RIEMANN HYPOTHESIS PROOF

IDRISS OLIVIER BADO*

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Abstract

The main contribution of this paper is to achieve the proof of Riemann hypothesis. The key idea is based on new formulation of the problem

$$\zeta(s) = \zeta(1-s) \Leftrightarrow re(s) = \frac{1}{2}$$

. This proof is considered as a great discovery in mathematic.

1 Introduction

The Riemann Hypothesis is a conjecture concerning the zeros of the Riemann zeta function. First proposed by Bernhard Riemann in 1859 [1], the hypothesis has yet to be tested despite great efforts for over 100 years. The hypothesis remains one of Hilbert's unsolved problems [2], as well as being a Millennium Prize Problem [3].To fully understand the Riemann Hypothesis we must first introduce the Riemann zeta function $\zeta(s)$. zeta functin is define as $\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}$ $\forall s \in \mathbb{C}$ such as $re(s) \in]1, +\infty[$ Another form of the zeta function which links prime to zeta is given by Euler $re(s) \in]1, +\infty[$

$$\zeta(s) = \prod_{p \in P} \frac{1}{1 - p^{-s}}$$

. By simple integration we can see that :

$$\forall s, re(s) \in]1, +\infty[, \ln(\zeta(s)) = s \int_2^{+\infty} \frac{\pi(u)du}{u(u^s - 1)}$$

In order to use zeta function to explain the Riemann Hypothesis, we must first extend the domain of the function to all complex values of s . through the

*olivier.bado@ensea.edu.ci,newdiscoveryresearch.blogspot.com

method of analytic continuation. First, we define the Dirichlet eta function, which converges for any complex number s with $re(s) \in]0, +\infty[$, and is given by the following Dirichlet series: $re(s) \in]1, +\infty[$

$$\zeta(s) = \prod_{p \in P} \frac{1}{1 - p^{-s}}$$

Zeta function has been extended to all of complex by using Dirichlet eta $\zeta(s) = \frac{\eta(s)}{1-2^s}$ with $\eta(s) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n^s}$ function through the method of analytic continuation Riemann's hypothesis attracts attention from eminent mathematicians because many mathematical theorems rely on its validity: a proof of this conjecture would be a real achievement in mathematics For example, it would improve the Miller-Rabin primality test - an algorithm to determine whether a given number is prime - is based on the generalized Riemann hypothesis]. If it turns out to be wrong it would be an upheaval. The Riemann hypothesis can also be vital to improve the error term in the prime number theorem [3] . This theorem attempts to estimate the rate at which prime numbers appear, or more precisely, the rate at which they become less common. The prime number theorem states that the function of counting prime numbers $\pi(x) \simeq \frac{x}{\ln x}$ .This approximation was done by Gauss and Legendre . By using Riemann zeta function Hadamard and de la Vallee-Poussin in 1896 prove the estimation

$$\pi(x) = \int_2^x \frac{dt}{\ln t} + \bigcirc(xe^{-a\sqrt{\ln x}})$$

The Riemann Hypothesis will brushing up the error term and we would have :

$$\pi(x) = \int_2^x \frac{dt}{\ln t} + \bigcirc(\sqrt{x} \ln x)$$

Many have tried to prove Riemann's hypothesis, but without success. Several statements implying the hypothesis of riemann have emerged. Here we will show some examples statements equivalent to the Riemann hypothesis all offering a different way of approach the proof of the hypothesis. the first equivalent statement done by Robin asserts that $\sum_{d|n} d < e^{\gamma} \ln_2 n, \forall n > 5040$ [4] where

$$\gamma = \lim_{n \mapsto +\infty} [\sum_{n \geq 1} \frac{1}{n} - \ln n]$$

In continuation with Robin works Jeff Lagarias shows that Riemann hypothesis will follow if

$$\sum_{d|n} d \leq \sum_{n \geq 1} \frac{1}{n} + e^{\sum_{n \geq 1} \frac{1}{n}} \ln(\sum_{n \geq 1} \frac{1}{n})$$

[5]

1.1 Principle of the proof

Our proof is based on the following assertion,

$$\zeta(s) = \zeta(1-s) \Leftrightarrow re(s) = \frac{1}{2}$$

From the Functional relationship $\zeta(s) = \frac{\eta(s)}{1-2^s}$ We deduce

$$\zeta(s) - \zeta(1-s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1} B_n(s)}{n}$$

where

$$B_n(s) = \frac{n^{1-s}}{1-2^{1-s}} - \frac{n^s}{1-2^s}$$

Then the formulation becomes :

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1} B_n(s)}{n} = 0 \Leftrightarrow re(s) = \frac{1}{2}$$

. For the proof we must first establish the following theorem

2 Theorem

It exists a $\mathbb{C}$- module E containing B_n and a monomorphism of $\mathbb{C}$- module On E which zero is B_n

2.1 Proof

As far as the proof is concerned, let us consider a complex number such as $re(s) \in]0,1[$ and let's designate by $\mathcal{F}(\mathbb{C} \mapsto \mathbb{C})$the set of complex functions and A set defined as follows

$$A = \{f \in \mathcal{F}(\mathbb{C} \mapsto \mathbb{C}) : \forall s \in \mathbb{C}, re(s) \in]0,1[, f(s) + f(1-s) = 0 \Rightarrow f = 0\}$$

Let E be a $\mathbb{C}$- module formed by providing stability property by addition and subtraction to A

Let Θ a complex function $\Theta(s) = 1 - s, \forall s \in \mathbb{C}$ Let T be an application on E define as follow : $\forall f \in E, Tf = f + f \circ \Theta$ It is clear that T is monomorphism of $\mathbb{C}$- module Furthermore $TB_n(s) = (B_n + B_n \circ \Theta)(s) = 0, \forall s$

then $TB_n = 0$

2.2 The proof of Riemann hypothesis

Let consider

$$F(s) = T(\overline{s} + s - 1 + \zeta(s) - \zeta \circ \Theta(s))$$

and suppose that

$$\zeta(s) = \zeta(1 - s)$$

so

$$F(s) = T(\overline{s} + s - 1)$$

As

$$T(\overline{s} + s - 1) = T(\overline{id} + id - 1)(s)$$

$$T(\overline{s} + s - 1) = (\overline{id} + id - 1)(s) + (\overline{id} + id - 1) \circ \Theta(s) = 0$$

$$T(\overline{id} + id - 1)(s) = 0$$

so

$$(\overline{id} + id - 1)(s) = 0$$

hence $\overline{s} + s = 1$ for the following $re(s) = \frac{1}{2}$ Reciprocally Let s such as $re(s) = \frac{1}{2}$ then $\overline{s} + s = 1$ so

$$F(s) = T(\zeta - \zeta \circ \Theta)(s) = T(\zeta(s) - \zeta \circ \Theta(s)) = T \sum_{n=1}^{+\infty} \frac{(-1)^{n-1} B_n(s)}{n}$$

As T is monomorphism of $\mathbb{C}$-module . Providing a topology on E which make T continuous then

$$T \sum_{n=1}^{+\infty} \frac{(-1)^{n-1} B_n(s)}{n} = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1} T B_n(s)}{n} = 0$$

$F(s) = T(\zeta - \zeta \circ \Theta)(s) = 0$

$$(\zeta - \zeta \circ \Theta)(s) = 0$$

then $\zeta(s). = \zeta(1 - s)$

2.3 corollary

Let s be a complex number such as $re(s) \in]0, 1[$ then

$$\zeta(s) = 0 \Leftrightarrow re(s) = \frac{1}{2}$$

2.4 Proof

Let s be a complex number such as $re(s) \in]0, 1[$ and suppose that $\zeta(s) = 0$, according to the functional relation $\zeta(s) = 2^s \pi^{s-1} sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)$ then $\zeta(1-s) = 0$.
We have in this case $\zeta(s) = \zeta(1-s)$ then $re(s) = \frac{1}{2}$
Reciprocally ,let suppose that $re(s) = \frac{1}{2}$ then $\zeta(s) = \zeta(1-s)$ finally $\zeta(s) = 0$

3 Conclusion

In this article we have proved Riemann's hypothesis which is such an important problem. our fundamental result derives its essence from the new formulation of the problem. This opens a new trajectory for science.

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5 References

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