*Near completion*

# Bifurcations and the Dynamic Content of Particle Physics

Ervin Goldfain

Advanced Technology and Sensor Group, Welch Allyn Inc, Skaneateles Falls, NY

*Abstract*

We have recently conjectured that the flow from the ultraviolet (UV) to the infrared (IR) sector of any multivariable field theory approaches chaotic dynamics in a universal way. A key assumption of this conjecture is that the flow evolves in far-from-equilibrium conditions and it implies that the end-point attractor of effective field theories replicates the geometry of multifractal sets. Our conclusions are further reinforced here in the framework of nonlinear dynamical systems and bifurcation theory. In particular, it is found that steady-state perturbations near the IR attractor induce formation of Dark Matter structures while oscillatory perturbations lead to the field content of the Standard Model.

**Key words**: Bifurcations, Dynamical Systems, Strange Attractors, Center Manifold Theory, Normal Forms, Standard Model, Cantor Dust.

## 1. Introduction and motivation

The Renormalization Group (RG) is a well-established framework for the analysis of complex physical systems at both ends of the energy scale. Over the years, the principles and methods of RG have found a wide range of applications, from critical behavior in Statistical Physics and Condensed Matter to perturbative and non-perturbative models in Quantum Field Theory (QFT) [ ]. An appealing feature of RG equations is that they resemble the evolution equations of dynamical systems [ ]. In particular, the Callan-

Symanzik equation stems from the independence of QFT from its subtraction point, which is on par with self-similarity of autonomous flows approaching attractors. In the Wilsonian formulation of the RG, the flow in coupling space is associated with the trajectory of QFT towards a subspace of relevant and marginal operators. Conventional wisdom asserts that the attractors of the RG flow consist of a finite number of isolated fixed points (FP). There is now mounting evidence that this assumption is too restrictive, that RG flows – echoing the onset of turbulence in fluid mechanics – may evolve towards limit cycles or tori as well as strange attractors, the latter denoting invariant sets having chaotic structure [ ].

The goal of this work is to extrapolate the conventional RG paradigm to a framework which minimizes the potential loss of generality due to simplifying assumptions. To this end, we posit that all trajectories connecting the UV and IR sectors of a generic field theory are characterized by the following initial conditions:

a) a large count of independent or coupled variables,
b) a large count of independent or coupled control parameters,
c) far-from-equilibrium settings,
d) non-perturbative and non-integrable dynamics.

In our view, the motivation for this extended framework is that the combined use of a) to d) enable a more realistic picture of complex dynamics that is likely to define the UV to IR flow. This view is backed up by many examples. For instance, integrable dynamical systems are isomorphic to free, non-interacting theories, which are unable to account for the arrow of time in transient regimes, the physics of self-organization and complex evolution outside equilibrium [ ]. Another instance is provided by Sakharov's nonequilibrium conditions for baryogenesis and the observed baryon asymmetry of the Universe [ ].

This paper expands on the line of research initiated and developed in [ ] and is organized as follows: the interpretation of RG flows as autonomous dynamical systems is detailed in section 2. Section 3 delves into the universal theory of flows evolving in far-from-equilibrium conditions and their reduction to normal form equations. The bifurcations generated by these equations and their connection to the structure of the Standard Model (SM) and Dark Matter form the topic of next three sections. Conclusions are summarized in the last section. For reader's convenience, a glossary of text abbreviations is also included.

We caution the reader on the introductory and tentative nature of our work. The intent is to draw attention to the many unexplored implications of nonlinear science and complexity theory on the dynamics of the SM and beyond. Independent research is needed to reject or expand the body of ideas discussed here.

## 2. RG flows as autonomous dynamical systems

The RG flow in the space of couplings $g \in \Gamma$ is a continuous map $\beta = R \times \Gamma \to \Gamma$ called the "beta function" and associated with [ ]

$$\beta(g) = \mu \frac{dg}{d\mu} = \frac{dg}{d(\log \mu / \mu_0)} \tag{1}$$

such that

$$\beta(0, g) = g \tag{2}$$

$$\beta(\tau, \beta(s, g)) = \beta(\tau + s, g) \tag{3}$$

where the "RG time" is $\tau = \log(\mu/\mu_0)$ and $\mu$ is the RG scale. A FP (equilibrium or conformal point) of (1) is a coupling $g_0 \in \Gamma$ for which $\beta(R, g_0) = g_0$. The FP of the RG flow correspond to zero or infinite correlation lengths and are accordingly classified as "trivial" or "non-trivial". The existence of FP reflects the asymptotic approach towards scale-invariance and it relates to the self-similarity of fractal structures [see e.g. 7]. A subset $\mathrm{I} \subset \Gamma$ is an invariant set of the flow if

$$\beta(R, \mathrm{I}) = \bigcup_{\tau \in \mathbf{R}} \beta(\tau, \mathrm{I}) \subset \Gamma \tag{4}$$

Likewise, the continuous time flow of autonomous dynamical systems is described by the differential equation

$$\frac{dx(\tau)}{d\tau} = f(x(\tau)) \tag{5}$$

where $x \in R^n$ and $f : R^n \to R^n$ is a function on the $n$-dimensional phase space $R^n$ [ ]. There are two ways of relating (5) to a map iteration of the phase space onto itself, namely,

a) Working in discrete "time" ($\tau \to \tau_0$) turns (5) into

$$x_{n+1} = x_n + \tau_0 f(x_n) = F(x_n)\ , \ \ x_n = x(n\tau_0) \tag{6}$$

b) If (1) has periodic solutions $x(T) = x(0) = x_0$ for some $T > 0$, one takes a hyperplane $R^{n-1}$ of dimension $n-1$ transverse to the orbit $\tau \to x(\tau)$ through $x_0$ and evaluates the distribution of neighboring intersections of the orbit with this hyperplane (the method of Poincaré sections).

Many dynamical systems and maps are dependent on a number of control parameters $\lambda \in R^m$. In this case, (5) and (6) take the form

$$\frac{dx(\tau)}{d\tau} = f(x(\tau), \lambda(\tau)) \tag{7}$$

$$x_{n+1} = x_n + \tau_0 f(x_n, \lambda) = F(x_n, \lambda) \tag{8a}$$

Of particular interest is the long-term evolution of (6)-(8), which reflects the behavior of the large $k^{th}$ iterate of the flow in phase-space, $\left\{F^k(x)\right\}$, $k >> 1$. By definition, a *period-k* FP of map (6) satisfies the condition

$$x^*_{n+k} = F^{(k)}(\lambda, x^*_n) = x^*_n \tag{8b}$$

Some flows may converge to specific attractors like a FP or a periodic orbit or erratically wander inside a bounded region $(\mathrm{I})$. If all iterates remain "trapped" in $(\mathrm{I})$ for $x \in \mathrm{I}$, then $(\mathrm{I})$ forms an invariant set [ ]. Moreover, if $(\mathrm{I})$ has a fine structure, or if there is sensitive dependence on initial conditions (two nearby points get farther apart under a large number of iterates of $f$), then $(\mathrm{I})$ represents a strange set.

## 3. Flows in far-from-equilibrium field theory

Quantum Field Theories are known to become scale-invariant at large distances. Viewed in the context of conformal field theory, this property is typically associated with the FP structure of the RG flow [ ]. Starting from this observation, we conjecture below that all field theories evaluated at sufficiently low-energy scales emerge from an underlying system of high-energy entities called primary variables. Let the UV sector of field theory be described by a large set of such variables $x \equiv \{x_i\}$, $i = 1, 2, ..., n$, $n >> 1$, whose mutual coupling and dynamics is far-from-equilibrium. The specific nature of the UV variables is irrelevant to our context, as they can take the form of irreducible objects such as, but not limited to, spinors, quaternions, twistors, strings, branes, loops, knots, bits of information and so on.

The downward flow of $x \equiv \{x_i\}$ may be mapped to a system of ordinary differential equations having the universal form

$$x'_\tau = f(x(\tau), \lambda(\tau), D(\tau)) \tag{9}$$

Here, $\lambda, \tau, D$ denote, respectively, the control parameters vector $\lambda = \{\lambda_j\}$, $j = 1, 2, ...m$, the evolution parameter and the dimension of the embedding space. If the dimension of the embedding space is taken to be independent variable or control parameter, the system (9) further reduces to

$$x'_\tau = f(x(\tau), \lambda(\tau)) \tag{10}$$

It is reasonable to assume that the flow (9) or (10) occurs in the presence of non-vanishing perturbations induced by far-from-equilibrium conditions. These may surface, for example, from primordial density fluctuations in the early Universe or from unbalanced vacuum fluctuations in the UV regime of QFT.

To make explicit the effect of perturbations, we resolve $x(\tau)$ into a reference stable state $x_s(\tau)$ and a deviation generated by perturbations, i.e.,

$$x(\tau) = x_s(\tau) + y(\tau) \tag{11}$$

Direct substitution in (10) yields the set of homogeneous equations

$$y'_\tau = f(\{x_s + y\}, \lambda) - f(\{x_s\}, \lambda) \tag{12}$$

Further expanding around the reference state leads to

$$y'_\tau = \sum_j L_{ij}(x_s, \lambda) y_j + h_i(\{y_j\}, \lambda) \tag{13}$$

where $L_{ij}$ and $h_i$ denote, respectively, the coefficients of the linear and nonlinear contributions induced by departures from the reference state. Here, $L_{ij}$ represents a $n \times n$ matrix dependent on the reference state and on the control parameters vector. Under the assumption that parameters $\lambda$ stay close to their critical values $(\lambda = \lambda_c)$, it can be shown that (13) undergoes bifurcations and its behavior can be mapped to a closed set of universal equations referred to as normal forms [ ]. If, at $\lambda = \lambda_c$ perturbations are non-oscillatory (steady-state), the normal form equations are

$$z'_\tau = (\lambda - \lambda_c) - uz^2 \tag{14a}$$

$$z'_\tau = (\lambda - \lambda_c)z - uz^3 \tag{14b}$$

$$z'_\tau = (\lambda - \lambda_c)z - uz^2 \tag{14c}$$

Instead, if perturbations are oscillatory at $\lambda = \lambda_c$, the normal form equation is given by

$$z'_\tau = [(\lambda - \lambda_c) + i\omega_0]z - uz|z|^2 \tag{15}$$

where $\omega_0$ is the frequency of perturbations at the bifurcation point and both $u$ and $z$ are complex-valued. Furthermore, it can be shown that (15) belongs to a rich spectrum of Andronov-Hopf bifurcation scenarios involving limit cycles [ ].

We end the section with the following observation: of particular interest is to augment the conditions a) to d) of section 1 with the assumption that (9) and (10) exhibit memory effects. These effects may be naturally attributed to a non-local dynamic regime that is far-from-equilibrium and whose characterization requires fractional calculus instead of ordinary calculus on smooth manifolds [ ]. It is reasonable to conjecture that (9) and (10) evolve in low-fractality conditions defined by arbitrarily small deviations from four-dimensionality of ordinary spacetime [ ]. Under these assumptions, the condition $\varepsilon = 4 - D << 1$ describes the minimal fractal manifold (MFM) geometry of spacetime near the IR attractor of (9) and (10), whereby $\varepsilon$ takes on the role of leading control parameter [9]. One then naturally proceeds with the identification $\varepsilon \Rightarrow \lambda$, $\varepsilon_c \Rightarrow \lambda_c = 0$ in (14) and (15), which shows that the four-dimensional spacetime represents the asymptotic limit of the MFM at the critical point $D = 4$.

In summary, the outcome of this analysis is that the multivariable dynamics (9) and (10) reduces in the long-run to a lower dimensional system of universal equations with the emerging variable $z$ playing the role of an effective order parameter. Moreover, if (9) and (10) carry low-amplitude non-local effects, the leading control parameter near the IR attractor may be assumed to be $\varepsilon = 4 - D << 1$ [ ].

## 4. Universal bifurcations of the normal form equations

We now proceed with the evaluation of (14) and (15) noting that, while (14) contains a true scalar order parameter, (15) embodies the dynamics of a complex order parameter.

One is led to suspect that (15) may shed light on the quantum field structure of the SM, while (14) may instead provide clues on the dynamic content of Dark Matter. We next explore this insight starting with (15) and moving on to (14) afterwards. Given that (14) and (15) are particular embodiments of the real and complex Ginzburg-Landau equations, with their vast range of possible solutions, we limit the ensuing analysis to few representative scenarios which are well documented in the literature [ ].

**4.1 Equation (15)**

It is known that, near a bifurcation point, dynamic variables typically evolve on a fast time scale and can be adiabatically eliminated [ ]. Thus, it makes sense to study the behavior of (15) when $\varepsilon << 1$ is the leading control parameter and perturbations are slow, that is, $\omega_0 << 1$. One is immediately led to the so-called Stuart-Landau equation, which represents a generic model of nonlinear dynamics near the onset of Andronov-Hopf bifurcations [ ]. Substituting the amplitude and phase of the order parameter $z(\tau) = \rho(\tau)\exp[i\theta(\tau)]$ and taking $u = u_r + iu_i$ in (15) yields

$$\rho'_\tau = \rho(\varepsilon - \rho^2) \tag{16a}$$

$$\theta'_\tau = \omega_0 - u_i\rho^2 \approx -u_i\rho^2 \tag{16b}$$

Two cases are of interest here, namely,

a) If $u$ is real ($u_i = 0$), the dynamics of the phase (16b) is trivial, i.e. $\theta'_\tau = 0$ . The equation of motion for the amplitude assumes the normal form of a pitchfork bifurcation. This bifurcation is supercritical if $u = u_r > 0$ and subcritical otherwise. In the former instance, the stable branch $\rho = 0$ of the diagram $(\rho, \varepsilon)$ becomes unstable for $\varepsilon > 0$ and splits into a symmetrical pair of stable branches at $\varepsilon = 0$ defined by $\rho_{1,2} = \pm\sqrt{\varepsilon / u_r}$ [ ].

b) If fast perturbations do not completely vanish and $\omega_0 = const. \neq 0$, the equation of motion for the amplitude stays the same as before but the phase evolves according to $\theta'_\tau = \omega_0 - u_i \rho^2$. To fix ideas, we look at the case where $u_i = 0,\ u_r = 1$. In addition to the stable equilibrium at the origin ($\rho = 0$) for $\varepsilon \leq 0\ (D \geq 4)$, an extra stable equilibrium develops at $\rho_0(\varepsilon) = \sqrt{\varepsilon}$ for $\varepsilon > 0$. This equilibrium is in the form of a limit cycle, that is, a closed orbit of radius $\rho_0(\varepsilon)$. All orbits starting outside or inside this cycle except the origin asymptotically approach the cycle as $\tau \to +\infty$. The transition induced by continuously tuning $\varepsilon$ represents a supercritical Andropov-Hopf bifurcation.

The two cases described by a) and b) are far from exhausting the vast array of solutions of the Ginzburg-Landau equation including, for example, coherent structures, topological defects and various forms of spatiotemporal chaos. Despite this limitation, both pitchfork and Andronov-Hopf bifurcations shed light into the basic mechanisms leading (14) and (15) to transition from a laminar regime to turbulence, and from order to chaotic behavior. The interested reader is referred to the large database of articles dealing with bifurcations of the Ginzburg-Landau equation [ ]. In [ ], for example, $\varepsilon$ assumes the role of leading bifurcation parameter and corresponds to the Reynolds number associated with the onset of turbulence.

This analysis supports the viewpoint that $\varepsilon$ and the MFM geometry of space-time near the electroweak scale ($M_{EW}$) play a critical role in the physics of SM. In particular, our research shows that [ ],

- The SM may be configured as a multifractal set, with all field components acting as primary generators of this set. Renormalization flow analysis of the generic Landau-Ginzburg-Wilson model reveals the connection between the dimensional parameter $\varepsilon$, low-scale particle masses $m_i = O(m)$ and the far ultraviolet scale $\Lambda_{UV} >> m$ via

$$\varepsilon = O\left(m^2 / \Lambda_{UV}^2\right) \tag{17}$$

  Renormalization flow consists of continuous variations in scale starting from $\Lambda_{UV}$ down to $\Lambda < \Lambda_{UV}$. These changes automatically imply that $\varepsilon$ is scale-dependent, which means that the dimensional flow is a consequence of the renormalization process.

- The MFM geometry of space-time near or above $M_{EW}$ explains the repetitive flavor structure of SM. Its mass spectrum satisfies a "closure" relationship replicating the construction of multifractal sets, namely

$$\sum_{i=1}^{16}\left(\frac{m_i}{M_{EW}}\right)^2 = 1 \tag{18}$$

  The number of SM flavors is constrained by anomaly cancelation [ ] and by the closure relationship (18). One can argue that it may also be fixed by demanding marginal stability of the perturbative RG flow [ ].

- MFM naturally mixes widely separated scales of particle physics and cosmology. The electroweak scale ($M_{EW}$), the cosmological constant scale ($\Lambda_{cc}^{1/4}$) and the far ultraviolet scale ($\Lambda_{UV}$) satisfy the constraint

$$\frac{\Lambda_{cc}^{1/4}}{M_{EW}} = \frac{M_{EW}}{\Lambda_{UV}} \tag{19}$$

Both (18) and (19) match experimental observations. It is also worth noting that (19) complies with the latest results from gravitational wave astronomy [ ].

- MFM accounts for the emergence of continuous and discrete symmetries in QFT and offers unforeseen solutions to the puzzles associated with the Faddeev-Popov ghosts in quantum gauge theories and the cosmological constant problem [ ].

### 4.2 Equation (14)

As a continuous dynamical system, equation (14b) displays pitchfork bifurcations of the type defined by (16). A similar behavior shows up if (14) is first turned into a set of iterated maps following the procedure detailed in (6) to (8). Specifically, (14a) to (14c) represent quadratic and cubic maps evolving towards chaos via the period doubling scenario [ ]. In the long run, one ends up with an unbounded proliferation of orbits confined to the Feigenbaum attractor. It is conceivable that these densely packed orbits undergo chaotic mixing and diffusion and may coalesce into some form of topological condensate on scales significantly lower than the electroweak scale. To a certain extent, this process is similar to Bose-Einstein condensation (BEC) at low temperature, which suggests a direct connection to the superfluid model of Dark Matter put forward in [ ]. Expanding on the body of ideas advanced in [ ], this analogy is further developed in section 6.

## 5. Flavor replication in the Standard Model

The paradigm outlined in paragraph 4.1 hints that the family structure of the Standard Model unfolds from letting (15) develop sequential bifurcations. One possible scenario is that the gluon octet emerges as twofold replica of the electroweak boson quartet as in

$$\begin{pmatrix}\gamma & W^{+} & W^{-} & Z\end{pmatrix} \rightarrow (g_i)_{i=1,2,...,8} \tag{20}$$

Likewise, color quartets may surface as twofold replicas of lepton doublets, namely,

$$\begin{pmatrix}\nu_e & e\end{pmatrix} \rightarrow \begin{pmatrix} u_R & d_R \\ u_G & d_G \end{pmatrix} \tag{21a}$$

$$\begin{pmatrix}\nu_\mu & \mu\end{pmatrix} \rightarrow \begin{pmatrix} c_R & c_R \\ s_G & s_G \end{pmatrix} \tag{21b}$$

$$\begin{pmatrix}\nu_\tau & \tau\end{pmatrix} \rightarrow \begin{pmatrix} t_R & t_G \\ b_R & b_G \end{pmatrix} \tag{21c}$$

Two observations are in order [ ]

a) charge conservation constrains the number of independent flavors generated through bifurcations. For example, taking $R, G, B$ to denote the triplet of independent color states, color conservation prohibits formation of the independent state $B$ since $R+G+B=1$, by definition.

b) the approach to chaos through successive bifurcations creates a natural mixing of orbits. As a result of this mixing, the transition $U(1)\times SU(2)\rightarrow SU(3)$ suggests that leptons and quarks are allowed to couple through electroweak fields but forbids leptons to couple to gluon fields.

Finally, the Higgs scalar arises as topological condensate of gauge bosons having anti-parallel spins. The simplest combination of weakly-coupled gauge bosons condensing into a spin-zero state is given by [ ]

$$\Phi_c = \frac{1}{4}[(W^+ + W^- + Z + \gamma + g) + (W^+ + W^- + Z + \gamma + g)] \tag{22}$$

## 6. Cantor Dust as underlying content of Dark Matter

The recent model of Dark Matter as a superfluid phase consisting of axion-like particles offers a number of appealing features [ ]. In this proposal, Dark Matter particles undergo BEC and give rise to the superfluid phase inside the galactic cores. The superfluid collective excitations behave as phonons and their coherence properties induce long-range forces. In turn, these forces are able to mimic the predictions of Modified Newtonian Dynamics (MOND) on galactic scales.

It has been long known that superfluidity can be analyzed via the Ginzburg-Landau theory. Specifically, in the weak-coupling approximation, the typical superfluid model consists of a self-interacting complex-scalar field endowed with global $U(1)$ symmetry [49-51]. In [ ], this model was built from the dimensional parameter $\varepsilon = 4 - D << 1$ and led to a superfluid picture of Dark Matter bypassing the axion paradigm and referred to as *Cantor Dust*. Here, we pursue an alternative strategy in which superfluid phonons are described by the order parameter $z$ in (14) and are governed by the effective Lagrangian [ ]

$$L = F\,(\dot{z} - \frac{(\partial_i z)^2}{2m}) \tag{23}$$

To this end, we turn our focus to (14b) and proceed with the following assumptions:

a) $z$ represents a complex-valued parameter, whose behavior on cosmological scales ( $M << M_{EW}$ ), can be reasonably well approximated by that of a scalar, that is, $z = z_r + i z_i$ , in which $|z_i| << |z_r|$.

b) to account for the nearly-vanishing contribution of diffusive transport in the dynamics of the cosmological fluid, equation (14b) is supplemented with a space-dependent diffusion term driven by $\eta = O(\varepsilon) << 1$ [ ].

c) to streamline the analysis, the model is considered in 1+1 space-time, with $\varepsilon$ set to be independent of space-time coordinates.

Under these assumptions, (14b) takes the form of the real Ginzburg-Landau equation [ ]

$$z'_{\tau} = \varepsilon z - u z |z|^2 + \eta \frac{\partial^2 z}{\partial x^2} \tag{24}$$

We next look for the stationary solution of (24) and study its linear stability. Taking the stationary solution in the form

$$z = z_0 \exp(ikx)\ , \quad z_0 \in R \tag{25}$$

and substituting it into (24) yields

$$uz_0^2 = \varepsilon - \eta k^2 \tag{26}$$

It is apparent that the scalar amplitude $z_0$ asymptotically decouples from the diffusion coefficient $\eta$ in the *far IR* region of vanishing wavenumbers $k << 1$. In this case, (25) describes a diffusion-free spatial wave extended over large (mesoscopic) distances on the order $O(k^{-1})$. By contrast, $uz_0^2 << 1$ occurs in the limit of large wavenumbers $k^2 = O(\varepsilon / \eta) = O(1)$, as (9) or (10) approach criticality in the *near IR* region centered about the electroweak scale $M_{EW}$. These observations suggest that there may be a limited overlap region of Dark Matter structures with SM fields near $M_{EW}$ and that these structures tend to decouple from the SM as the flow (9) or (10) evolve towards mesoscopic scales $M << M_{EW}$.

To evaluate the stability of (26), we add to (25) a nearly-vanishing perturbation $y(x,\tau)$,

$$z = z_0 \exp(ikx) + y(x,\tau) \tag{27}$$

and linearize (24) to obtain [ ]

$$y'_\tau = \varepsilon y - uz_0^2[2y + \bar{y}\exp(2ikx)] + \eta \frac{\partial^2 y}{\partial x^2} \tag{28}$$

in which $\bar{y}$ stands for the complex conjugate of $y$. We assume next that (28) has a solution that represents a superposition of harmonic waves

$$y(x,\tau) = a(\tau)\exp(ik_1 x) + b(\tau)\exp(ik_2 x) \tag{29}$$

with $2k = k_1 + k_2$ . Substituting (29) into (28) leads to

$$a'_\tau = a(\varepsilon_1 - 2\varepsilon_0) - \varepsilon_0 \bar{b} \tag{30a}$$

$$\overline{b'_\tau} = \bar{b}(\varepsilon_2 - 2\varepsilon_0) - \varepsilon_0 a \tag{30b}$$

where

$$\varepsilon_0 = \varepsilon - \eta k^2 \tag{31a}$$

$$\varepsilon_1 = \varepsilon - \eta k_1^2 \tag{31b}$$

$$\varepsilon_2 = \varepsilon - \eta k_2^2 \tag{31c}$$

On the basis of (27) to (31), it can be shown that the stability of (26) depends on the magnitude of the wavevector $k$ and the proximity of $\varepsilon$ to its value at the bifurcation point $\varepsilon_{cr}$. In particular, the farther away $\varepsilon$ lies from $\varepsilon_{cr}$, the larger the instability range of (26) is to the perturbation $y(x,t)$. This result hints that flow (9) or (10) "settles down" in the far IR region of mesoscopic scales $M << M_{EW}$ and provides clues on the self-contained nature of Dark Matter structures emerging from (24).

## 7. Concluding remarks

…

## Glossary of abbreviations

UV = ultraviolet

IR = infrared

RG = Renormalization Group

QFT = Quantum Field Theory

FP = fixed point

SM = Standard Model of particle physics

MFM = minimal fractal manifold

BEC =Bose-Einstein condensation

MOND = Modified Newtonian Dynamics

## **References**

[1] Guckenheimer, J. and Holmes, P., "*Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*", Springer, Berlin (1983).

[2] Nicolis, G., "Physics of far-from-equilibrium systems and self-organization" in "*The New Physics*" (edited by Paul Davies), Cambridge Univ. Press, (1990), pp.316-347.

[3] Kuznetsov, Y. A., "*Elements of Applied Bifurcation Theory*", Springer Series in "Applied Mathematical Sciences", vol.112, (1998).

[4] Collet, P. and Eckmann, J-P., "*Iterated Maps on the Interval as Dynamical Systems*", Birkhäuser, Boston, (1980).

[5] Jackson, E. A., "*Perspectives of Nonlinear Dynamics*", Cambridge Univ. Press., (1992).

[6] Falconer, K. J., "*The Geometry of Fractal Sets*", Cambridge Univ. Press, (1988).

[7] Khristensen, K. and Moloney, N. R., "*Complexity and Criticality*", Imperial College Press, (2005).

[8] http://www.prespacetime.com/index.php/pst/article/view/1244

[9] Available at the following sites:

http://www.aracneeditrice.it/aracneweb/index.php/pubblicazione.html?item=9788854889972

https://www.researchgate.net/publication/278849474_Introduction_to_Fractional_Field_Theory_consolidated_version

…