistency.

The philosophy of Cantor, Fraenkel, Gödel, Hilbert, Kronecker, Russell, Zermelo and other great mathematicians was based on macroscopic experience in which the notions of infinitely small, infinitely large, continuity and standard division are natural. However, as noted above, those notions contradict the existence of elementary particles and are not natural in quantum theory. The illusion of continuity arises when one neglects the discrete structure of matter.

However, since classical mathematics is a special degenerate case of finite one, foundational problems in this mathematics do not have a fundamental role and classical mathematics can be treated only as a technique which in many cases (but not all of them) describes reality with a high accuracy.

References

[1] F. Lev, *Modular Representations as a Possible Basis of Finite Physics*, J. Math. Phys. **30**, 1985-1998 (1989); *Finiteness of Physics and its Possible Consequences*, J. Math. Phys. **34**, 490-527 (1993); *Why is Quantum Theory Based on Complex Numbers?* Finite Fields and Their Applications **12**, 336-356 (2006); *Finite Mathematics, Finite Quantum Theory And A Conjecture On The Nature Of Time*, arxiv:1711.04607 (2018).

[2] F. Lev, *Massless Elementary Particles in a Quantum Theory over a Galois Field*, Theor. Math. Phys. **138**, 208-225 (2004); *de Sitter Symmetry and Quantum Theory*, Phys. Rev. **D85**, 065003 (2012); *Finite Quantum Theory and Applications to Gravity and Particle Theory*, arxiv 1104.4647 (2016).