A study of the sum of three or more consecutive natural numbers

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Abstract

It holds that every product of natural numbers can also be written as a sum. The inverse does not hold when 1 is excluded from the product. For this reason, the investigation of natural numbers should be done through their sum and not through their product. Such an investigation is presented in the present article. We prove that primes play the same role for odd numbers as the powers of 2 for even numbers, and vice versa. The following theorem is proven: "Every natural number, except for 0 and 1, can be uniquely written as a linear combination of consecutive powers of 2 with the coefficients of the linear combination being -1 or +1." This theorem reveals a set of symmetries in the internal order of natural numbers which cannot be derived when studying natural numbers on the basis of the product. From such a symmetry a method for identifying large prime numbers is derived. We prove a factorization test for the natural numbers.

Keywords: Number theory, Composite numbers, Prime numbers.

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1   Introduction

It holds that every product of natural numbers can also be written as a sum. The inverse (i.e. each sum of natural numbers can be written as a product) does not hold when 1 is excluded from the product. This is due to prime numbers p which can be written as a product only in the form of $p = 1 \cdot p$. For this reason, the investigation of natural numbers should be done through their sum and not through their product. Such an investigation is presented in the present article.

The sum of consecutive natural numbers has been studied in the past [1, 2]. In this paper we studying the sum of 3 or more consecutive natural numbers.

We prove that each natural number can be written as a sum of three or more consecutive natural numbers except of the powers of 2 and the prime numbers. Each power of 2 and each prime number cannot be written as a sum of three or more consecutive natural numbers. Primes play the same role for odd numbers as the powers of 2 for even numbers, and vice versa.

We prove a theorem which is analogous to the fundamental theorem of arithmetic, when we study the positive integers with respect to addition: ''Every natural number, with the exception of 0 and 1, can be written in a unique way as a linear combination of consecutive powers of 2, with the coefficients of the linear combination being -1 or +1.'' This theorem reveals a set of symmetries in the internal order of natural numbers which cannot be derived when studying natural numbers on the basis of the product. From such a symmetry a method for identifying large prime numbers is derived.

In the last chapter we prove a factorization test for the odd numbers. In the study of natural numbers, if we focus in the sum, the parameter which determines the minimum number of operations required for the factorization of an odd number, is highlighted.

2 The sequence $\mu(k,n)$

We consider the sequence of natural numbers

$$\mu(k,n)= k+(k+1)+(k+2)+...+(k+n)=\frac{(n+1)(2k+n)}{2}$$

$$k \in \mathbb{N}^* = \{1,2,3,...\}$$

$$n \in A = \{2,3,4,...\}$$
(2.1)

For the sequence $\mu(k,n)$ the following theorem holds:

Theorem 2.1. *For the sequence $\mu(k,n)$ the following hold*:

1. $\mu(k,n) \in \mathbb{N}^*$.

2. *No element of the sequence is a prime number.*

3. *No element of the sequence is a power of 2.*

4. *The range of the sequence is all natural numbers that are not primes and are not powers of 2.*

Proof. 1. $\mu(k,n) \in \mathbb{N}^*$ as a sum of natural numbers.

2. $n \in A = \{2,3,4,...\}$ and therefore it holds that

$$n \geq 2$$
$$n+1 \geq 3$$

Also we have that

$$2k + n \geq 4$$

$$\frac{2k+n}{2} \geq \frac{3}{2} > 1$$

since $k \in \mathbb{N}^*$ and $n \in A = \{2, 3, 4, ...\}$. Thus, the product

$$\frac{(n+1)(2k+n)}{2} = \mu(k,n)$$

is always a product of two natural numbers different than 1, thus the natural number $\mu(k,n)$ cannot be prime.

3. Let that the natural number

$$\mu(k,n) = \frac{(n+1)(2k+n)}{2}$$

is a power of 2. Then, it exists $\lambda \in \mathbb{N}$ such as

$$\frac{(n+1)(2k+n)}{2} = 2^\lambda$$

and equivalently

$$(n+1)(2k+n) = 2^{\lambda+1}. \tag{2.2}$$

Equation (2.2) can hold if and only if there exist $\lambda_1, \lambda_2 \in \mathbb{N}$ such as

$$n+1 = 2^{\lambda_1} \wedge 2k+n = 2^{\lambda_2}$$

and equivalently

$$\left. \begin{array}{l} n = 2^{\lambda_1} - 1 \\ n = 2^{\lambda_2} - 2k \end{array} \right\}. \tag{2.3}$$

We eliminate n from equations (2.3) and we obtain

$$2^{\lambda_1} - 1 = 2^{\lambda_2} - 2k$$

$$2k - 1 = 2^{\lambda_2} - 2^{\lambda_1},$$

which is impossible since the first part of the equation is an odd number and the second part is an even number. Thus, the range of the sequence $\mu(k,n)$ does not include the powers of 2.

4. We now prove that the range of the sequence $\mu(k,n)$ includes all natural numbers that are not primes and are not powers of 2. Let a random natural number N which is not a prime nor a power of 2. Then, N can be written in the form

$$N = \chi\psi$$

where at least one of the χ,ψ is an odd number ≥ 3. Let χ be an odd number ≥ 3 . We will prove that there are always exist $k \in \mathbb{N}$ and $n \in A = \{2,3,4,...\}$ such as

$$N = \chi \cdot \psi = \mu(k,n).$$

We consider the following two pairs of k and n :

$$\chi \leq 2\psi - 1, \chi, \psi \in \mathbb{N}$$
$$k = k_1 = \frac{2\psi + 1 - \chi}{2} \tag{2.4}$$
$$n = n_1 = \chi - 1$$

$$\chi \geq 2\psi + 1, \chi, \psi \in \mathbb{N}$$
$$k = k_2 = \frac{\chi + 1 - 2\psi}{2} \cdot \tag{2.5}$$
$$n = n_2 = 2\psi - 1$$

For every $\chi, \psi \in \mathbb{N}$ it holds either the inequality $\chi \leq 2\psi - 1$ or the inequality $\chi \geq 2\psi + 1$. Thus, for each pair of naturals (χ, ψ), where χ is odd, at least one of the pairs (k_1, n_1), (k_2, n_2) of equations (2.4), (2.5) is defined. We now prove that when the natural number k_1 of equation (2.4) is $k_1 = 0$ then the natural number k_2 of equation (2.5) is $k_2 = 1$ and additionally it holds that $n_2 > 2$. For $k_1 = 0$ from equations (2.4) we take

$$\chi = 2\psi + 1$$

and from equations (2.5) we have that

$$k_2 = \frac{(2\psi + 1) + 1 - 2\psi}{2} = 1$$
$$n_2 = 2\psi - 1$$

and because $\psi \geq 2$ we obtain

$$k_2 = 1$$
$$n_2 = 2\psi - 1 \geq 3 > 2 \cdot$$

We now prove that when $k_2 = 0$ in equations (2.5), then in equations (2.4) it is $k_1 = 1$ and $n_1 > 2$. For $k_2 = 0$, from equations (2.5) we obtain

$$\chi = 2\psi - 1$$

and from equations (2.4) we get

$$k_1 = \frac{2\psi + 1 - (2\psi - 1)}{2} = 1.$$
$$n_1 = \chi - 1 = 2\psi - 2 \geq 2$$

We now prove that at least one of the k_2 and k_2 is positive. Let

$$k_1 < 0 \wedge k_2 < 0.$$

Then from equations (2.4) and (2.5) we have that

$$2\psi + 1 - \chi < 0 \wedge \chi + 1 - 2\psi < 0. \tag{2.6}$$

Taking into account that $\chi > 1$ is odd, that is $\chi = 2\rho + 1, \rho \in \mathbb{N}$, we obtain from inequalities (2.6)

$$2\psi + 1 - (2\rho - 1) < 0 \wedge (2\rho + 1) + 1 - 2\psi < 0$$

$$2\psi - 2\rho < 0 \wedge 2\rho - 2\psi + 2 > 0$$

$$\psi < \rho \wedge \psi > \rho + 1$$

which is absurd. Thus, at least one of k_1 and k_2 is positive.

For equations (2.4) we take

$$\mu(k_1, n_1) = \frac{(n_1 + 1)(2k_1 + n_1)}{2}$$

$$\mu(k_1, n_1) = \frac{(\chi - 1 + 1)\left(2\dfrac{2\psi + 1 - \chi}{2} + \chi - 1\right)}{2} = \frac{\chi \cdot 2 \cdot \psi}{2} = \chi\psi = N.$$

For equations (2.5) we obtain

$$\mu(k_2, n_2) = \frac{(n_2 + 1)(2k_2 + n_2)}{2}$$

$$\mu(k_2, n_2) = \frac{(2\psi - 1 + 1)\left(2\dfrac{\chi + 1 - 2\psi}{2} + 2\psi - 1\right)}{2} = \frac{2\psi\chi}{2} = \chi\psi = N.$$

Thus, there are always exist $k \in \mathbb{N}^*$ and $n \in A = \{2, 3, 4, \ldots\}$ such as

$N = \chi\psi = \mu(k,n)$ for every N which is not a prime number and is not a power of 2.$\square$

Example 2.1. For the natural number $N = 40$ we have

$N = 40 = 5\cdot 8$

$\chi = 5$

$\psi = 8$

and from equations (2.4) we get

$$k = k_1 = \frac{16+1-5}{2} = 6$$

$n = n_1 = 5-1 = 4$

thus, we obtain

$$40 = \mu(6,4) \ .$$

Example 2.2. For the natural number $N = 51$,

$N = 51 = 3\cdot 17 = 17\cdot 3$

there are two cases. First case:

$N = 51 = 3\cdot 17$

$\chi = 3$

$\psi = 17$

and from equations (2.4) we obtain

$$k = k_1 = \frac{34+1-3}{2} = 16$$

$n = n_1 = 3-1 = 2$

thus,

$$51 = \mu(16,2).$$

Second case:

$N = 51 = 17\cdot 3$

$\chi = 17$

$\psi = 3$

and from equations (2.5) we obtain

$$k = k_2 = \frac{17+1-6}{2} = 6$$

$n = n_2 = 6-1 = 5$

thus,

$$51 = \mu(6,5).$$

The second example expresses a general property of the sequence $\mu(k,n)$. The more composite an odd number that is not prime (or an even number that is not a power of 2) is, the more are the $\mu(k,n)$ combinations that generate it.

Example 2.3.

$$135 = 15 \cdot 9 = 27 \cdot 5 = 9 \cdot 15 = 45 \cdot 3 = 5 \cdot 27 = 3 \cdot 45$$

$$135 = \mu(2,14) = \mu(9,9) = \mu(11,8) = \mu(20,5) = \mu(25,4) = \mu(44,2)$$

a. $135 = 9 \cdot 15 = \mu(2,14) = \mu(11,8)$

$$135 = 2+3+4+\ldots\ldots+15+16 = 11+12+13\ldots..+18+19.$$

b. $135 = 5 \cdot 27 = \mu(9,9) = \mu(25,4)$

$$135 = 9+10+11+\ldots..+17+18 = 25+26+27+28+29.$$

c. $135 = 3 \cdot 45 = \mu(20,5) = \mu(44,2)$

$$135 = 20+21+22+23+24+25 = 44+45+46.$$

In the transitive property of multiplication, when writing a composite odd number or an even number that is not a power of 2 as a product of two natural numbers, we use the same natural numbers $\chi, \psi \in \mathbb{N}$:

$$\Phi = \chi \cdot \psi = \psi \cdot \chi.$$

On the contrary, the natural number Φ can be written in the form $\Phi = \mu(k,n)$ using different natural numbers $k \in \mathbb{N}^*$ and $n \in A = \{2,3,4,\ldots\}$, through equations (2.4), (2.5). This difference between the product and the sum can also become evident in example 2.3:

$$135 = 3 \cdot 45 = 45 \cdot 3$$

$$135 = 44+45+46 = 20+21+22+23+24+25.$$

From Theorem 2.1 the following corollaries are derived:

Corollary 2.1. 1. *Every natural number which is not a power of 2 and is not a prime can be written as the sum of three or more consecutive natural numbers.*

2. *Every power of 2 and every prime number cannot be written as the sum of three or more consecutive natural numbers.*

Proof. Corollary 2.1 is a direct consequence of Theorem 2.1.□

Corollary 2.2. *The sequence* $\mu(k,n)$ *can be written as a difference of two triangle numbers.*

Proof. From equation (2.1) we obtain

$$\mu(k,n) = \mu(1,k+n-1) - \mu(1,k-2) = \frac{(k+n+1)(k+n)}{2} - \frac{k(k-1)}{2} \quad . \ \square \tag{2.7}$$

$$k \in \mathbb{N}^*, n \in A = \{2,3,4,...\}$$

3   The concept of rearrangement

In this paragraph, we present the concept of rearrangement of the composite odd numbers and even numbers that are not power of 2. Moreover, we prove some of the consequences of the rearrangement in the Diophantine analysis. The concept of rearrangement is given from the following definition:

Definition 3.1. *We say that the sequence* $\mu(k,n), k \in \mathbb{N}^*, n \in A = \{2,3,4,...\}$ *is rearranged if there exist natural numbers* $k_1 \in \mathbb{N}^*, n_1 \in A$, $(k_1,n_1) \neq (k,n)$ *such as*

$$\mu(k,n) = \mu(k_1,n_1). \tag{3.1}$$

From equation (2.1) written in the form of

$$\mu(k,n) = k + (k+1) + (k+2) + ... + (k+n)$$

two different types of rearrangement are derived: The "compression", during which n decreases with a simultaneous increase of k. The «decompression», during which n increases with a simultaneous decrease of k. The following theorem provides the criterion for the rearrangement of the sequence $\mu(k,n)$.

Theorem 3.1. 1. *The sequence* $\mu(k_1,n_1), (k_1,n_1) \in \mathbb{N}^* \times A$ *can be compressed*

$$\mu(k_1,n_1) = \mu(k_1 + \varphi, n_1 - \omega) \tag{3.2}$$

if and only if there exist $\varphi, \omega \in \mathbb{N}^*, \omega \leq n_1 - 2$ *which satisfies the equation*

$$\omega^2 - (2k_1 + 2n_1 + 1 + 2\varphi)\omega + 2(n_1 + 1)\varphi = 0$$

$$\varphi, \omega \in \mathbb{N}^* \qquad . \tag{3.3}$$

$$\omega \leq n_1 - 2$$

2. *The sequence* $\mu(k_2,n_2), (k_2,n_2) \in \mathbb{N}^* \times A$ *can be decompressed*

$$\mu(k_2,n_2) = \mu(k_2 - \varphi, n_2 + \omega) \tag{3.4}$$

if and only if there exist $\varphi, \omega \in \mathbb{N}^*, \varphi \leq k_2 - 1$ *which satisfies the equation*

$$\omega^2 + (2k_2 + 2n_2 + 1 - 2\varphi)\omega - 2(n_2 + 1)\varphi = 0$$
$$\varphi, \omega \in \mathbb{N}^* \qquad\qquad . \qquad\qquad (3.5)$$
$$\varphi \leq k_2 - 1$$

3. *The odd number* $\Pi \neq 1$ *is prime if and only if the sequence*

$$\mu(k, n) = \Pi \cdot 2^l$$
$$l, k \in \mathbb{N}^*, n \in A \qquad\qquad\qquad (3.6)$$

cannot be rearranged.

4. *The odd* Π *is* *prime if and only if the sequence*

$$\mu\left(\frac{\Pi + 1}{2}, \Pi - 1\right) = \Pi^2 \qquad\qquad\qquad (3.7)$$

cannot be rearranged.

Proof. 1, 2. We prove part 1 of the corollary and similarly number 2 can also be proven. From equation (4.1) we conclude that the sequence $\mu(k_1, n_1)$ can be compressed if and only if there exist $\varphi, \omega \in \mathbb{N}^*$ such as

$$\mu(k_1, n_1) = \mu(k_1 + \varphi, n_1 - \omega).$$

In this equation the natural number $n_1 - \omega$ belongs to the set $A = \{2, 3, 4, ...\}$ and thus $n_1 - \omega \geq 2 \Leftrightarrow \omega \leq n_1 - 2$.

Next, from equations (2.1) we obtain

$$\mu(k_1, n_1) = \mu(k_1 + \varphi, n_1 - \omega)$$

$$\frac{(n_1 + 1)(2k_1 + n_1)}{2} = \frac{(n_1 - \omega + 1)\left[2(k_1 + \varphi) + n_1 - \omega\right]}{2}$$

and after the calculations we get equation (3.3).

3. The sequence (3.6) is derived from equations (2.4) or (2.5) for $\chi = \Pi$ and $\psi = 2^l$. Thus, in the product $\chi\psi$ the only odd number is Π. If the sequence $\mu(k, n)$ in equation (3.6) cannot be rearranged then the odd number Π has no divisors. Thus, Π is prime. Obviously, the inverse also holds.

4. First, we prove equations (3.7). From equation (2.1) we obtain:

$$\mu\left(\frac{\Pi+1}{2},\Pi-1\right) = \frac{(\Pi-1+1)\left(2\frac{\Pi+1}{2}+\Pi-1\right)}{2} = \Pi^2.$$

In case that the odd number Π is prime in equations (2.4), (2.5) the natural numbers χ,ψ are unique $\chi = \Pi \wedge \psi = \Pi$, and from equation (2.5) we get

$$k = \frac{\Pi+1}{2} \wedge n = \Pi-1.$$

Thus, the sequence

$$\mu(k,n) = \mu\left(\frac{\Pi+1}{2},\Pi-1\right)$$

cannot be rearranged. Conversely, if the sequence

$$\mu\left(\frac{\Pi+1}{2},\Pi-1\right) = \Pi^2 = \Pi\cdot\Pi$$

cannot be rearranged the odd number Π cannot be composite and thus Π is prime.□

We now prove the following corollary:

Corollary 3.1. 1. *The odd number* Φ,

$$\Phi = \Pi^2 = \mu\left(\frac{\Pi+1}{2},\Pi-1\right)$$

$\Pi = odd$ (3.8)

$\Pi \neq 1$

is decompressed and compressed if and only if the odd number Π *is composite.*

2. *The even number* α_1,

$$\alpha_1 = 2^l\Pi = \mu\left(2^l - \frac{\Pi-1}{2},\Pi-1\right)$$

$\Pi = odd$ (3.9)

$3 \leq \Pi \leq 2^l - 1$

$l \in \mathbb{N}, l \geq 2$

cannot be decompressed, while it compresses if and only if the odd number Π *is composite.*

3. *The even number* α_2,

$$\alpha_2 = 2^l \Pi = \mu\left(\frac{\Pi+1}{2} - 2^l, 2^{l+1} - 1\right)$$

$$\Pi = odd \qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.10)$$

$$\Pi \geq 2^{l+1} + 1$$

$$l \in \mathbb{N}^*$$

cannot be compressed, while it decompresses if and only if the odd number Π is composite.

4. *Every even number that is not a power of can be written either in the form of equation* (3.9) *or in the form of equation* (3.10).

Proof. 1. It is derived directly through number (4) of Theorem 3.1. A second proof can be derived through equations (2.4), (2.5) since every composite odd Π can be written in the form of $\Pi = \chi\psi$, $\chi, \psi \in \mathbb{N}$, χ, ψ odds.

2, 3. Let the even number α,

$$\alpha = 2^l \Pi$$

$$\Pi = odd . \qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.11)$$

$$l \in \mathbb{N}^*$$

From equation (2.4) we obtain

$$k = \frac{2 \cdot 2^l + 1 - \Pi}{2} = 2^l - \frac{\Pi - 1}{2} \qquad\qquad\qquad\qquad (3.12)$$

$$n = \Pi - 1$$

and since $k, n \in \mathbb{N}, k \geq 1 \wedge n \geq 2$ we get

$$\frac{2 \cdot 2^l + 1 - \Pi}{2} \geq 1$$

$$\Pi - 1 \geq 2$$

and equivalently

$$3 \leq \Pi \leq 2^{l+1} - 1.$$

In the second of equations (3.12) the natural number n obtains the maximum possible value of $n = \Pi - 1$, and thus the natural number k takes the minimum possible value in the first of equations (3.12). Thus, the even number

$$\alpha_1 = \mu\left(2^l - \frac{\Pi - 1}{2}, \Pi - 1\right)$$

cannot decompress. If the odd number Π is composite then it can be written in the form of $\Pi = \chi\psi$, $\chi, \psi \in \mathbb{N}^*$, χ, ψ odds, $\chi, \psi < \Pi$, $\alpha_1 = 2^l \chi\psi$. Therefore, the natural number $\alpha_1 = 2^l \chi\psi$

decompresses since from equations (3.11) it can be written in the form of $\alpha_1 = \mu(k,n)$ with $n = \chi - 1 < \Pi - 1$. Similarly, the proof of 3 is derived from equations (2.5).

4. From the above proof process it follows that every even number that is not a power of 2 can be written either in the form of equation (3.9) or in the form of equation (3.10). $\square$

By substituting $\Pi = P = prime$ in equations of Theorem 3.1 and of corollary 3.1 four sets of equations are derived, each including infinite impossible diophantine equations.

Example 3.1. The odd number $P = 999961$ is prime. Thus, combining (1) of Theorem 3.1 with (1) of corollary 3.1 we conclude that there is no pair $(\omega, \varphi) \in \mathbb{N}^2$ with $\omega \leq 999958$ which satisfies the diophantine equation

$$\omega^2 - (2999883 + 2\varphi)\omega + 1999922\varphi = 0.$$

We now prove the following corollary:

Corollary 3.2. *The square of every prime number can be uniquely written as the sum of consecutive natural numbers.*

Proof. For $\Pi = P = prime$ in equation (3.5) we obtain

$$P^2 = \mu\left(\frac{P+1}{2}, P-1\right). \tag{3.13}$$

According with 4 of Theorem 3.1 the odd P^2 cannot be rearranged. Thus, the odd can be uniquely written as the sum of consecutive natural numbers, as given from equation (3.13). $\square$

Example 3.2. The odd $P = 17$ is prime. From equation (3.13) for $P = 17$ we obtain

$$289 = \mu(9,16)$$

and from equation (2.1) we get

$$289 = 9+10+11+12+13+14+15+16+17+18+19+20+21+22+23+24+25$$

which is the only way in which the odd number 289 can be written as a sum of consecutive natural numbers.

4 Natural numbers as linear combination of consecutive powers of 2

According to the fundamental theorem of arithmetic, every natural number can be uniquely written as a product of powers of prime numbers. The previously presented study reveals a correspondence between odd prime numbers and the powers of 2. Thus, the question arises whether there exists a theorem for the powers of 2 corresponding to the fundamental theorem of arithmetic. The answer is given by the following theorem:

Theorem 4.1. *Every natural number, with the exception of* 0 *and* 1, *can be uniquely written as a linear combination of consecutive powers of* 2, *with the coefficients of the linear combination being* -1 *or* +1.

Proof. Let the odd number Π as given from equation

$$\Pi = \Pi(v, \beta_i) = 2^{v+1} + 2^v \pm 2^{v-1} \pm 2^{v-2} \pm .... \pm 2^1 \pm 2^0 = 2^{v+1} + 2^v + \sum_{i=0}^{v-1} \beta_i 2^i$$

$$\beta_i = \pm 1, i = 0,1,2,........, v-1 \qquad\qquad (4.1)$$

$$v \in \mathbb{N}$$

From equation (4.1) for $v = 0$ we obtain

$$\Pi = 2^1 + 2^0 = 2 + 1 = 3.$$

We now examine the case where $v \in \mathbb{N}^*$. The lowest value that the odd number Π of equation (4.1) can obtain is

$$\Pi_{min} = \Pi(v) = 2^{v+1} + 2^v - 2^{v-1} - 2^{v-1} - ....2^1 - 1$$

$$\Pi_{min} = \Pi(v) = 2^{v+1} + 1. \qquad\qquad (4.2)$$

The largest value that the odd number Π of equation (4.1) can obtain is

$$\Pi_{max} = \Pi(v) = 2^{v+1} + 2^v + 2^{v-1} + ...2^1 + 1$$

$$\Pi_{max} = \Pi(v) = 2^{v+2} - 1. \qquad\qquad (4.3)$$

Thus, for the odd numbers $\Pi = \Pi(v, \beta_i)$ of equation (4.1) the following inequality holds

$$\Pi_{min} = 2^{v+1} + 1 \le \Pi(v, \beta_i) \le 2^{v+2} - 1 = \Pi_{max}. \qquad\qquad (4.4)$$

The number $N(\Pi(v, \beta_i))$ of odd numbers in the closed interval $\left[2^{v+1} + 1, 2^{v+2} - 1\right]$ is

$$N(\Pi(v, \beta_i)) = \frac{\Pi_{max} - \Pi_{min}}{2} + 1 = \frac{(2^{v+2} - 1) - (2^{v+1} + 1)}{2} + 1$$

$$N(\Pi(v, \beta_i)) = 2^v. \qquad\qquad (4.5)$$

The integers $\beta_i, i = 0,1,2,........, v-1$ in equation (4.1) can take only two values, $\beta_i = -1 \lor \beta_i = +1$, thus equation (4.1) gives exactly $2^v = N(\Pi(v, \beta_i))$ odd numbers. Therefore, for every $v \in \mathbb{N}^*$ equation (4.1) gives all odd numbers in the interval $\left[2^{v+1} + 1, 2^{v+2} - 1\right]$.

We now prove the theorem for the even numbers. Every even number α which is a power of 2 can be uniquely written in the form of $\alpha = 2^v, v \in \mathbb{N}^*$. We now consider the case where the

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even number α is not a power of 2. In that case, according to corollary 3.1 the even number α is written in the form of

$$\alpha = 2^l \Pi, \Pi = \text{odd}, \Pi \neq 1, l \in \mathbb{N}^*. \tag{4.6}$$

We now prove that the even number α can be uniquely written in the form of equation (4.6). If we assume that the even number α can be written in the form of

$$\alpha = 2^l \Pi = 2^{l'} \Pi'$$
$$l \neq l' (l > l')$$
$$\Pi \neq \Pi' \tag{4.7}$$
$$l, l' \in \mathbb{N}^*$$
$$\Pi, \Pi' = odd$$

the we obtain

$$2^l \Pi = 2^{l'} \Pi'$$
$$2^{l-l'} \Pi = \Pi'$$

which is impossible, since the first part of this equation is even and the second odd. Thus, it is $l = l'$ and we take that $\Pi = \Pi'$ from equation (4.7). Therefore, every even number α that is not a power of 2 can be uniquely written in the form of equation (4.6). The odd number Π of equation (4.6) can be uniquely written in the form of equation (4.1), thus from equation (4.6) it is derived that every even number α that is not a power of 2 can be uniquely written in the form of equation

$$\alpha = \alpha(l, v, \beta_i) = 2^l \left(2^{v+1} + 2^v + \sum_{i=0}^{v-1} \beta_i 2^i \right)$$
$$l \in \mathbb{N}^*, v \in \mathbb{N} \tag{4.8}$$
$$\beta_i = \pm 1, i = 0,1,2,\ldots\ldots, v-1$$

and equivalently

$$\alpha = \alpha(l, v, \beta_i) = 2^{l+v+1} + 2^{l+v} + \sum_{i=0}^{v-1} \beta_i 2^{l+i}$$
$$l \in \mathbb{N}^*, v \in \mathbb{N} \qquad . \tag{4.9}$$
$$\beta_i = \pm 1, i = 0,1,2,\ldots\ldots, v-1$$

For 1 we take

$$1 = 2^0$$
$$1 = 2^1 - 2^0$$

thus, it can be written in two ways in the form of equation (4.1). Both the odds of equation (4.1) and the evens of the equation (4.8) are positive. Thus, 0 cannot be written either in the form of equation (4.1) or in the form of equation (4.8). $\square$

In order to write an odd number $\Pi \neq 1, 3$ in the form of equation (4.1) we initially define the $\nu \in \mathbb{N}^*$ from inequality (4.4). Then, we calculate the sum

$$2^{\nu+1} + 2^{\nu}.$$

If it holds that $2^{\nu+1} + 2^{\nu} < \Pi$ we add the $2^{\nu-1}$, whereas if it holds that $2^{\nu+1} + 2^{\nu} > \Pi$ then we subtract it. By repeating the process exactly ν times we write the odd number Π in the form of equation (4.1). The number of ν steps needed in order to write the odd number Π in the form of equation (4.1) is extremely low compared to the magnitude of the odd number Π, as derived from inequality (4.4).

Example 4.1. For the odd number $\Pi = 23$ we obtain from inequality (4.4)

$$2^{\nu+1} + 1 < 23 < 2^{\nu+2} - 1$$

$$2^{\nu+1} + 2 < 24 < 2^{\nu+2}$$

$$2^{\nu} < 12 < 2^{\nu+1}$$

thus $\nu = 3$. Then, we have

$$2^{\nu+1} + 2^{\nu} = 2^4 + 2^3 = 24 > 23 \text{ (thus } 2^2 \text{ is subtracted)}$$

$$2^4 + 2^3 - 2^2 = 20 < 23 \text{ (thus } 2^1 \text{ is added)}$$

$$2^4 + 2^3 - 2^2 + 2^1 = 22 < 23 \text{ (thus } 2^0 = 1 \text{ is added)}$$

$$2^4 + 2^3 - 2^2 + 2^1 + 1 = 23.$$

Fermat numbers F_s can be written directly in the form of equation (4.1), since they are of the form $\Pi_{\min}$,

$$F_s = 2^{2^s} + 1 = \Pi_{\min}\left(2^s - 1\right) = 2^{2^s} + 2^{2^s - 1} - 2^{2^s - 2} - 2^{2^s - 3} - \ldots\ldots - 2^1 - 1.$$
$$s \in \mathbb{N}$$
(4.10)

Mersenne numbers M_p can be written directly in the form of equation (4.1), since they are of the form $\Pi_{\max}$,

$$M_p = 2^p - 1 = \Pi_{\max}\left(p - 2\right) = 2^{p-1} + 2^{p-2} + 2^{p-3} + \ldots\ldots + 2^1 + 1.$$
$$p = prime$$
(4.11)

In order to write an even number α that is not a power of 2 in the form of equation (4.1), initially it is consecutively divided by 2 and it takes of the form of equation (4.6). Then, we write the odd number Π in the form of equation (4.1).

Example 4.2. By consecutively dividing the even number $\alpha = 368$ by 2 we obtain

$\alpha = 368 = 2^4 \cdot 23$.

Then, we write the odd number $\Pi = 23$ in the form of equation (4.1),

$23 = 2^4 + 2^3 - 2^2 + 2^1 + 1$,

and we get

$368 = 2^4 \left(2^4 + 2^3 - 2^2 + 2^1 + 1 \right)$

$368 = 2^8 + 2^7 - 2^6 + 2^5 + 2^4$.

This equation gives the unique way in which the even number $\alpha = 368$ can be written in the form of equation (4.9).

From inequality (4.4) we obtain

$2^{\nu+1} + 1 \leq \Pi \leq 2^{\nu+2} - 1$

$2^{\nu+1} < 2^{\nu+1} + 1 \leq \Pi \leq 2^{\nu+2} - 1 < 2^{\nu+2}$

$2^{\nu+1} < \Pi < 2^{\nu+2}$

$(\nu+1)\log 2 < \log \Pi < (\nu+2)\log 2$

from which we get

$\dfrac{\log \Pi}{\log 2} - 1 < \nu + 1 < \dfrac{\log \Pi}{\log 2}$

and finally

$$\nu + 1 = \left[\dfrac{\log \Pi}{\log 2} \right] \tag{4.12}$$

where $\left[\dfrac{\log \Pi}{\log 2} \right]$ the integer part of $\dfrac{\log \Pi}{\log 2} \in \mathbb{R}$.

We now give the following definition:

Definition 4.1. *We define as the conjugate of the odd*

$\Pi = \Pi(\nu, \beta_i) = 2^{\nu+1} + 2^\nu + \displaystyle\sum_{i=0}^{\nu-1} \beta_i 2^i$

$\beta_i = \pm 1, i = 0,1,2,\ldots\ldots,\nu-1$ \hfill (4.13)

$\nu \in \mathbb{N}^*$

the odd Π^*,

$$\Pi^* = \Pi^*\left(v,\gamma_j\right) = 2^{v+1} + 2^v + \sum_{j=0}^{v-1}\gamma_j 2^j$$

$$\gamma_i = \pm 1, \, j = 0,1,2,........,v-1 \qquad\qquad (4.14)$$

$$v \in \mathbb{N}^*$$

for which it holds

$$\gamma_k = -\beta_k \, \forall k = 0,1,2,........,v-1. \qquad\qquad (4.15)$$

For conjugate odds, the following corollary holds:

Corollary 4.1. *For the conjugate odds* $\Pi = \Pi\left(v,\beta_i\right)$ *and* $\Pi^* = \Pi^*\left(v,\gamma_i\right)$ *the following hold:*

1. $\left(\Pi^*\right)^* = \Pi.$ $\qquad\qquad (4.16)$

2. $\Pi + \Pi^* = 3\cdot 2^{v+1}.$ $\qquad\qquad (4.17)$

3. Π *is divisible by* 3 *if and only if* Π^* *is divisible by* 3.

Proof. 1. The 1 of the corollary is an immediate consequence of definition 4.1.

2. From equations (4.13), (4.14) and (4.15) we get

$$\Pi + \Pi^* = \left(2^{v+1} + 2^v\right) + \left(2^{v+1} + 2^v\right)$$

and equivalently

$$\Pi + \Pi^* = 3\cdot 2^{v+1}.$$

3. If the odd Π is divisible by 3 then it is written in the form $\Pi = 3x, x = odd$ and from equation (4.17) we get $3x + \Pi^* = 3\cdot 2^{v+1}$ and equivalently $\Pi^* = 3\left(2^{v+1} - x\right)$. Similarly we can prove the inverse. $\square$

5 The T symmetry and a method for defining large prime numbers

We now give the following definition:

Definition 5.1. *Define as "symmetry" every specific algorithm which determines the signs of* $\beta_i = \pm 1, i = 0,1,2,........,v-1$ *in equation* (4.1):

$$\Pi = \Pi\left(v,\beta_i\right) = 2^{v+1} + 2^v \pm 2^{v-1} \pm 2^{v-2} \pm........\pm 2^1 \pm 2^0 = 2^{v+1} + 2^v + \sum_{i=0}^{v-1}\beta_i 2^i$$

$$\beta_i = \pm 1, i = 0,1,2,........,v-1 \qquad\qquad .$$

$$v \in \mathbb{N}$$

Next, we develop a specific symmetry, the *T symmetry.*

If the natural number v, in the equation (4.1), is not a prime and is not a power of 2, the equation (2.1) gives

$$v = \mu(k,n) = \frac{(n+1)(2k+n)}{2} = k + (k+1) + (k+2) + \ldots\ldots + (k+n)$$

(5.1)

$$k \in \mathbb{N}^*, n \in A = \{2,3,4,\ldots\}$$

We define the odd number $T_1 = \Pi(v = \mu(k,n)) = T_1(k,n)$ as follows: In the right side of equation (4.1), from left to right, we take k signs -1, and then (k+1) signs +1, (k+2) signs -1, (k+3) signs +1 etc., according to the right side of equation (5.1). After making some calculations we have

$$T_1 = T_1(k,n) = 2^{\frac{(n+1)(2k+n)}{2}+1} + \left(\sum_{j=0}^{n} (-1)^j \times 2^{\frac{(n+1)(2k+n)}{2}+1-\sum_{i=0}^{j}(k+i)} \right) - (-1)^n$$

$$= 2^{\mu(k,n)+1} + \left(\sum_{j=0}^{n} (-1)^j \times 2^{\mu(k,n)+1-\mu(k,j)} \right) - (-1)^n$$

(5.2)

$$= 2^{\mu(k,n)+1} + \left(\sum_{j=0}^{n-1} (-1)^j \times 2^{\mu(k,n)+1-\mu(k,j)} \right) + (-1)^n$$

$$k \in \mathbb{N}^*, n \in A$$

and

$$T_1^* = T_1^*(k,n) = 3 \times 2^{\mu(k,n)+1} - T_1(k,n)$$

and equivalently

$$T_1^* = T_1^*(k,n) = 2^{\frac{(n+1)(2k+n)}{2}+2} - \left(\sum_{j=0}^{n} (-1)^j \times 2^{\frac{(n+1)(2k+n)}{2}+1-\sum_{i=0}^{j}(k+i)} \right) + (-1)^n$$

$$= 2^{\mu(k,n)+2} - \left(\sum_{j=0}^{n} (-1)^j \times 2^{\mu(k,n)+1-\mu(k,j)} \right) + (-1)^n$$

(5.3)

$$= 2^{\mu(k,n)+2} - \left(\sum_{j=0}^{n-1} (-1)^j \times 2^{\mu(k,n)+1-\mu(k,j)} \right) - (-1)^n$$

$$k \in \mathbb{N}^*, n \in A$$

We write the equation (5.1) in the form

$$v = \mu(k,n) = \frac{(n+1)(2k+n)}{2} = (k+n) + (k+n-1) + (k+n-2) + \dots + k \quad .$$
(5.4)

$$k \in \mathbb{N}^*, n \in A$$

We define the odd number $T_2 = \Pi(v = \mu(k,n)) = T_2(k,n)$ by the same way as we defined $T_1 = \Pi(v = \mu(k,n)) = T_1(k,n)$ but the signs in equation (4.1) are now determined according to the right side of equation (5.4), (k+n) signs -1, (k+n-1) signs +1, (k+n-2) signs -1, (k+n-3) signs +1 etc. After making some calculations we have

$$T_2 = T_2(k,n) = 2^{\frac{(n+1)(2k+n)}{2}+1} + \left(\sum_{j=0}^{n} (-1)^j \times 2^{\frac{(n+1)(2k+n)}{2}+1-\sum_{i=0}^{j}(k+n-i)} \right) - (-1)^n$$

$$= 2^{\mu(k,n)+1} + \left(\sum_{j=0}^{n} (-1)^j \times 2^{\mu(k,n)+1-\mu(k+n-j,j)} \right) - (-1)^n$$
(5.5)

$$= 2^{\mu(k,n)+1} + \left(\sum_{j=0}^{n-1} (-1)^j \times 2^{\mu(k,n)+1-\mu(k+n-j,j)} \right) + (-1)^n$$

$$k \in \mathbb{N}^*, n \in A$$

and

$$T_2^* = T_2^*(k,n) = 3 \times 2^{\mu(k,n)+1} - T_2(k,n)$$

and equivalently

$$T_2^* = T_2^*(k,n) = 2^{\frac{(n+1)(2k+n)}{2}+2} - \left(\sum_{j=0}^{n} (-1)^j \times 2^{\frac{(n+1)(2k+n)}{2}+1-\sum_{i=0}^{j}(k+n-i)} \right) + (-1)^n$$

$$= 2^{\mu(k,n)+2} - \left(\sum_{j=0}^{n} (-1)^j \times 2^{\mu(k,n)+1-\mu(k+n-j,j)} \right) + (-1)^n \qquad .$$
(5.6)

$$= 2^{\mu(k,n)+2} - \left(\sum_{j=0}^{n-1} (-1)^j \times 2^{\mu(k,n)+1-\mu(k+n-j,j)} \right) - (-1)^n$$

$$k \in \mathbb{N}^*, n \in A$$

Equations (5.2), (5.3), (5.5) and (5.6) define the T symmetry.

A method for the determination of large prime numbers emerges from the study we presented. This method is completely different from previous methods [3-7]. For the T symmetry holds:

"*There are pairs* $(k,n) \in \mathbb{N}^* \times A$,

$$n \neq 3 + 4L, L \in \mathbb{N} \ ,$$
(5.7)

for which one or more of $T_1(k,n)$, $T_1^*(k,n)$, $T_2(k,n)$, $T_2^*(k,n)$ *are prime numbers."*

We will present three examples:

1. The number

$T_2(11,5) = 2^{82} + 2^{66} - 2^{51} + 2^{37} - 2^{24} + 2^{12} - 2^1 + 1 = 4\,835777\,063183\,149145\,526271$ is a prime.

The number

$T_1^*(11,5) = 2^{83} - 2^{71} + 2^{59} - 2^{46} + 2^{32} - 2^{17} + 2^1 - 1 = 9\,669045\,950065\,986429\,124609$ is a prime.

2. The number

$T_1(23,4) = 2^{126} + 2^{103} - 2^{79} + 2^{54} - 2^{28} + 2^1 - 1 = 85\,070601\,871438\,813228\,787070\,915221$
389313 is a prime.

3. The number $T_1^*(80,2) = 2^{245} - 2^{164} + 2^{83} - 2^1 + 1 = 56\,539106\,072908\,298546\,665496\,639747$
195212 032793 441072 154605 920000 979840 794623 (74 digits) is a prime.

The number D of digits of the primes calculated by the method is of order

$$D = D(k,n) = (\mu(k,n)+1)\log 2 = \left(\frac{(n+1)(2k+n)}{2}+1\right)\log 2.\qquad (5.8)$$

The smallest prime number given by the method is $T_1(2,2) = 2^{10} + 2^8 - 2^5 + 2^1 - 1 = 1249$. Also, it doesn't give prime numbers Fermat and Mersenne.

We now cite some remarkable properties of the T symmetry. When the numbers of the T symmetry are not primes, with high probability, one or more of them are the product of a set of small primes with a large prime (with ratio of the number of digits at least 3:1 in the decimal system). We give an example for n=4 and k=1, 2, 3,......., 23.

Example 5.1.

1. $T_1^*(1,4) = 5 \times 21107$.

2. $T_1(2,4) = 3 \times 853291$

   $T_2(2,4) = 3 \times 709651$

   $T_2^*(2,4) = 3 \times 1\,387501$.

3. $T_1^*(3,4) = 126\,337279$ (9 digits) *is a prime*

   $T_2^*(3,4) = 133\,701391$ (9 digits) *is a prime.*

4. $T_2^*(4,4) = 3 \times 13 \times 109\ 913929$.

5. $T_2(5,4) = 68853\ 174209$ (11 digits) *is a prime*

$\quad T_2^*(5,4) = 19 \times 7226\ 592421$.

6. $T_2(6,4) = 3 \times 7 \times 104817\ 455293$.

7. $T_2(7,4) = 37 \times 1\ 902785\ 687213$

$\quad T_2^*(7,4) = 11 \times 12\ 791196\ 555101$.

8. $T_1(8,4) = 3 \times 47 \times 16\ 032473\ 358917$.

9. $T_1^*(11,4) = 1301 \times 113403\ 483925\ 962179$.

10. $T_1^*(12,4) = 3 \times 13 \times 121\ 071540\ 832866\ 439273$

$\quad T_2(12,4) = 3 \times 7 \times 112\ 439012\ 815828\ 430653$

$\quad T_2^*(12,4) = 3 \times 89 \times 17\ 686630\ 918247\ 456093$.

11. $T_1^*(13,4) = 5 \times 30221\ 300928\ 544913\ 175347$

$\quad T_2^*(13,4) = 2239 \times 67\ 492251\ 451483\ 773121$.

12. $T_1^*(14,4) = 3 \times 19 \times 107 \times 792\ 844025\ 087630\ 419877$

13. $T_1^*(15,4) = 23 \times 6\ 727832\ 337541\ 722681\ 821273$.

14. $T_2(16,4) = 3 \times 825\ 294146\ 583166\ 134057\ 740971$.

15. $T_2(17,4) = 3541 \times 22\ 374527\ 052572\ 768094\ 438269$.

16. $T_1^*(18,4) = 3 \times 73 \times 2903 \times 7\ 975677\ 388569\ 543733\ 588379$.

17. $T_2^*(19,4) = 11 \times 641 \times 23012\ 234740\ 860744\ 903766\ 035421$.

18. $T_1^*(20,4) = 3 \times 6\ 643069 \times 260\ 536928\ 672371\ 642740\ 686521$.

19. $T_2(21,4) = 7 \times 79 \times 150\ 229208\ 340754\ 381651\ 561471\ 195673$ (33 digits).

20. $T_1(22,4) = 3 \times 29 \times 1259 \times 24\ 270828\ 201501\ 431550\ 885053\ 400181$ (32 digits)

$\quad T_1^*(22,4) = 3 \times 1933 \times 916\ 866933\ 835909\ 456002\ 715952\ 336617$ (33 digits).

21. $T_1(23,4)$ = 85 070601 871438 813228 787070 915221 389313 (38 digits) *is a prime*

   $T_1^*(23,4)$ =18269 × 9313 108178 842029 359502 101081 537291 (34 digits)

   $T_2^*(23,4)$ =19 × 89 × 100615 720181 338817 896100 110722 568301 (36 digits).

   For

$$n = 3 + 4L, L \in \mathbb{N} \qquad (5.9)$$

the numbers of the T symmetry have 3 as a factor. In these cases, we factorize the numbers of the T symmetry in order to identify the ones which are the product of a set of small primes with a large prime (with ratio of the number of digits at least 3:1 in the decimal system). We give an example for L=0 and k=1, 2, 3,…….., 33.

Example 5.2.

1. $T_2^*(1,3)$ =3 × 1327.

2. $T_2^*(2,3)$ =3 × 21523.

3. $T_2^*(4,3)$ =3 × 5 570891.

4. $T_1(5,3)$ =3 × 46 115669.

5. $T_2^*(7,3)$ =3 × 5 × 4579 065839.

6. $T_2^*(9,3)$ =3^2 × 1 954448 845369.

7. $T_1(10,3)$ =3 × 73 × 643264 201901.

8. $T_1(11,3)$ = 3 × 5 × 23 × 6 530142 193943.

9. $T_1(12,3)$ = 3^2 × 4004 176893 145543.

10. $T_2(13,3)$ =3 × 7 × 31 × 107 × 151 × 54806 826689.

11. $T_1(14,3)$ = 3 × 11 × 279513 180897 836063.

12. $T_1^*(16,3)$ =3^3 × 7 × 19 × 73 × 18014 329790 791679.

13. $T_1(17,3)$ =3 × 12593 073364 077934 630229.

14. $T_1(18,3)$ =3^2 × 9239 × 7 269488 227993 959889.

15. $T_1(19,3)$ =3 × 5 × 73 × 331 × 26 683841 696377 422587

$T_1^*(19,3) = 3 \times 7^2 \times 127 \times 269 \times 337 \times 11429\ 204013\ 400937$

$T_2^*(19,3) = 3 \times 5 \times 557 \times 2315\ 117990\ 184578\ 945803.$

16. $T_2(20,3) = 3 \times 23 \times 89 \times 683 \times 9041 \times 4080\ 688125\ 380017.$

17. $T_1^*(21,3) = 3 \times 47 \times 178481 \times 196\ 765246\ 663328\ 879957$

$T_2^*(21,3) = 3^2 \times 83 \times 23473 \times 282\ 403703\ 315945\ 507251.$

18. $T_2^*(22,5) = 3 \times 5273 \times 5\ 008417\ 809828\ 231066\ 746851.$

19. $T_2^*(23,3) = 3 \times 5^2 \times 137 \times 211 \times 379 \times 1542\ 751819\ 716389\ 148523.$

20. $T_2(24,3) = 3 \times 2731 \times 7487 \times 8191 \times 20\ 183749\ 276015\ 547071.$

21. $T_1(25,3) = 3 \times 13 \times 269 \times 636697 \times 24291\ 809038\ 323309\ 475541.$

22. $T_1(26,3) = 3 \times 47 \times 829 \times 22210\ 374525\ 858205\ 252016\ 927831$

$T_2(26,3) = 3 \times 5 \times 13 \times 29 \times 43 \times 83 \times 113 \times 127 \times 151 \times 59\ 359638\ 928368\ 977041.$

23. $T_1^*(27,3) = 3 \times 233 \times 1103 \times 2089 \times 48091 \times 1072\ 567317\ 671651\ 381903$

$T_2^*(27,3) = 3^2 \times 5 \times 23 \times 139 \times 257 \times 6\ 560737 \times 342\ 482665\ 485076\ 269161.$

24. $T_1^*(28,3) = 3^2 \times 7 \times 11 \times 31 \times 151 \times 331 \times 1237\ 940038\ 132458\ 773513\ 502719.$

25. $T_2^*(30,3) = 3^2 \times 5801 \times 288383 \times 22600\ 831355\ 114079\ 948328\ 119407.$

26. $T_1^*(31,3) = 3 \times 7 \times 23 \times 83 \times 89 \times 109 \times 599479 \times 23\ 353056\ 263230\ 084539\ 231539.$

27. $T_1(32,3) = 3 \times 37 \times 392\ 397684\ 468660\ 613729\ 344084\ 167488\ 872817$ (39 digits).

28. $T_1(33,3) = 3^2 \times 23 \times 857 \times 3\ 928422\ 863348\ 787826\ 215906\ 015441\ 564473$ (37 digits)

$T_2^*(33,3) = 3^4 \times 20\ 286419 \times 848\ 220926\ 630659\ 241732\ 391340\ 317419$ (33 digits).

Fermat and Mersenne, for odds $N \neq 3$ of the form $N = 2^n + 1 = 2^n + 1^n, n \in \mathbb{N}^*$ and $N = 2^n - 1 = 2^n - 1^n, n \in \mathbb{N}^*$, respectively, chose the values of $n \in \mathbb{N}$ for which the odd N, firstly, does not have 3 as a factor ($n = 2^s, s \in \mathbb{N}$ and $n = prime$, respectively). This has as a consequence that the Fermat and Mersenne numbers are not divisible by 3, that is, they are not divisible by ⅓ of the odd numbers (that are smaller than N). This non-divisibility by 3, is a property of the numbers of the T symmetry for n=5. Consequently, the odds $T_1(k,5)$, $T_1^*(k,5)$,

$T_2(k,5)$, $T_2^*(k,5)$, $k \in \mathbb{N}^*$ are not divisible by ⅓ of the odd numbers (that are smaller than $T_1(k,5)$, $T_1^*(k,5)$, $T_2(k,5)$, $T_2^*(k,5)$, $k \in \mathbb{N}^*$). Because of this, the method is particularly efficient for n=5. We give an example for n=5 and for small values of k, k=1, 2, 3,........,18.

Example 5.3.

1. $T_2(2,5)$ =270 500807 (9 digits) *is a prime.*

2. $T_2(3,5)$ =17246 461711 (11digits) *is a prime*

   $T_1^*(3,5)$ =32342 343169 (11 digits) *is a prime.*

3. $T_1^*(4,5)$ =2 132417 969153 (13 digits) *is a prime.*

4. $T_1^*(8,5)$ =36 821571 153497 669633 (20 digits) *is a prime.*

5. $T_2(9,5)$ =1180 663669 517502 645247 (22 digits) *is a prime.*

6. $T_1(10,5)$ =75631 614682 207162 007551 (23 digits) *is a prime.*

7. $T_2(11,5)$ =4 835777 063183 149145 526271 (25 digits) *is a prime*

   $T_1^*(11,5)$ =9 669045 950065 986429 124609 (25 digits) *is a prime.*

8. $T_1^*(12,5)$ =618 894471 001773 327207 104513 (27 digits) *is a prime.*

9. $T_2(16,5)$ =5192 299334 412545 020553 193752 494079 (34 digits) *is a prime.*

10. $T_1^*(18,5)$ =42 535214 735633 635683 576920 453379 260417 (38 digits) *is a prime.*

From the identity of the Euclidean division, we have that the equations

$n = 3 + 4L, L \in \mathbb{N}^*$
$n = 2 + 4L, L \in \mathbb{N}$
$n = 1 + 4L, L \in \mathbb{N}^*$
$n = 4L, L \in \mathbb{N}^*$

give all values for $n \in A = \{2,3,4,...\}$.

For $n = 2 + 4L, L \in \mathbb{N}$ the numbers of T symmetry give prime numbers only for even values of k:

$(k,n) = (2S, 2 + 4L)$
$S \in \mathbb{N}^*, L \in \mathbb{N}$.

(5.10)

For $n = 1 + 4L, L \in \mathbb{N}^*$ the numbers of T symmetry give prime numbers for both, even and odd values of k:

$$(k,n) = (S, 1+4L)$$
$$S, L \in \mathbb{N}^* \qquad . \qquad\qquad\qquad\qquad (5.11)$$

For $n = 4L, L \in \mathbb{N}^*$ the numbers of T symmetry give prime numbers only for odd values of k:

$$(k,n) = (2S-1, 4L)$$
$$S, L \in \mathbb{N}^* \qquad . \qquad\qquad\qquad\qquad (5.12)$$

The values of sequence μ(k,n) for the pairs (k,n) of equations (5.10), (5.11) and (5.12), are odd numbers. So the numbers of T symmetry give prime numbers only in cases where the sequence

$$\mu(k,n) = \frac{(n+1)(2k+n)}{2}$$

In equations (5.2), (5.3), (5.5) and (5.6) is an odd number.

From the above study it emerges that the method is applied in two ways:

a. *We factorize the numbers of the T symmetry and identify the ones that are products of a set of prime numbers with a comparatively larger prime number.*

b. *We identify the prime numbers of the T symmetry, via a primality test, when the equations (5.10), (5.11), (5.12) hold.*

We suggest, in both cases, that a specific $n \in A = \{2, 3, 4, ...\}$ should be chosen, and then the values k=1, 2, 3… can be given in equations (5.2), (5.3), (5.5) and (5.6). The method may be further investigated for the form of the pairs $(k,n) \in \mathbb{N}^* \times A$ in equations (5.2), (5.3), (5.5) and (5.6).

The observations above have high theoretical interest, but they have not been completely proved. During the application of the method, it is necessary, a primality test to be done, for all possible primes of T symmetry.

6. A factorization test

The corollary gives a factorization test for the odd numbers $\Pi \geq 9$.

Corollary 6.1.1. *Every odd number $\Pi \geq 9$ is composite if and only if there exists an odd number f such that*

$$(2h-1)^2 + 8\Pi = f^2$$

$$h \in \mathbb{N}^*, f \in \mathbb{N}, f = odd \tag{6.1}$$

$$\sqrt{8\Pi} < f \le \sqrt{8\Pi + 1} + 2h - 2$$

and then

$$\Pi = \mu(h,c) = \frac{(c+1)(2h+c)}{2} \tag{6.2}$$

$$c = \frac{f - (2h+1)}{2} \in A \quad . \tag{6.3}$$

2. The biggest number S=S(Π) of operations required for the factorization of the odd number Π depends on the value of the parameter $h \in \mathbb{N}^*$ and derives from the equation

$$S = S\left(\Pi = \mu(h,c)\right) = \frac{\left[\sqrt{8\Pi + 1}\right] - \left[\sqrt{8\Pi}\right] + 2h - 2}{2} = h - 1 \quad . \tag{6.4}$$

Proof. 1. According to theorem 2.1 every composite odd number Π can be written in the form of the equation

$$\Pi = \mu(h,c) = \frac{(c+1)(2h+c)}{2}$$

$$(h,c) \in \mathbb{N}^* \times A$$

and we have

$$c^2 + (2h+1)c + 2h - 2\Pi = 0 \quad . \tag{6.5}$$

This equation is of second order with respect to $c \in A$ and the determinant D of the equation (6.5) is a square of a natural number:

$$D = (2h-1)^2 + 8\Pi = f^2. \tag{6.6}$$

From the equations (6.5) and (6.6) we have

$$c = \frac{f - (2h+1)}{2} \in A \text{ which is the equation (6.3).}$$

In equation (6.3) the natural number $(2h+1)$ is odd and consequently f is also odd. From the equation (6.6) we have that $8\Pi < f^2$ and finally $\sqrt{8\Pi} < f$. From the equation (6.2) we have

$$\frac{\Pi}{c+1} - \frac{c}{2} = h \ge 1$$

and so

$$c \leq \frac{\sqrt{8\Pi+1}-3}{2}$$

and in combination with the equation (6.3) we have

$$\frac{f-(2h+1)}{2} \leq \frac{\sqrt{8\Pi+1}-3}{2}$$

And finally we have $f \leq \sqrt{8\Pi+1}+2h-2$.

2. From the inequality of relation (6.1) we have that the odd number f belongs to the closed interval $\Delta=\Delta(\Pi)$:

$$f \in \Delta = \Delta\left(\Pi = \mu(h,c)\right) = \left[\left[\sqrt{8\Pi}\right]+1, \left[\sqrt{8\Pi+1}\right]+2h-2\right] . \tag{6.7}$$

Consequently the biggest number of operation required is

$$S = S\left(\Pi = \mu(h,c)\right) = \frac{\left[\sqrt{8\Pi+1}\right]-\left[\sqrt{8\Pi}\right]+2h-2}{2} = h-1$$

in the case where the number f takes all odd values in the interval Δ. □

The factorization of the odd number Π can be done by two ways:

By giving to the natural number h the values h=1, 2, 3… in the equation (6.1) until we have an odd value for f. Then, from equation (6.3) we calculate $c \in A$ and take the odd Π factorized in the form $\Pi = \mu(h,c) = \frac{(c+1)(2h+c)}{2}$ (2 factors).

By giving to the odd number f values $f > \sqrt{8\Pi}$ until we have a natural number $h \in \mathbb{N}^*$ from equation (6.1). Then, from the equation (6.3) we calculate $c \in A$ and take the odd Π factorized in the form $\Pi = \mu(h,c) = \frac{(c+1)(2h+c)}{2}$ (2 factors).

From the equation (6.4) we conclude that the biggest number of operations required S=S(Π) for the factorization of the odd number Π is minimized when the number h takes the smallest possible value in the rearrangements of Π=μ(h, c) (see chapter 3). By comparing our factorization test with the sieve of Eratosthenes: the number of operations required for the factorization of an odd number Π by the sieve of Eratosthenes is approximately $\frac{\sqrt{\Pi}}{\ln\sqrt{\Pi}}$.

Therefor, the factorization test is efficient for the odd numbers Π for which we have

$$S\left(\Pi = \mu(h,c)\right) < \frac{\sqrt{\Pi}}{\ln\sqrt{\Pi}}$$

and equivalently

$$h < \frac{\sqrt{\Pi}}{\ln\sqrt{\Pi}} + 1 \ . \tag{6.8}$$

As we can conclude from equation (6.4). The test is very effective for the odd numbers Π for which we have

$$h << \frac{\sqrt{\Pi}}{\ln\sqrt{\Pi}} + 1 \ . \tag{6.9}$$

From equations (2.4) and (2.5), where k=h and n=c, we have that parameter h takes small values, and equivalently *the test is effective when an odd number Π is a product of two odd numbers χ and ψ, $\chi<\psi$ and ψ be about twice as high as χ:*

$$\frac{\psi}{2\chi} \sim 1 \ . \tag{6.10}$$

This observation leads to the concept of *"rearrangement multiplier": If the odd number Π cannot be factorized by the test then we multiply it by an odd ξ (*rearrangement multiplier*) so that $\xi\Pi$ is product of two odd numbers χ and ψ, $\chi<\psi$ and ψ be about twice as high as χ. Then we factorize $\xi\Pi$ by the test. In factorization of $\xi\Pi$ the biggest factor of Π is appeared.* Next we can see seven examples.

Example 6.1. We apply the test with the first way mentioned above for the odd Π=289 034935 520125 533903. The test factorizes in just 1 operation: 289 034935 520125 533903 = 12021541821×24043083643.

Example 6.2. We apply the test with the first way mentioned above for the odd Π= 499999 994488 500015 187991. The test factorizes in 26 operations: 499999 994488 500015 187991 = 499999 997257×999999.

Example 6.3. We apply the test with the first way mentioned above for the odd Π=4988 007042 023480 817299 187879 152021 079829 121957. The test factorizes in 8040399872594 operations: Π= 4988 007042 023480 817299 187879 152021 079829 121957= 49939 999213 188599 848703×99879 998410 296399 952219. We apply the test with the second way mentioned above for the odd Π=4988 007042 023480 817299 187879 152021 079829 121957. The test factorizes in 324 operations: Π= 4988 007042 023480 817299 187879 152021 079829 121957= 49939 999213 188599 848703×99879 998410 296399 952219.

Example 6.4. We apply the test with the first way mentioned above for the odd Π=107262 463439 540776 796592 199985 646769 019834 926564 739147 021788 491549 774112 240589 533735 307367 547289 463131 370341 744570 421106 676687 468135 967135 411548 323345 072121 (156 digits). The test factorizes in 5 operations:
Π=46316835694926478169428394003475163141307993866256225615783033 60316 52518559

751×2315841784746323908471419700173758157065399693312811280789151680158262592799871.

Example 6.5. We apply the test with the first way mentioned above for the odd Π= 1 339385 758982 834151 185531 311325 002263 201756 014631 917009 304687 985462 938813 906170 153116 497973 519619 822659 493341 146941 433531 483931 607115 392554 498072 196841 413588 751691 149190 545360 298668 824573 067195 292298 103755 984923 583067 909903 930689 138499 780608 667013 060760 007914 870267 859719 075299 638934 789671 311367 995385 (301digits). The test factorizes in 5 operations:

Π=16366953039480709350065948484137995761083210230215323947416456840480668982023372774416350461629520785754433420637800355046086282729426965266642637946 95×81834765197403546750329742420689978805416051151076619737082284202403344910116863872081752308147603928772167103189001775230431413647134826333213189 7343.

Example 6.6. We apply the test with the first way mentioned above for the odd Π=11917. The test factorizes in 334 operations which is an extremely high number for such a small number. We apply the test with the second way mentioned above for the odd Π=11917. The test factorizes in 214 operations.

For ξ=21 we can factorizes 21×Π=21×11917= 250257 and by applying the test by the first way mentioned above we can take, in 7 operations performed, ξΠ= 250257 =701×357. The odd number 701 is the biggest factor of the odd Π=11917.

For ξ=83 we can factorizes 83×Π=83×11917=989111 and by applying the test by the first way mentioned above we can take, in 5 operations performed, ξΠ= 989111=701×1411. The odd number 701 is the biggest factor of the odd Π=11917.

Example 6.7. We apply the test with the first way mentioned above for the odd Π=374757029. The test factorizes in 182504 operations which is an extremely high number for such a small number. We apply the test with the second way mentioned above for the odd Π=374757029. The test factorizes in 157169 operations.

For ξ=717 we can factorizes 717×Π=717×374757029= 268700789793 and by applying the test by the first way mentioned above we can take, in 1021 operations performed, ξΠ= 268700789793 =367049×732057. The odd number 367049 is the biggest factor of the odd Π=374757029.

For ξ=719 we can factorizes 719×Π=719×374757029= 269450303851 and by applying the test by the first way mentioned above we can take, in just 1 operation performed, ξΠ= 269450303851 =367049×734099. The odd number 367049 is the biggest factor of the odd Π=374757029.

For ξ=721 we can factorizes 721×Π=721×374757029=270199817909 and by applying the test by the first way mentioned above we can take, in 1022 operations performed, ξΠ=

270199817909 =367049×736141. The odd number 367049 is the biggest factor of the odd Π=374757029.

For ξ=179 we can factorizes 179×Π=179×374757029= 67081508191 and by applying the test by the first way mentioned above we can take, in 766 operations performed, ξΠ=67081508191= 367049×184801. The odd number 367049 is the biggest factor of the odd Π=374757029.

We prove now the following corollary for the rearrangement multiplier:

Corollary 6.2. *For every odd number Π, Π=χψ, χ<ψ, χ, ψ odd numbers, the rearrangement multipliers ξ are always pairs* (ξ_1, ξ_2), $\xi_1 < \xi_2$ *for which the following equations hold*

$$\xi_2 = 4\xi_1 \pm 1$$
$$\xi_1 \in \mathbb{N}^*, \xi_1 = odd \tag{6.11}$$

$$\xi_1 \sim \frac{\xi_2 \pm 1}{4}$$
$$\xi_1, \xi_2 \in \mathbb{N}^* \qquad \qquad . \tag{6.12}$$
$$\xi_1, \xi_2 = odd \ numbers$$

Prof. For the odd number Π, Π=χψ, χ<ψ, χ, ψ odd numbers, we have

$$\xi\Pi=\xi\chi\psi. \tag{6.13}$$

From equation (6.4) for the odd number ξΠ and from equations (2.4), (2.5) we have that the odd number ξΠ is factorized by the test in the following cases:

For $2\xi\chi \sim \psi$ and we have

$$\xi = \xi_1 \sim \frac{\psi}{2\chi} = \frac{\Pi}{2\chi^2} = \frac{\psi^2}{2\Pi} \ . \tag{6.14}$$

For $\xi\chi \sim 2\psi$ and we have

$$\xi = \xi_2 \sim \frac{2\psi}{\chi} = \frac{2\Pi}{\chi^2} = \frac{2\psi^2}{\Pi} \ . \tag{6.15}$$

By cancelling χ and ψ from the equations (6.14), (6.15) we have

$$\xi_2 \sim 4\xi_1 \ . \tag{6.16}$$

In relation (6.16) the numbers ξ_1 and ξ_2 are odd and consequently we have $\xi_2 = 4\xi_1 \pm 1$ which is exactly the equation (6.11). The equation (6.12) results from relation (6.11) whereas the numbers ξ_1 and ξ_2 are odd. □

For the odd numbers of the examples 6.1, 6.2, 6.3, 6.4 and 6.5 we have that the relation (6.10) holds. In examples 6.6 and 6.7 we can see the consequences of the corollary 6.2.

We prove now the following corollary:

Corollary 6.3. *Every odd number* $\Pi \geq 9$ *is composite if and only if there exist natural numbers N and h so that to have*

$$\Pi = \frac{N(N-1)}{2} - \frac{h(h-1)}{2}$$

$$\frac{\sqrt{8\Pi}+1}{2} < N \leq \frac{\sqrt{8\Pi+1}}{2} + h \ . \tag{6.17}$$

$$h, N \in \mathbb{N}^*$$

Proof. We set

$$f = 2N - 1$$
$$N \in \mathbb{N}^* \tag{6.18}$$

in equation (6.1) and by making calculations we take the equation (6.17). □

The equations (6.1) and (6.17) are equivalent. From the equation (6.17) and by our test we have that the odd number Π can be written as a difference of *two triangle numbers. Consequently the test can factorize odd numbers which cannot be factorized by tests that are based on differences and on sums of natural number squares.* We underline that our test gives the odd number Π as a product of two factors, so many as the factors of the sequence μ(h,c) are. The test can also applied to even numbers.

As in the case of equations (6.1) our test can be applied in two ways:

For the natural number h we give values h=1,2,3,… in equation (6.17) until a natural number N to find out,

$$N \in \left(\frac{\sqrt{8\Pi}+1}{2}, \frac{\sqrt{8\Pi+1}}{2} + h \right] \ .$$

Next, from the equation (6.18) we calculate the odd number f and from equation (6.3) we calculate $c \in A$, and take the odd number Π factorized in the form $\Pi=\mu(h,c)$.

We give N values, $N > \dfrac{\sqrt{8\Pi}+1}{2}$ in equation (6.17) until a natural number $h \in \mathbb{N}^*$ to find out.

Next, from equation (6.18) next we calculate the odd number f and from equation (6.3) we calculate $c \in A$, and take the odd number Π factorized in the form $\Pi=\mu(h,c)$.

From corollaries 6.2 and 6.3 we can find out the following results for the usage of the test:

1. *The test factorizes odd numbers Π for which the equation (6.10) holds. The number of calculations required for the factorization depends on the internal structure of the odd number Π (from the quotient $\psi/2\chi$) and it is independent from the size of the odd number Π.*

2. *The test can factorize odd numbers for which the equation (6.10) doesn't hold, in the following way: From equation (6.11) we have that for every odd number candidate multiplier ξ_1 which is rejected, immediately all odd candidate multipliers of the form $\xi_2 = 4\xi_1 \pm 1$ are rejected. Consequently, every trial we do for a candidate multiplier ξ_1 is equivalent to three trials. If $\xi_1 = 1$ is rejected, then the candidate multipliers $\xi_2 = 3,5$ are rejected, too. So we go on by testing as candidate multiplier the next odd number which is $\xi_1 = 7$. If $\xi_1 = 7$ is rejected, then immediately the candidate multipliers $\xi_2 = 27, 29$ are rejected, too. Then we can test as candidate multiplier $\xi_1 = 9$ from $\xi_1 = 9,13,15,17,19,21,23,25$ and so on. The initial value of ξ_1 depended on the value of the odd number Π. In the same way we have a descending sequence of candidate ξ_2 which are rejected by relation (6.12).*

3. *Using a rearrangement multiplier, the test can factorize odd numbers Q which cannot be factorized by tests that are based on differences and on sums of natural number squares. The initial values of ξ_1 and ξ_2 are depended on the value of odd Q.*

The above presented factorization test may be combined with factorization tests that are based on differences and on sums of natural numbers squares. This combination could solve the problem of natural numbers factorization. For example, classical tests experience a difficulty in factorizing the odd number Π (301 digits) of example 6.5 ($\sim 1.2 \times 10^8$ operations). On the contrary, they can easily factorize the odd numbers

1 636695 303948 070935 006594 848413 799576 108321 023021 532394 741645 684048 066898 202337 277441 635046 162952 078575 443342 063780 035504 608628 272942 696526 664263 794695 $= 3^2 \times 5 \times 23 \times 61 \times 1237 \times 149173 \times 783\ 077369 \times 1\ 730874\ 200754\ 908897\ 438581 \times 103\ 649580\ 891650\ 334533\ 217110\ 728686\ 722681\ 891397\ 035344\ 668562\ 358179\ 839502\ 136821\ 566545\ 795337\ 979963\ 740075\ 089413$

and

818347 651974 035467 503297 424206 899788 054160 511510 766197 370822 842024 033449 101168 638720 817523 081476 039287 721671 031890 017752 304314 136471 348263 332131 897343 $= 3^2 \times 7 \times 167 \times 499 \times 1163 \times 2657 \times 155377 \times 1621\ 324657 \times 13455\ 809771 \times 9\ 202419\ 446683 \times 57912\ 614113\ 275649\ 087721 \times 3\ 388098\ 290567\ 587377\ 052016\ 525627\ 948593 \times 8241\ 594690\ 167137\ 359552\ 274418\ 432855\ 740327.$

Thus, by combining the two tests we obtain the factorization of the odd number Π:

Π= 1 339385 758982 834151 185531 311325 002263 201756 014631 917009 304687 985462 938813 906170 153116 497973 519619 822659 493341 146941 433531 483931 607115 392554

498072 196841 413588 751691 149190 545360 298668 824573 067195 292298 103755 984923 583067 909903 930689 138499 780608 667013 060760 007914 870267 859719 075299 638934 789671 311367 995385 (301digits) = ($3^2 \times 5 \times 23 \times 61 \times 1237 \times 149173 \times 783\ 077369 \times 1$ 730874 200754 908897 438581 $\times$ 103 649580 891650 334533 217110 728686 722681 891397 035344 668562 358179 839502 136821 566545 795337 979963 740075 089413)$\times$($3^2 \times 7 \times 167$ $\times 499 \times 1163 \times 2657 \times 155377 \times 1621\ 324657 \times 13455\ 809771 \times 9\ 202419\ 446683 \times 57912$ 614113 275649 087721 $\times$ 3 388098 290567 587377 052016 525627 948593 $\times$ 8241 594690 167137 359552 274418 432855 740327).

The tests that are based on differences and on sums of natural number squares are highly efficient when the following relationships hold

$$\chi << \psi \Leftrightarrow \frac{\psi}{2\chi} >> \frac{1}{2} \tag{6.19}$$

or

$$\chi \sim \psi \sim \sqrt{\Pi} \Leftrightarrow \frac{\psi}{2\chi} \sim \frac{1}{2} \tag{6.20}$$

for the odd numbers $\Pi=\chi\psi$, $\chi<\psi$, χ, ψ odd numbers. Therefore, there is an emerging need of studying the association between the two tests, based on relationships (6.10) and (6.19), (6.20). This study can provide the initial values of rearrangement multipliers ξ_1 and ξ_2 for the odd numbers Q that cannot be factorized by the classical tests. In some first applications that have been conducted, the determination of the initial value of ξ_1 was based on the statistical estimation of the relationship between ξ_1 and χ in equation (6.14) for $\Pi=Q$,

$$\xi_1 \sim \frac{Q}{2\chi^2} \ . $$
$$\xi_1 = \xi_1(\chi) \tag{6.21}$$

The results we have set out, as well as the applications of Chapters 5 and 6, can be further explored. This is expected because this is the first time we study the natural numbers by their sum and not by their product.

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