$\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued Gravity as a Grand Unified Field Theory *

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Abstract

It is argued how $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued Gravity (real-complex-quaterno-octonionic Gravity) naturally can describe a Grand Unified Field theory of Einstein's gravity with a Yang-Mills theory containing the Standard Model group $SU(3) \times SU(2) \times U(1)$. In particular, it leads to a $[SU(4)]^4$ symmetry group revealing the possibility of extending the standard model by introducing additional gauge bosons, heavy quarks and leptons, and an extra *fourth* family of fermions. We finalize by displaying the analog of the Einstein-Hilbert action for $\mathbf{R}\otimes\mathbf{C}\otimes\mathbf{H}\otimes\mathbf{O}$-valued gravity via the use of matrices, and which is based on "coloring" the graviton; i.e. by attaching internal indices to the metric $g_{\mu\nu}$. In the most general case, $U(16)$ arises as the isometry group, while $U(8)$ is the isometry group in the split-octonion case.

Keywords: Nonassociative Geometry, Clifford algebras, Quaternions, Octonionic Gravity, Unification, Strings.

*Dedicated to the loving memory of Zita Lechter, who loved life, family and bridge with a passion

1 A brief Introduction

Exceptional, Jordan, Division, Clifford and Nonassociative algebras are deeply related and essential tools in many aspects in Physics. See for instance the work on Jordan algebras by [1], [12], [14]. Exceptional algebras in [2], [16], [30]. Nonassociative algebras [3]. Octonions and other Division algebras [4], [5], [7], [6], [15], and Clifford algebras in [26], [34].

The exceptional E_8 group was proposed long ago [28] as a candidate for a grand unification model building in $D = 4$. The supersymmetric E_8 model was studied as a fermion family and grand unification model [28] under the assumption that there is a vacuum gluino condensate but this condensate is *not* accompanied by a dynamical generation of a mass gap in the pure E_8 gauge sector. Clifford algebras and E_8 are key ingredients in Smith's $D_4 - D_5 - E_6 - E_7 - E_8$ grand unified model in $D = 8$ [18].

Exceptional Jordan Matrix Models based on the compact E_6 involve a *double* number of the required physical degrees of freedom inherent in a complex-valued action [12]. This led Ohwashi to construct an interacting pair of mirror universes within the compact E_6 matrix model and equipped with a $Sp(4, \mathbf{H})/Z_2$ symmetry based on the quaternionic valued symplectic group. The interacting picture resembles that of the bi-Chern-Simons gravity models. A construction of nonassociative Chern-Simons membranes and 3-branes based on the large N limit of Exceptional Jordan algebras was put forward by [13]. More recently the construction of *Exceptional Periodicity* (EP) based on semi-simple rank-3 Jordan algebras has been generalized to rank-3 T-algebras (ternary Vinberg algebras) of special type. This allowed the authors in [17] to explore the Geometry of Exceptional Super Yang-Mills Theories in connection to *bosonic* M-theory in $D = 27$.

A complexification of ordinary gravity (not to be confused with Hermitian-Kahler geometry) has been known for a long time. Complex gravity requires a metric $g_{\mu\nu} = g_{(\mu\nu)} + ig_{[\mu\nu]}$ such that $g_{\nu\mu} = (g_{\mu\nu})^*$, so the diagonal components of the metric $g_{z_1 z_1} = g_{z_2 z_2} = g_{\tilde{z}_1 \tilde{z}_1} = g_{\tilde{z}_2 \tilde{z}_2}$ are real. A treatment of a non-Riemannan geometry based on a complex tangent space and involving a symmetric $g_{(\mu\nu)}$ plus antisymmetric $g_{[\mu\nu]}$ metric component was first proposed by Einstein-Strauss [11] (and later on by [20]) in their unified theory proposal of Electromagentism with gravity by identifying the EM field strength $F_{\mu\nu}$ with the antisymmetric metric $g_{[\mu\nu]}$ component.

Borchsenius [19] proceeded to formulate the quaternionic extension of Einstein-Strauss unified theory of gravitation with EM by incorporating appropriately the $SU(2)$ Yang-Mills field strength into the degrees of a freedom of a quaternionc-valued metric. Oliveira and Marques [21] later on provided the Octonionic Gravitational extension of Borchsenius theory involving two interacting $SU(2)$ Yang-Mills fields and where the exceptional group G_2 was realized naturally as the automorphism group of the octonions. The non-Desarguesian geometry of the Moufang projective plane to describe Octonionic QM was discussed by [15].

The authors [23] showed how one could generalize Octonionic Gravitation into an Extended Relativity theory in Clifford spaces, involving polyvector-valued (Clifford-algebra valued) coordinates and fields, where in addition to the speed of light there is also an invariant length scale (set equal to the Planck scale) in the definition of a generalized metric distance in Clifford spaces encoding, lengths, areas, volumes and hyper-volumes metrics. An overview of the basic features of the Extended Relativity in Clifford spaces can be found in [23].

The purpose of this work is to advance further the Octonionic Gravitational construction of [21], [22], and show how $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued Gravity naturally can describe a grand unified field theory of Einstein's gravity with a Yang-Mills theory containing the Standard Model group $SU(3) \times SU(2) \times U(1)$.

This work is organized as follows. In section **2** we review the algebraic structure of octonions and discuss the octonionic realizations of $SO(8), SO(7), G_2$, $SU(3), GL(8, R)$. In section **3** we present all the steps involved in the construction of $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued Gravity, and show how the $U(4) \times U(4) \times U(4) \times U(4)$ symmetry is encoded in the $\mathrm{C} \otimes \mathrm{H} \otimes \mathrm{O}$ piece of the connection. In section **4** we display the analog of the Einstein-Hilbert action for $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued gravity via the use of matrices, and based on "coloring" the graviton; i.e. by attaching internal indices to the metric $g_{\mu\nu}$. In the most general case, $U(16)$ arises as the isometry group, while $U(8)$ is the isometry group in the split-octonion case. The introduction of matter fields and solutions to the generalized Einstein field equations for the $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued Gravitational theory will be the subject of future investigation.

2 Octonions, Clifford and Lie Algebras

This introductory section is very important in order to understand some of the arguments in the next section. For this reason we deem it necessary.

2.1 Octonionic Realizations of $SO(8), SO(7), G_2, SU(3)$

Given an octonion $\mathbf{X}$ it can be expanded in a basis (e_o, e_a) as

$$\mathbf{X} = x^o\ e_o\ +\ x^a\ e_a,\ a = 1, 2, \cdots, 7. \tag{2.1}$$

where e_o is the identity element. The Noncommutative and Nonassociative algebra of octonions is determined from the relations

$$e_o^2 = e_o,\ e_o e_a = e_a e_o = e_a,\ e_a e_b = -\delta_{ab} e_o\ + C_{abc} e_c,\ a, b, c = 1, 2, 3, ...7. \tag{2.2}$$

The non-vanishing values of the fully antisymmetric structure constants C_{abc} is chosen to be **1** for the following 7 sets of index triplets (cycles) [7]

$$(124),\ (235),\ (346),\ (457),\ (561),\ (672),\ (713) \tag{2.3}$$

Each cycle represents a quaternionic subalgebra. The values of C_{abc} for the other combinations are zero. The latter 7 sets of index triplets (cycles) correspond to the 7 lines of the Fano plane.

The octonion conjugate is defined

$$\bar{\mathbf{X}} = x^o \ e_o \ - \ x^m \ e_m. \tag{2.4}$$

and the norm

$$N(\mathbf{X}) \ = \ < \ \mathbf{X} \ \mathbf{X} \ > \ = \ Real \ (\bar{\mathbf{X}} \ \mathbf{X}) \ = (x_o \ x_o \ + x_k \ x_k). \tag{2.5}$$

The inverse

$$\mathbf{X}^{-1} \ = \ \frac{\bar{\mathbf{X}}}{N(\mathbf{X})}, \quad \mathbf{X}^{-1}\mathbf{X} = \mathbf{X}\mathbf{X}^{-1} = 1. \tag{2.6}$$

The non-vanishing associator is defined by

$$\{\mathbf{X}, \mathbf{Y}, \mathbf{Z}\} = (\mathbf{X}\mathbf{Y})\mathbf{Z} - \mathbf{X}(\mathbf{Y}\mathbf{Z}) \tag{2.7}$$

In particular, the associator

$$\{e_i, e_j, e_k\} \ = \ d_{ijkl} \ e_l, \quad d_{ijkl} \ = \ \epsilon_{ijklmnp} \ c^{mnp}, \ i, j, k.... = 1, 2, 3, ...7 \tag{2.8}$$

There are **no** matrix *representations* of the Octonions due to the non-associativity, however Dixon has shown how many Lie algebras can be obtained from the left/right action of the octonion algebra on itself [7]. By an algebra acting on itself, one does not mean that quantum mechanical operators and states are expressed in the same algebra. An example of an algebra action on itself is the Clifford algebra. The Clifford vector generators γ^μ acting on themselves give

$$\gamma^\mu \ \gamma^\nu \ = \ \frac{1}{2} \ \{\gamma^\mu, \gamma^\nu\} \ + \frac{1}{2} \ [\gamma^\mu, \gamma^\nu] \ = \ g_{\mu\nu} \ \mathbf{1} \ + \ \gamma^{[\mu\nu]} \tag{2.9}$$

and which in turn lead to the unit element $\mathbf{1}$, and the *bivector* $\gamma^{[\mu\nu]}$ generators of the 2^D-dim Clifford algebra $Cl(D)$.

$\mathbf{O}_L$ and $\mathbf{O}_R$ are identical, isomorphic to the matrix algebra $R(8)$ of 8×8 real matrices. The 64-dimensional bases are of the form $\mathbf{1}, e_{La}, e_{Lab}, e_{Labc}$, or $\mathbf{1}, e_{Ra}, e_{Rab}, e_{Rabc}$, where, for example, if $\mathbf{x} \in \ \mathbf{O}$, then $e_{Lab}[\mathbf{x}] = e_a(e_b\mathbf{x})$, and $e_{Rab}[\mathbf{x}] = (\mathbf{x}e_a)e_b$.

Focusing on the $left$ actions, Dixon found [7]

- $so(8) \ : \ \{e_{La}; \ e_{Lab} \mid a, b = 1, \cdots, 7\}$ giving a total of 7+21 = 28 generators.

- $so(7) \ : \ \{e_{Lab} \mid a, b = 1, \cdots, 7\}$ giving a total of 21 generators.
- $so(6) \ : \ \{e_{Lpq} \mid p, q = 1, \cdots, 6\}$ giving a total of 15 generators.
- The Lie algebra g_2

$$g_2 \ : \ \{e_{Lab} - e_{Lcd} \mid e_a \ e_b \ - \ e_c \ e_d \ = \ 0, \quad a, b, c, d = 1, \cdots, 7\} \tag{2.10a}$$

g_2 is the 14-dim Lie algebra of G_2, the automorphism group of **O**. The 14 generators are [7]

$$e_{L24} - e_{L56}; \quad e_{L56} - e_{L37}; \quad e_{L35} - e_{L67}; \quad e_{L67} - e_{L41}$$

$$e_{L46} - e_{L71}; \quad e_{L71} - e_{L52}; \quad e_{L57} - e_{L12}; \quad e_{L12} - e_{L63}$$

$$e_{L61} - e_{L23}; \quad e_{L23} - e_{L74}; \quad e_{L72} - e_{L34}; \quad e_{L34} - e_{L15}$$

$$e_{L13} - e_{L45}; \; e_{L45} - e_{L26} \tag{2.10b}$$

The $su(3)$ Lie algebra is a subalgebra of g_2 which leaves invariant one of the imaginary units of the octonions. In particular if one chooses e_7, $su(3)$ is the Lie algebra of $SU(3)$ which is the stability group of e_7 (a subgroup of G_2). The 8 generators of $su(3)$ are determined from the conditions

•

$$su(3) \; : \; \{ e_{Lpq} - e_{Lrs} \mid e_p e_q - e_r e_s = 0, \quad p, q, r, s = 1, \cdots, 6 \}$$

from which one obtains the following 8 generators [7]

$$e_{L24} - e_{L56}; \quad e_{L35} - e_{L41}; \quad e_{L46} - e_{L52}$$

$$e_{L12} - e_{L63}; \quad e_{L61} - e_{L23}; \quad e_{L34} - e_{L15}$$

$$e_{L13} - e_{L45}, \quad e_{L45} - e_{L26} \tag{2.11}$$

Notice that *some* of the generators of (2.11) are given by suitable linear combinations of the generators in eq-(2.10b)[1].

• The generator of the $U(1)$ Lie algebra is [7]

$$e_{L45} \; + \; e_{L13} \; + \; e_{L26} \tag{2.12}$$

and commutes with all the 8 generators of $SU(3)$. The 7-dim round sphere can be identified as the coset $S^7 \sim SO(8)/SO(7)$. The 7-dim squashed sphere can be identified as the coset $SO(7)/G_2$. Compactifications of 11-dim M-theory on 7-dim manifolds of exceptional holonomy G_2 have been extensively studied over the years

• 8×8 matrix realizations of the left/right actions. From the structure constants of the Octonion algebra one can associate to the left action of e_a on e_o and e_b

$$e_{La} \; [e_o] \; = \; e_a \; e_o \; = \; e_a, \;\; e_{La} \; [e_b] \; = \; e_a \; e_b \; = \; C_{abc} \; e_c \tag{2.13}$$

the following 8×8 *antihermitian* matrix $\mathbf{M}_{La} : e_{La} \leftrightarrow \mathbf{M}_{La}$, and whose entries are given by

$$(M^L_a)_{bc} = C_{abc}, \; a, b, c = 1, 2, \cdots, 7; \; (M^L_a)_{00} = 0, \; (M^L_a)_{0c} = \delta_{ac}, \; (M^L_a)_{c0} \; = \; -\delta_{ac} \tag{2.14}$$

[1]The $su(3)$ algebra is obtained from the intersection of the $so(6)$ and g_2 algebra. G. Dixon, private communication

Due to the non-associativity of the Octonions one has $e_1 e_2 = e_4$, but $\mathbf{M}_{L1}\mathbf{M}_{L2} \neq \mathbf{M}_{L4}$!, otherwise the generators in the above equations would have been trivially zero. As said previously, there are **no** matrix representations of the non-associative Octonion algebra, and as a result one has that

$$\mathbf{M}_{La}\ \mathbf{M}_{Lb}\ \neq\ C_{abc}\ \mathbf{M}_{Lc} \tag{2.15}$$

Given the antihermian 8×8 matrices in eq-(2.14) the $g_2, su(3), \cdots$ algebras are realized in terms of the commutators of the generators given by eqs-(2.10, 2.11). For example, in the $su(3)$ algebra case, the commutator of the first two $su(3)$ generators (2.11) is

$$[e_{L24} - e_{L56},\ e_{L35} - e_{L41}]\ \leftrightarrow\ [M_{L2}M_{L4} - M_{L5}M_{L6},\ M_{L3}M_{L5} - M_{L4}M_{L1}]\ =$$

$$M_{L2}[M_{L4}, M_{L3}]M_{L5}\ -\ M_{L5}[M_{L6}, M_{L3}]M_{L5}\ +\ \cdots \tag{2.16}$$

The commutators of the 8 $su(3)$ generators $\mathbf{L}_\alpha$ are given by

$$[\mathbf{L}_\alpha,\ \mathbf{L}_\beta]\ =\ f_{\alpha\beta\sigma}\ \mathbf{L}_\sigma, \quad \alpha, \beta, \sigma = 1, 2, \cdots, 7, 8 \tag{2.17}$$

where $f_{\alpha\beta\sigma}$ are the antisymmetric structure constants of the $su(3)$ algebra. The 8-dim adjoint representation of $su(3)$ can be implemented in terms of 8 *antihermitian* 8×8 matrices $\mathbf{T}_\alpha = (T_\alpha)_{\beta\sigma} = f_{\alpha\beta\sigma}$. Since the commutators of two *antihermitain* matrices is *antihermitian*, the (antisymmetric) structure constants $f_{\alpha\beta\sigma}$ are real-valued, and there are *no* i factors in the right hand side of eq-(2.17). It is not difficult to verify that the commutators in eq-(2.16) are indeed the same as those in eq-(2.17) [2]. Similarly one could have written the Lie algebra generators in terms of the *right* action of the Octonion algebra on itself.

2.2 Octonionic realization of $GL(8, R)$

The *combined* left and right action of the algebra acting on itself [9] is defined as

$$e_{La}\ e_{Rb}\ [\mathbf{x}]\ =\ e_{La}\ (\mathbf{x}\ e_{Rb}); \quad e_{Rb}\ e_{La}\ [\mathbf{x}]\ =\ (e_{La}\ \mathbf{x})\ e_{Rb}) \tag{2.18}$$

Based on this left/right action, the authors [9] were able to find an octonionic realization (*not* a representation) of the Lie algebra $gl(8, R)$ based on the generators (8×8 matrices)

$$\mathbf{1},\ L_a,\ R_b,\ L_a R_a,\ [L_a, R_b], \quad a, b = 1, 2, \cdots, 7 \tag{2.19}$$

[2] Despite that $\mathbf{M}_{La}\mathbf{M}_{Lb} \neq C_{abc}\mathbf{M}_{Lc}; \mathbf{M}_{Lb}\mathbf{M}_{La} \neq C_{bac}\mathbf{M}_{Lc}$ it is still possible to have $[\mathbf{M}_{La}, \mathbf{M}_{Lb}] = 2C_{abc}\mathbf{M}_{Lc}$

obeying the relations

$$L_a\, L_b \;=\; -\delta_{ab} + C_{abc}\, L_c - [R_a,\; L_b],\;\; R_a\, R_b \;=\; -\delta_{ab} + C_{abc}\, R_c - [L_a,\; R_b],$$

$$[L_a,\; L_b] \;=\; f_{abc} L_c \;-2\, [R_a,\; L_b],\;\; [R_a,\; R_b] \;=\; f_{abc} R_c \;-2\, [L_a,\; R_b],$$

$$[R_a,\; L_b] \;=\; [L_a,\; R_b] \;=\; -\; [R_b,\; L_a] \;=\; -\; [L_b,\; R_a]$$

$$[R_a,\; L_a] = 0, \quad a = 1, 2, \cdots, 7 \tag{2.20}$$

there is *no* sum over a in the eq-(2.20), and the structure constants are $f_{abc} = 2C_{abc}$.

There are $7+7 = 14$ generators : L_a, R_b. There are 7 generators $L_a R_a$ (no sum over a). There are $7 \times 6 = 42$ generators $[L_a, R_b](a \neq b)$. Combined with the unit 8×8 matrix $\mathbf{1}$, it gives a total of $1+7+7+7+42 = 64$ generators, and which matches the dimension of the Lie algebra $gl(8, R)$.

The modified composition $\odot$ defined as

$$L_a \odot L_b \;=\; L_a\; L_b \;+\; [R_a, L_b] \;\Rightarrow\; L_a \odot L_b \;-\; L_b \odot L_a \;=\; f_{abc}\; L_c \tag{2.21}$$

$$R_a \odot R_b \;=\; R_a\; R_b \;+\; [L_a, R_b] \;\Rightarrow\; R_a \odot R_b \;-\; R_b \odot R_a \;=\; f_{abc}\; R_c \tag{2.22}$$

allows closure $[L_a, L_b]_\odot, [R_a, R_b]_\odot$ where $f_{abc} = 2C_{abc}$.

2.3 Clifford Algebraic Realization of $SU(N)$

• The dim $Cl(0,6) = 64$, is same as the dim of $gl(8, R)$. $\mathbf{O}_L \simeq \mathbf{O}_R \simeq Cl(0,6)$.

The $u(4)$ algebra can also be realized in terms of $so(8)$ generators, and in general, $u(N)$ algebras admit realizations in terms of $so(2N)$ generators Given the Weyl-Heisenberg "superalgebra" involving the N fermionic creation and annihilation (oscillators) operators

$$\{a_i, a_j^\dagger\} \;=\; \delta_{ij}, \quad \{a_i, a_j\} \;=\; 0,\; \{a_i^\dagger, a_j^\dagger\} \;=\; 0; \quad i, j \;=\; 1, 2, 3, ...\; N. \tag{2.23}$$

one can find a realization of the $u(N)$ algebra bilinear in the oscillators as $E_i{}^j \;=\; a_i^\dagger\; a_j$ and such that the commutators

$$[E_i{}^j,\; E_k{}^l] \;=\; a_i^\dagger\; a_j\; a_k^\dagger\; a_l \;-\; a_k^\dagger\; a_l\; a_i^\dagger\; a_j \;=$$

$$a_i^\dagger\; (\delta_{jk} - a_k^\dagger\; a_j)\; a_l \;-\; a_k^\dagger\; (\delta_{li} - a_i^\dagger\; a_l)\; a_j \;=\; a_i^\dagger\; (\delta_{jk})\; a_l \;-\; a_k^\dagger\; (\delta_{li})\; a_j \;=$$

$$\delta_k^j\; E_i{}^l - \; \delta_i^l\; E_k{}^j. \tag{2.24}$$

reproduce the commutators of the Lie algebra $u(N)$ since

$$-a_i^\dagger\; a_k^\dagger\; a_j\; a_l \;+\; a_k^\dagger\; a_i^\dagger\; a_l\; a_j \;=\; -\, a_k^\dagger\; a_i^\dagger\; a_l\; a_j \;+\; a_k^\dagger\; a_i^\dagger\; a_l\; a_j \;=\; 0. \tag{2.25}$$

due to the anti-commutation relations (2.23) yielding a double negative sign $(-)(-) = +$ in (2.25). Furthermore, one also has an explicit realization of the Clifford algebra $Cl(2N)$ Hermitian generators by defining the even-number and odd-number generators as

$$\Gamma_{2j} \;=\; \frac{1}{2}\;(a_j \;+\; a_j^\dagger); \quad \Gamma_{2j-1} \;=\; \frac{1}{2i}\;(a_j \;-\; a_j^\dagger). \tag{2.26}$$

The Hermitian generators of the $so(2N)$ algebra are defined as usual $\Sigma_{mn} = \frac{i}{4}[\Gamma_m, \Gamma_n]$ where $m, n = 1, 2, ...2N$. Therefore, the $u(4), so(8), Cl(8)$ algebras admit an explicit realization in terms of the fermionic Weyl-Heisenberg oscillators $a_i, a_j^\dagger$ for $i, j = 1, 2, 3, 4$.

Having overviewed the basics of Octonions, Clifford and Lie Algebras, in the next section we will extend the notion of a real-valued metric, connection and curvature, to the case where they are R ⊗ C ⊗ H ⊗ O-valued. In this fashion we can show how one can extract ordinary gravity and Yang-Mills theory from the different pieces involved. Ordinary gravity will appear in the R-piece, whereas Yang-Mills theory appears in the C ⊗ H ⊗ O piece.

3 R ⊗ C ⊗ H ⊗ O-valued Gravity and Grand Unification

In the first part of this section we follow all the steps in the construction of R ⊗ C ⊗ H ⊗ O-valued Gravity, and in the second part we explain how the $U(4) \times U(4) \times U(4) \times U(4)$ symmetry arises from the C ⊗ H ⊗ O piece of the connection.

3.1 R ⊗ C ⊗ H ⊗ O-valued Gravity

Dixon [8] many years ago published a monograph pointing out the key role that the composition algebra **R ⊗ C ⊗ H ⊗ O** had in the architecture of the Standard Model. More recently, it has been shown how this algebra acting on itself allows to find the Standard Model particle representations [10]. For this reason we shall construct a gravitational theory based on a **R ⊗ C ⊗ H ⊗ O**-valued metric defined as

$$\mathbf{g}_{\mu\nu}(x^\mu) \;=\; g_{(\mu\nu)}(x^\mu) + g_{\mu\nu}^{IA}(x^\mu)\,(q_I \otimes e_A), \;\; q_I = q_o, q_1, q_2, q_3; \;\; e_A = e_o, e_1, e_2, \cdots, e_7 \tag{3.1}$$

where the ordinary $4D$ spacetime coordinates are $x^\mu, \mu = 0, 1, 2, 3$, and $g_{(\mu\nu)}$ is the standard Riemannian metric. The extra "internal" $C \otimes H \otimes O$-valued metric components are explicitly given by

$$(g_{(\mu\nu)} + i g_{[\mu\nu]})^{oo}, \quad (g_{[\mu\nu]} + i g_{(\mu\nu)})^{ko}, \quad (g_{[\mu\nu]} + i g_{(\mu\nu)})^{oa}, \quad (g_{(\mu\nu)} + i g_{[\mu\nu]})^{ka} \quad (3.2)$$

$k = 1,2,3; a = 1,2,\cdots,7$. The index o is associated with the real units q_o, e_o. The bar conjugation amounts to $i \rightarrow -i;\ q_k \rightarrow -q_k;\ e_a \rightarrow -e_a$, so that $\bar{\mathbf{g}}_{\mu\nu} = \mathbf{g}_{\nu\mu}$.

The generalization of the line interval in $4D$ considered in [21], [22] based on the metric (3.1) is defined by

$$ds^2 \ = \ < \ \mathbf{g}_{\mu\nu} \ dx^\mu \ dx^\nu \ > \ = \ (\ g_{(\mu\nu)} \ + \ g^{oo}_{(\mu\nu)} \ ) \ dx^\mu \ dx^\nu, \ \ \mu,\nu = 0,1,2,3 \quad (3.3)$$

where the operation $< \cdots >$ denotes taking the *real* components of the terms inside the $<>$. From eq-(3.3) one learns that the $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued metric leads to a line interval involving two metrics, respectively, $g_{(\mu\nu)}$ and $g^{oo}_{(\mu\nu)} = h_{(\mu\nu)}$ [3].

The $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued affinity $\mathbf{\Upsilon}^\rho_{\mu\nu}$ (μ,ν,ρ are spacetime indices ranging from $0,1,2,3$) is given by

$$\mathbf{\Upsilon}^\rho_{\mu\nu} \ = \Gamma^\rho_{\mu\nu}(g_{\mu\nu}) \ + \ \mathbf{\Theta}^\rho_{\mu\nu} \ = \ \Gamma^\rho_{\mu\nu}(g_{\mu\nu}) \ + \ \delta^\rho_\mu \ \mathbf{A}_\nu \ =$$

$$\Gamma^\rho_{\mu\nu}(g_{\mu\nu}) + \delta^\rho_\mu \ \left(\ A^{oo}_\nu \ (q_o \otimes e_o) + A^{ia}_\nu \ (q_i \otimes e_a) + A^{io}_\nu \ (q_i \otimes e_o) + A^{oa}_\nu \ (q_o \otimes e_a) \ \right) \quad (3.4)$$

Thus we have decomposed the $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued affinity $\mathbf{\Upsilon}^\rho_{\mu\nu}$ into a real-valued "external" part Γ plus an "internal" part $\mathbf{\Theta}^\rho_{\mu\nu}$. The base spacetime connection can be chosen to be given by the torsionless Christoffel connection

$$\Gamma^\rho_{\mu\nu} \ = \ \Gamma^\rho_{\nu\mu} \ = \ \frac{1}{2} \ g^{\rho\sigma} \ (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \quad (3.5)$$

but the 'internal" part $\mathbf{\Theta}^\rho_{\mu\nu} = \delta^\rho_\mu \mathbf{A}_\nu$ of the connection is completely *independent* on the metric like in the Palatini formulation of gravity.

The $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued curvature tensor $\mathbf{R}^\sigma_{\rho\mu\nu} = \mathcal{R}^\sigma_{\rho\mu\nu} + \mathbf{\Omega}^\rho_{\sigma\mu\nu}$, involving the base spacetime and internal space curvature is defined by

$$\mathbf{R}^\sigma_{\rho\mu\nu} \ = \ \mathbf{\Upsilon}^\sigma_{\rho\mu,\nu} \ - \ \mathbf{\Upsilon}^\sigma_{\rho\nu,\mu} \ + \ \mathbf{\Upsilon}^\sigma_{\tau\nu} \ \mathbf{\Upsilon}^\tau_{\rho\mu} \ - \ \mathbf{\Upsilon}^\sigma_{\tau\mu} \ \mathbf{\Upsilon}^\tau_{\rho\nu}. \quad (3.6)$$

$$\mathbf{R}^\sigma_{\rho\mu\nu} \ = \ \mathcal{R}^\sigma_{\rho\mu\nu}(\Gamma^\rho_{\mu\nu}) \ + \ \delta^\sigma_\rho \ \mathbf{F}_{\mu\nu}. \quad (3.7)$$

where $\mathcal{R}^\sigma_{\rho\mu\nu}(\Gamma^\rho_{\mu\nu})$ is the base spacetime Riemannian curvature associated to the symmetric Christoffel connection $\Gamma^\rho_{\mu\nu}$.

The "internal" space $\mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued curvature is

$$\mathbf{\Omega}^\rho_{\sigma\mu\nu} \ = \ \delta^\rho_\sigma \ \mathbf{F}_{\mu\nu} \quad (3.8)$$

with

$$\mathbf{F}_{\mu\nu} \ = \ \mathbf{A}_{\mu,\nu} - \mathbf{A}_{\nu,\mu} \ - \ [\ \mathbf{A}_\mu \ , \ \mathbf{A}_\nu \ ]. \quad (3.9)$$

[3] In a *bimetric* theory of gravity two separate metrics are introduced

and where the field $\mathbf{A}_\mu$ can be read directly in terms of the internal space affinity from the relation

$$\mathbf{\Theta}^\rho_{\mu\nu} \ = \ \delta^\rho_\mu \ \mathbf{A}_\nu \tag{3.10}$$

There are 32 complex-valued fields (64-real valued fields)

$$\mathbf{A}_\mu \ = \ \{A^{oo}_\mu, A^{io}_\mu, A^{oa}_\mu, A^{ia}_\mu\} \tag{3.11}$$

and based on the *identity* [4]

$$[A \otimes B, \ C \otimes D] \ = \ \frac{1}{2} \ \{A, C\} \otimes [B, D] \ + \ \frac{1}{2} \ [A, C] \otimes \{B, D\} \tag{3.12a}$$

the commutators in eq-(3.9) become

$$[q_I \otimes e_A, \ q_J \otimes e_B] \ = \ \frac{1}{2} \ \{q_I, q_J\} \otimes [e_A, e_B] \ + \ \frac{1}{2} \ [q_I, q_J] \otimes \{e_A, e_B\} \tag{3.12b}$$

which lead to the following explicit components for $\mathbf{F}_{\mu\nu}$

$$F^{oo}_{\mu\nu} \ = \ \partial_\mu A^{oo}_\nu \ - \ \partial_\nu A^{oo}_\mu \tag{3.13}$$

$$F^{oc}_{\mu\nu} \ = \ \partial_\mu A^{oc}_\nu \ - \ \partial_\nu A^{oc}_\mu \ + \ (A^{oa}_\mu \ A^{ob}_\nu \ - \delta_{ij} \ A^{ia}_\mu \ A^{jb}_\nu) \ C^c_{ab} \tag{3.14}$$

$$F^{ko}_{\mu\nu} \ = \ \partial_\mu A^{ko}_\nu \ - \ \partial_\nu A^{ko}_\mu \ + \ (A^{io}_\mu \ A^{jo}_\nu \ - \delta_{ab} \ A^{ia}_\mu \ A^{jb}_\nu) \ f^k_{ij} \tag{3.15}$$

$$F^{kc}_{\mu\nu} \ = \ \partial_\mu A^{kc}_\nu \ - \ \partial_\nu A^{kc}_\mu \ + \ A^{oa}_\mu \ A^{kb}_\nu \ C^c_{ab} \ + \ A^{io}_\mu \ A^{jc}_\nu \ f^k_{ij} \tag{3.16}$$

The next step is to establish the Gravity/Gauge correspondence (not unlike the AdS/CFT correspondence) which in essence amounts to embed the 12 Gauge Fields of the Standard Model $SU(3) \times SU(2) \times U(1)$ into the fields appearing inside the internal connection $\mathbf{\Theta}^\rho_{\mu\nu} = \delta^\rho_\mu \mathbf{A}_\nu$.

Eqs-(3.13-3.16) yield the following 32 complex-valued non-vanishing field strengths

$$F^{oo}_{\mu\nu}, \quad F^{ko}_{\mu\nu}, \quad F^{oc}_{\mu\nu}, \quad F^{kc}_{\mu\nu}, \quad k = 1, 2, 3; \quad c = 1, 2, \cdots, 7 \tag{3.17}$$

Given the $U(1)$ Maxwell field

$$\mathcal{F}_{\mu\nu} \ = \ \partial_\mu \mathcal{A}_\nu \ - \ \partial_\nu \mathcal{A}_\mu \tag{3.18}$$

the Maxwell kinetic term in the Standard Model action is embedded as follows

[4] Similar relations follow for the anti-commutator, by having instead the combinations $\{A, C\} \otimes \{B, D\}$, and $[A, C] \otimes [B, D]$

$$\mathcal{F}_{\mu\nu}\ \mathcal{F}^{\mu\nu}\ \subset\ F^{oo}_{\mu\nu}\ (F^{\mu\nu}_{oo})^* \tag{3.19}$$

Given the $SU(2)$ field strength

$$\mathcal{F}^k_{\mu\nu}\ =\ \partial_\mu \mathcal{A}^k_\nu\ -\ \partial_\nu \mathcal{A}^k_\mu\ +\ \mathcal{A}_\mu{}^i\ \mathcal{A}^j_\nu\ \epsilon^k_{ij} \tag{3.20}$$

the $SU(2)$ Yang-Mills term is embedded as

$$\mathcal{F}^i_{\mu\nu}\ \mathcal{F}^{\mu\nu}_i\ (i=1,2,3)\ \subset\ (F^{ko}_{\mu\nu})\ (F^{\mu\nu}_{ko})^*\ (k=1,2,3) \tag{3.21}$$

Since the $SU(2)$ algebra is isomorphic to the algebra of quaternions, the embedding (3.21) is very natural. The chain of subgroups

$$SO(8)\ \supset\ SO(7)\ \supset\ G_2\ \supset\ SU(3) \tag{3.22}$$

related to the round and squashed seven-spheres : $S^7 \simeq SO(8)/SO(7), S^7_* \simeq SO(7)/G_2$, reflect how the $SU(3)$ group is embedded. The number of generators of $SO(8)$, $SO(7)$ are 28 and 21 respectively. There are $7+21=28$ complex-valued $(14;42$ real-valued) field strengths, respectively

$$F^{oc}_{\mu\nu},\quad F^{kc}_{\mu\nu},\quad k=1,2,3;\quad c=1,2,\cdots,7 \tag{3.23}$$

such that the $SU(3)$ Yang-Mills terms can be embedded into the contribution of the above $7+21=28$ complex-valued fields as follows

$$\mathcal{F}^\alpha_{\mu\nu}\ \mathcal{F}^{\mu\nu}_\alpha\ (\alpha=1,2,\ldots,7,8)\ \subset\ (F^{oc}_{\mu\nu})\ (F^{\mu\nu}_{oc})^*\ +\ (F^{kc}_{\mu\nu})\ (F^{\mu\nu}_{kc})^*\ (c=1,2,\ldots,7) \tag{3.24}$$

and where the $SU(3)$ field strength is given by

$$\mathcal{F}^\gamma_{\mu\nu}\ =\ \partial_\mu \mathcal{A}^\gamma_\nu\ -\ \partial_\nu \mathcal{A}^\gamma_\mu\ +\ \mathcal{A}_\mu{}^\alpha\ \mathcal{A}^\beta_\nu\ f^\gamma_{\alpha\beta} \tag{3.25}$$

3.2 Emergence of $U(4)\times U(4)\times U(4)\times U(4)$

Having explained how the $U(1), SU(2), SU(3)$ field strengths can be *embedded* into the 32 complex-valued non-vanishing field strengths of eq-(3.17), we shall show next how the $C\otimes H\otimes O_L$ algebra associated with the *internal* part of the connection $\mathbf{\Theta}^\rho_{\mu\nu} = \delta^\rho_\mu \mathbf{A}_\nu$ in eq-(3.4) can accommodate a grand unified group given by $SU(4)\times SU(4)\times SU(4)\times SU(4)$, and containing the $SU(3)\times SU(2)\times U(1)$ Standard Model group.

Given that the complex quaternionic algebra $C\otimes H$ is isomorphic to the Pauli spin algebra with the 2×2 matrices $q_0 = \mathbf{1}_{2\times 2}$, $q_k = i\sigma_k, k=1,2,3$, and the algebra of left actions of the octonionic algebra on itself [5] is represented by

[5] A similar construction follows for the algebra of right actions of the octonionic algebra on itself. Note that $\mathbf{O}_L \simeq \mathbf{O}_R$ [7]

the 8×8 matrices $e_{LA} = \mathbf{M}^L_A, A = 0, 1, \cdots, 7$, then the $4 \times 8 = 32$ generators $q_I \otimes e_{LA}$ of the $C \otimes H \otimes O_L$ algebra can be realized by 32 complex 16×16 matrices, which is tantamount to 64 real 16×16 matrices, and which is compatible with the fact that 64 $(2 \times 4 \times 8)$ is the dimension of the $C \otimes H \otimes O_L$ algebra.

Each complex 16×16 matrix, above, can be expanded in terms of the basis elements of the complex Clifford algebra $Cl(8, C)$ comprised of $2^8 = 256$ complex 16×16 matrices. However this is far too cumbersome. However, it is easier if we expand each of the above 32 complex 16×16 matrices in terms of the tensor products $\Gamma_M \otimes \mathbf{1}_{4\times 4}$, where $\Gamma_M (M = 1, 2, \cdots, 32 = 2^5)$ is the basis of the complex Clifford algebra $Cl(5, C)$ comprised of 32 complex 4×4 matrices, and $\mathbf{1}_{4\times 4}$ is the unit 4×4 matrix.

Therefore we end up having that the 32 complex 16×16 matrix generators $q_I \otimes e_{LA}$ of the $C \otimes H \otimes O_L$ algebra can be expanded in terms of a linear combination of the 32 $Cl(5, C)$ algebra generators Γ_M as follows

$$q_I \ \otimes \ e_{LA} \ = \ (\mathbf{M}^L_{IA})_{16\times 16} \ = \ \sum_{M=1}^{32} \ C^M_{IA} \ (\Gamma_M)_{4\times 4} \ \otimes \ \mathbf{1}_{4\times 4}, \tag{3.26}$$

where $I = 0, 1, 2, 3; A = 0, 1, 2, \cdots, 7$, and C^M_{IA} are complex numerical coefficients.

Let us recall the following isomorphisms among real and complex Clifford algebras [34]

$$Cl(2m+1, C) = Cl(2m, C) \oplus Cl(2m, C) \sim M(2^m, C) \oplus M(2^m, C) \ \Rightarrow$$
$$Cl(5, C) \ = \ Cl(4, C) \oplus Cl(4, C) \tag{3.27}$$

where $M(2^m, C)$ is the $2^m \times 2^m$ matrix algebra over the complex numbers (some authors [7] use the different notation $\mathbf{C}(2^m)$).

Also one has

$$Cl(4, C) \sim M(4, C) \sim Cl(4, 1, R) \sim Cl(2, 3, R) \sim Cl(0, 5, R) \tag{3.28}$$

$$Cl(4, C) \sim M(4, C) \sim Cl(3, 1, R) \oplus \mathbf{i}\ Cl(3, 1, R) \sim M(4, R) \oplus \mathbf{i}\ M(4, R) \tag{3.29}$$

$$Cl(4, C) \sim M(4, C) \sim Cl(2, 2, R) \oplus \mathbf{i}\ Cl(2, 2, R) \sim M(4, R) \oplus \mathbf{i}\ M(4, R) \tag{3.30}$$

where $M(4, R), M(4, C)$ is the 4×4 matrix algebra over the reals and complex numbers, respectively. $Cl(p, q, R)$ denotes a real Clifford algebra in $p + q$ dimensions and associated to a metric of signature $p - q$.

In [35] we showed, by recurring to the Weyl unitary "trick", how from each one of the $Cl(3, 1, R)$ commuting sub-algebras inside the $Cl(4, C)$ algebra one can also obtain the $u(p, q)$ algebras with the provision $p + q = 4$. Namely, the $u(p, q)$ algebra generators are given by suitable linear combinations of the $Cl(3, 1, R)$ generators. In particular, the $u(2, 2) = su(2, 2) \oplus u(1)$ algebra contains the conformal algebra in four dimensions $su(2, 2) \sim so(4, 2)$. When $p = 4, q = 0$, the algebra is $u(4) = u(1) \oplus su(4) \sim u(1) \oplus so(6)$.

To sum up, given that the algebra $M(4,C) \sim gl(4,C)$ is also the complexification of $u(4)$ ($sl(4,C)$ is the complexification of $su(4)$), and by virtue of eqs-(3.27), the $Cl(5,C)$ algebra can be decomposed into $four$ copies of $u(4)$

$$Cl(5,C) \ = \ Cl(4,C) \oplus Cl(4,C) \ \sim \ u(4) \oplus u(4) \oplus u(4) \oplus u(4) \tag{3.31}$$

The dimension of the four copies of $u(4)$ is $4 \times 16 = 64$ which matches the dimension of the $C \otimes H \otimes O_L$ algebra, as expected (64 is also the dimension of the real $Cl(6)$ algebra). Consequently, the $C \otimes H \otimes O_L$ algebra, by virtue of the decomposition in eq-(3.26), can accommodate a grand unified group given by

$$SU(4)_C \times SU(4)_F \times SU(4)_L \times SU(4)_R \ \subset \ U(4) \times U(4) \times U(4) \times U(4) \tag{3.32}$$

The gauge group $SU(3)_C \times SU(3)_F \times SU(3)_L \times SU(3)_R$ can naturally be embedded into the above $[SU(4)]^4$ group. The former group involving a unification of left-right $SU(3)_L \times SU(3)_R$ chiral symmetry, color $SU(3)_C$ and family $SU(3)_F$ symmetries in a maximal rank-8 subgroup of E_8 was proposed by [33] as a landmark for future explorations beyond the Standard Model (SM). This model is called the $SU(3)$-family extended SUSY trinification model [33]. Among the key properties of this model are the unification of SM Higgs and lepton sectors, a common Yukawa coupling for chiral fermions, the absence of the μ-problem, gauge couplings unification and proton stability to all orders in perturbation theory.

The standard model (SM) fermions (quarks, leptons) can be embedded into the fermionic matter belonging to the following $SU(4)_C \times SU(4)_F \times SU(4)_L \times SU(4)_R$ representations as follows

$$Q_{SM} \subset \mathbf{Q} = (4, 4, \bar{4}, 1), \quad Q^c_{SM} \subset \mathbf{Q}^c = (\bar{4}, \bar{4}, 1, 4), \tag{3.33}$$

$$L_{SM} \subset \mathbf{L} = (1, 4, \bar{4}, 1), \quad \mathbf{L}^c = (1, \bar{4}, 1, 4) \tag{3.34}$$

where the $\mathbf{Q}, \mathbf{Q}^c, \mathbf{L}, \mathbf{L}^c$ multiplets include the addition of heavy quarks (anti-quarks); leptons (anti-leptons), and an extra $fourth$ family of fermions (and their anti-particles). The first (left handed) quark family is

$$\mathbf{Q}_1 \equiv \begin{pmatrix} u_r & d_r & U_r & D_r \\ u_b & d_b & U_b & D_b \\ u_g & d_g & U_g & D_g \\ Q_u & Q_d & Q_U & Q_D \end{pmatrix} \tag{3.35}$$

where Q_u, Q_d, Q_U, Q_D, and $U_{r,b,g}, D_{r,b,g}$ are the additional quarks . As usual r, b, g stand for red, blue, green color. The charge conjugate multiplet containing the (right-handed) anti-quarks of the first family is

$$\mathbf{Q}^c_1 \ \equiv \ \begin{pmatrix} \overline{u_r} & \overline{u_b} & \overline{u_g} & \overline{Q_u} \\ \overline{d_r} & \overline{d_b} & \overline{d_g} & \overline{Q_d} \\ \overline{U_r} & \overline{U_b} & \overline{U_g} & \overline{Q_U} \\ \overline{D_r} & \overline{D_b} & \overline{D_g} & \overline{Q_D} \end{pmatrix} \tag{3.36}$$

By $\overline{u_r}$ one means $u_{\bar{r}}^c$, the up anti-quark with anti-red color, etc $\cdots$. Whereas $\overline{Q_u} = Q_u^c, \cdots$. And similar assignments for the remaining quark families.

The lepton multiplet will include the ordinary leptons (neutrino, electron, $\cdots$), plus the addition of charged $E_-, E_+, \cdots$, and neutral leptons $N_E, N_E^c, \cdots$. The first (left handed) lepton multiplet is comprised of $\{\nu_e, e_-, N_E, E_-\}$, and its (right handed) anti-multiplet is comprised of $\{\nu_e^c, e_+, N_E^c, E_+\}$. If necessary, one may also have to add extra fermions to cancel anomalies.

Concluding this section, the algebraic structure of $C \otimes H \otimes O_L$ associated with the *internal* part of the connection $\mathbf{\Theta}_{\mu\nu}^\rho = \delta_\mu^\rho \mathbf{A}_\nu$ in eq-(3.4) leads to the group $[SU(4)]^4$ and reveals the possibility of extending the standard model by introducing additional gauge bosons, heavy quarks and leptons, and a *fourth* family of fermions. The physical implications are enormous.

4 Gravitational Actions and Matrix Geometry

In this section we shall discuss how to construct analogs of the Einstein-Hilbert action for $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued gravity via the use of matrices based on "coloring" the graviton (by attaching internal indices to the metric $g_{\mu\nu}$).

4.1 $U(16)$ Matrix Geometry/Gravity

Following the arguments of the previous section, we may associate to the $\mathbf{R}\otimes\mathbf{C}\otimes\mathbf{H}\otimes\mathbf{O}$-valued metric $\mathbf{g}_{\mu\nu} \leftrightarrow \mathbf{G}_{\mu\nu} \equiv G_{\mu\nu}^{MN}$, the 16×16 matrix $\mathbf{G}_{\mu\nu}$ whose 16×16 entries are comprised of complex-valued rank-2 tensors. $M, N = 1, 2, \cdots, 16$; and $\mu, \nu = 0, 1, 2, 3$. For example, in $4D$, let us firstly write down the 16 matrix components of the 4×4 metric $g_{\mu\nu}$ given by : $g_{00}, g_{01}, \cdots, g_{03}, \cdots, g_{33}$; and then let us "color" each one of these 16 matrix components by attaching an internal 16×16-complex matrix at each entry. In this fashion one ends up with what one may coin as a "colored graviton" G_{uv}^{MN}.

The $4D$ line element is defined as

$$ds^2 \ = \ \frac{1}{32} \ \left(\ Trace_{16\times16} \ \{ \ G_{\mu\nu}^{MN} \ dx^\mu \ dx^\nu \ \} \ + \ complex \ conjugate \ \right) \quad (4.1)$$

and it is *invariant* under $\mathbf{U(16)}$ (unitary) transformations $\mathbf{G} \to \mathbf{U}\mathbf{G}\mathbf{U}^\dagger$ acting on the internal "color indices". Under unitary $U(16)$ transformations $\mathbf{U}^\dagger\mathbf{U} = \mathbf{U}\mathbf{U}^\dagger = \mathbf{1}$ acting on the matrix (and not on the coordinates) one has

$$Trace \ \{ \ \mathbf{G'}_{\mu\nu} \ dx^\mu \ dx^\nu \ \} \ = \ Trace \ \{ \ \mathbf{U} \ \mathbf{G}_{\mu\nu} \ \mathbf{U}^\dagger \ dx^\mu \ dx^\nu \ \} \ =$$

$$Trace \ \{ \ \mathbf{U}^\dagger \ \mathbf{U} \ \mathbf{G}_{\mu\nu} \ dx^\mu \ dx^\nu \ \} \ = \ Trace \ \{ \ \mathbf{G}_{\mu\nu} \ dx^\mu \ dx^\nu \ \} \qquad (4.2)$$

due to the unitary matrix $\mathbf{U}^\dagger = \mathbf{U}^{-1}$ condition, and the cyclic property of the trace. The line element is also invariant under Lorentz transformations acting on the spacetime coordinates. The rank of the $u(16)$ Lie algebra is 16 which agrees also with the rank of the Lie algebras corresponding to the $E_8 \times E_8, SO(32)$ groups associated with the anomaly-free heterotic string in $10D$. This is an interesting coincidence that deserves further scrutiny. For references on low energy Grand Unification based $SU(16)$ see [38], [32].

The absolute value of the determinant of the "colored graviton" G^{MN}_{uv} is

$$||det\ \mathbf{G}_{\mu\nu}|| \ = \ \sqrt{(det\ \mathbf{G}_{\mu\nu})\ (det\ \mathbf{G}_{\mu\nu})^*} \tag{4.3}$$

where the $det\ (G^{MN}_{\mu\nu})$ is given by the antisymmetrized sums of products of the determinants of the *blocks* of internal 16×16 complex matrices which comprise each single entry of $\mathbf{G}_{\mu\nu}$. The measure of integration is

$$d\mu(x) \ = \ d^4x\ \sqrt{||det\ \mathbf{G}_{\mu\nu}||} \ = \ d^4x\ \left(\sqrt{(det\ \mathbf{G}_{\mu\nu})\ (det\ \mathbf{G}_{\mu\nu})^*}\right)^{\frac{1}{2}} \tag{4.4}$$

and the generalized version of the $4D$ Einstein-Hilbert gravitational action is

$$\mathbf{S} \ = \ \frac{1}{16\pi G} \int d\mu(x)\ \frac{1}{32}\ (\ Trace_{16\times 16}\{\ \mathbf{G}^{\mu\nu}\ \mathbf{R}_{\nu\mu}\ \} \ + \ cc\ ) =$$

$$\frac{1}{16\pi G} \int d\mu(x)\ \frac{1}{32}\ (\ G^{\mu\nu}_{MN}\ R^{NM}_{\mu\nu} \ + \ cc\ ) \tag{4.5}$$

The above action is Lorentz invariant and $U(16)$-invariant because the $Trace(\mathbf{G}^{\mu\nu}\mathbf{R}_{\mu\nu})$ remains invariant under $\mathbf{G}^{\mu\nu} \rightarrow \mathbf{U}\mathbf{G}^{\mu\nu}\mathbf{U}^\dagger$, $\mathbf{U}\mathbf{R}_{\mu\nu}\mathbf{U}^\dagger$, due to the cyclic property.

We may add other terms to the action, like the analog of the cosmological constant, and quadratic curvature terms. This 16×16-complex matrix formulation ("color graviton" construction) of $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued gravity, based on the $U(16)$ (and Lorentz) invariant action (4.5) of the "colored graviton", may cast some light on the interplay between the rank-16 $e_8 \times e_8, so(32)$ Lie algebras in string theory, and normed division algebras. The coordinates x^μ are real-valued ones; promoting them to complex, quaternionic, octonionic valued ones is also possible and worth exploring.

4.2 Split Octonions and $U(8)$ Matrix Geometry/Gravity

In this subsection we explain how the split-octonion case yields an $U(8)$ isometry instead of $U(16)$. The generators of the split-octonionic algebra admit a realization in terms of the 4×4 Zorn matrices (in blocks of 2×2 matrices) by writing

$$u_o = \frac{1}{2}\ (e_o + ie_7), \quad u_o^* = \frac{1}{2}\ (e_o - ie_7)$$

$$u_i = \frac{1}{2}\ (e_i + i e_{i+3}), \quad u_i^* = \frac{1}{2}\ (e_i - i e_{i+3}) \tag{4.6}$$

$$u_o = \begin{pmatrix} 0 & 0 \\ 0 & \omega_o \end{pmatrix} \quad u_o^* = \begin{pmatrix} \omega_o & 0 \\ 0 & 0 \end{pmatrix}$$

$$u_i = \begin{pmatrix} 0 & 0 \\ \omega_i & 0 \end{pmatrix} \quad u_i^* = \begin{pmatrix} 0 & -\omega_i \\ 0 & 0 \end{pmatrix} \tag{4.7}$$

The quaternionic generators $\omega_o,\ \omega_i,\ i = 1,2,3$ obey the algebra $\omega_i\omega_j = \epsilon_{ijk}\omega_k - \delta_{ij}\omega_o$ and are related to the Pauli spin 2×2 matrices by setting $\sigma_i = i\ \omega_i$ and $\omega_o = \mathbf{1}_{2\times 2}$. The u_i, u_i^* behave like fermionic creation and annihilation operators corresponding to an exceptional (non-associative) Grassmannian algebra

$$\{u_i, u_j\} = \{u_i^*, u_j^*\} = 0, \quad \{u_i, u_j^*\} = -\delta_{ij}. \tag{4.8}$$

$$\frac{1}{2}[u_i, u_j] = \epsilon_{ijk}\ u_k^*, \quad \frac{1}{2}[u_i^*, u_j^*] = \epsilon_{ijk}\ u_k, \quad u_o^2 = u_o, \quad (u_o^*)^2 = u_o^*. \tag{4.9}$$

Unlike the octonionic algebra, the split-octonionic algebra contains zero divisors and therefore is *not* a division algebra.

The Zorn *modified* matrix product of

$$\mathbf{A} = \begin{pmatrix} A_o\ \omega_o & -A_i\ \omega^i \\ B_i\ \omega^i & B_o\ \omega_o \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} C_o\ \omega_o & -C_i\ \omega^i \\ D_i\ \omega^i & D_o\ \omega_o \end{pmatrix} \tag{4.10}$$

and which encodes the nonassociativity of octonions is defined by

$$\mathbf{A}\bullet\mathbf{B} \ = \ \begin{pmatrix} (A_oC_o + A_iD_i)\ \omega_o & -(A_oC_k + D_oA_k + \epsilon_{ijk}B_iD_j)\ \omega^k \\ (C_oB_k + B_oD_k + \epsilon_{ijk}A_iC_j)\ \omega^k & (B_oD_o + B_iC_i)\ \omega_o \end{pmatrix} \tag{4.11}$$

where we have used

$$\omega_i\ \omega_j = \epsilon_{ijk}\omega_k - \delta_{ij}\omega_o \ \Rightarrow\ \omega_i\ \omega_i = -\omega_o,\ \textit{for each}\ i = 1,2,3\ \Rightarrow$$

$$\vec{x}.\vec{y} = (x_i\omega_i)\ (y_i\omega_i) = -x_iy_i\ \omega_o. \tag{4.12}$$

Therefore, the multiplication product of the split-octonions generators u_o, u_o^*, u_i, u_i^* is reproduced in this Zorn matrix realization.

The split-octonionic-valued metric $\mathbf{G}_{\mu\nu} = G^{ab}_{\mu\nu}$ can be represented by a 4×4 (tensor-valued) Zorn matrix as [21]

$$\mathbf{G}_{\mu\nu} \ = \ \begin{pmatrix} (g_{\mu\nu} + i\ g_{[\mu\nu]})\ \omega_o & -\ s^i_{[\mu\nu]}\ \omega_i \\ r^i_{[\mu\nu]}\ \omega_i & (\ g_{\mu\nu}\ +\ i\ g_{[\mu\nu]}\ )\ \omega_o \end{pmatrix} \tag{4.13}$$

Therefore, the split-octonions case permits to *associate* to the $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued metric $\mathbf{g}_{\mu\nu}$ a metric $\mathbf{G}_{\mu\nu} = G^{MN}_{\mu\nu}$ whose entries are now comprised of internal 8×8 complex matrices, instead of 16×16 complex matrices. In this fashion one can rewrite the line element as follows

$$ds^2 \ = \ \frac{1}{16} \ \left(\ Trace_{8\times 8} \ \{ \ G^{MN}_{\mu\nu} \ dx^\mu \ dx^\nu \ \} \ + \ complex \ conjugate \ \right) \qquad (4.14)$$

The *isometry* group that leaves *invariant* the line element in eq-(4.14) is the unitary $U(8)$ group. Similarly, one can construct the analog of the action in eq-(4.5) for the split-octonion case.

4.3 The Standard Gravity and Yang-Mills Actions

We finalize this section by describing the relation to the standard gravity and Yang-Mills Actions. To begin with one can realize that there are problems with quadratic curvature actions like

$$\int \ < \ \mathbf{g}^{\mu\nu} \ \mathbf{g}^{\rho\sigma} \ \bar{\mathbf{F}}_{\mu\rho} \ \mathbf{F}_{\nu\sigma} \ >, \quad \int \ < \ \bar{\mathbf{R}}_{\mu\nu\rho\sigma} \ \mathbf{R}^{\mu\nu\rho\sigma} \ >, \quad \cdots ... \qquad (4.15)$$

(as usual $< \cdots >$ denotes taking the real part) because the composition algebra $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$ is non-commutative, non-associative, *and* non-alternative [7]. To raise the four indices in $< \bar{\mathbf{R}}\mathbf{R} >$ requires the product of 4 factors of the metric $\mathbf{g}$ making matters more problematic because the Moufang identities, like $(AB)(CA) = A(BC)A$ are *no* longer obeyed due to the loss of alternativity.

For the time being we shall discard the other metric components, and raise/lower spacetime indices with the base spacetime metric $g_{\mu\nu}$ to simplify things. Actions based on terms linear in the curvature $\int < \mathbf{R} >$ furnish the standard Einstein-Hilbert action $\int \mathcal{R}$ if one chooses for the integral measure $\sqrt{det \ |g_{\mu\nu}|}$. In doing so, we also may build quadratic curvature actions like

$$S_2[F] \ = \ \int \ < g^{\mu\nu} \ g^{\rho\sigma} \ \bar{\mathbf{F}}_{\mu\rho} \ \mathbf{F}_{\nu\sigma} > \ = \ \int \ g^{\mu\nu} \ g^{\rho\sigma} \ (F^{IA}_{\mu\rho})^* \ F^{JB}_{\nu\sigma} \ \delta_{AB} \ \delta_{IJ} \qquad (4.16)$$

($I = 0, 1, 2, 3; A = 0, 1, 2, 3, \cdots, 7$), and

$$S_2[\mathcal{R}] \ = \ \int \ c_1 \ \mathcal{R}^2 \ + c_2 \ (\mathcal{R}_{\mu\nu})^2 \ + c_3 \ (\mathcal{R}_{\mu\nu\rho\sigma})^2 \qquad (4.17)$$

An action linear in $\mathcal{R}$, and quadratic in $\mathbf{F}$

$$S \ = \ \frac{1}{16\pi G} \ \int \ d^4x \ \sqrt{|det \ g_{\mu\nu}|} \ \left(\ \mathcal{R} \ - \kappa^2 \ (F^{IA}_{\mu\nu}) \ (F^{\mu\nu}_{IA})^* \ \right) \qquad (4.18)$$

leads to the standard gravity-Yang-Mills action. κ is a length parameter, and the metric signature is chosen to be Lorentzian $(-,+,+,+)$. Earlier on in section **3** we found how the 32 complex-valued fields A_μ^{IA}, and field strengths $F_{\mu\nu}^{IA}$, have a one-to-one *correspondence* with the 64 real-valued fields associated with the $u(4) \oplus u(4) \oplus u(4) \oplus u(4)$ Lie algebra of the compact group $[U(4)]^4$, and which contains the Standard Model group. Thus, the correspondence

$$\frac{\kappa^2}{16\pi G} \sum_{I=0}^{3} \sum_{A=0}^{7} (F_{\mu\nu}^{IA})\ (F^{\mu\nu}_{IA})^* \ \leftrightarrow\ \frac{1}{4g^2} \sum_{m=1}^{m=4} \sum_{\alpha=1}^{16} \mathcal{F}_{\mu\nu}^{\alpha m}\ \mathcal{F}^{\mu\nu}_{\alpha m} \tag{4.19}$$

derived from eq-(4.18) allows to determine the $[U(4)]^4$ gauge coupling g (associated to the Yang-Mills Lagrangian in the right hand side of eq-(4.19)) from the relation

$$\frac{1}{4g^2} \ =\ \frac{\kappa^2}{16\pi G}\ \Rightarrow\ g^2\kappa^2 \ =\ 4\pi G \ =\ 4\pi L_P^2 \tag{4.20}$$

where L_P is the Planck scale.

5 Conclusions

To sum up, $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued Gravity naturally can describe a Grand Unified Field Theory of Einstein gravity with a $[SU(4)]^4$ Yang-Mills theory and leading to an extension of the Standard Model including a $fourth$ fermion family. The role of the extra metric element $h_{\mu\nu} = g^{oo}_{(\mu\nu)}$ found in eq-(3.3) within the context of *bimetric* theories of gravity (and dark energy) [36] deserves further scrutiny.

The introduction of matter fields and solutions to the generalized Einstein field equations for the $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued Gravitational theory based on the action (4.5) will be the subject of future investigations. For additional references on the role that Clifford and Division algebras have in grand unification see [35]. It is interesting to note that the net dimension of $R \times S^1 \times S^3 \times S^7$ is 12 as in F-theory [37]. The S^1, S^3, S^7 "spheres" correspond to the unit-norm complex, quaternion and octonion, respectively.

To finalize one must emphasize that the choice of the internal affinity $\mathbf{\Theta}^\rho_{\mu\nu} = \delta^\rho_\mu \mathbf{A}_\nu$ in section **3** was a very restrictive one. There are many more components for the internal affinity $\mathbf{\Theta}^\rho_{\mu\nu}$ in the most general case. Hence, the $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$-valued Gravitational theory is far richer in scope than the findings of this work. It is also warranted to incorporate Octonionic Ternary Gauge Field theories [35] into our program. In essence, one needs to explore deeper the question asked by many in the past : are there four forces in Nature due to the existence of four division algebras ?

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