

Cosine Function at Rational Argument and Infinite Product Representation

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"It is the spirit that quickeneth; the flesh profiteth nothing: the words that I speak unto you, they are spirit, and they are life." - John 6:63.

ABSTRACT. I used an identity for cosine function at rational argument involving finite sum of the gamma functions; hence, the representation of infinite product arose.

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1. INTRODUCTION

In present paper, I used the following identity [1, p. 10, Theorem 22]

$$\cos\left(\frac{p\pi}{q}\right) = \prod_{s=1}^q \frac{\Gamma^2\left(\frac{s}{q}\right)}{\Gamma\left(\frac{s}{q} - \frac{p}{q^2}\right)\Gamma\left(\frac{s}{q} + \frac{p}{q^2}\right)},$$

which enabled me to prove the infinite product representation for cosine function at rational argument

$$\cos\left(\frac{p\pi}{q}\right) = \prod_{\ell=0}^{\infty} \prod_{s=1}^q \left(1 - \frac{4p^2}{(2\ell q^2 + 2sq - q)^2}\right);$$

more specifically, I get

$$\begin{aligned} \frac{1}{2} &= \prod_{\ell=0}^{\infty} \left(1 - \frac{4}{(18\ell + 3)^2}\right) \left(1 - \frac{4}{(18\ell + 9)^2}\right) \left(1 - \frac{4}{(18\ell + 15)^2}\right) \\ &= \frac{5}{9} \cdot \frac{77}{81} \cdot \frac{221}{225} \cdot \frac{437}{441} \cdot \frac{725}{729} \cdot \frac{1085}{1089} \cdot \dots, \end{aligned}$$

$$\begin{aligned} \frac{\sqrt{2}}{2} &= \prod_{\ell=0}^{\infty} \left(1 - \frac{4}{(32\ell + 4)^2}\right) \left(1 - \frac{4}{(32\ell + 12)^2}\right) \left(1 - \frac{4}{(32\ell + 20)^2}\right) \left(1 - \frac{4}{(32\ell + 28)^2}\right) \\ &= \frac{12}{16} \cdot \frac{140}{144} \cdot \frac{396}{400} \cdot \frac{780}{784} \cdot \frac{1292}{1296} \cdot \frac{1932}{1936} \cdot \frac{2700}{2704} \cdot \frac{3596}{3600} \cdot \dots, \end{aligned}$$

$$\begin{aligned} \frac{1 + \sqrt{5}}{4} &= \prod_{\ell=0}^{\infty} \left(1 - \frac{4}{(50\ell + 5)^2}\right) \left(1 - \frac{4}{(50\ell + 15)^2}\right) \\ &\quad \left(1 - \frac{4}{(50\ell + 25)^2}\right) \left(1 - \frac{4}{(50\ell + 35)^2}\right) \left(1 - \frac{4}{(50\ell + 45)^2}\right) \\ &= \frac{21}{25} \cdot \frac{221}{225} \cdot \frac{621}{625} \cdot \frac{1221}{1225} \cdot \frac{2021}{2025} \cdot \frac{3021}{3025} \cdot \frac{4221}{4225} \cdot \frac{5621}{5625} \cdot \frac{7221}{7225} \cdot \frac{9021}{9025} \cdot \dots \end{aligned}$$

and so on.

2. PRELIMINARIES

I use the following classical formula, [2, Section 12.13; 3], which is a Corollary of the Weierstrass infinite product representation for the gamma function:

Corollary 2.1. *If k is a positive integer and $a_1 + a_2 + \dots + a_k = b_1 + b_2 + \dots + b_k$, where the a_j and b_j are complex numbers and no b_j is zero or a negative integer, then*

$$\prod_{\ell=0}^{\infty} \frac{(\ell + a_1) \cdot \dots \cdot (\ell + a_k)}{(\ell + b_1) \cdot \dots \cdot (\ell + b_k)} = \frac{\Gamma(b_1) \cdot \dots \cdot \Gamma(b_k)}{\Gamma(a_1) \cdot \dots \cdot \Gamma(a_k)}. \quad (2.1)$$

Proof. See [2, Section 12.13]. □

On the other hand, I will need the Theorem 22 [1, p. 10]

Theorem 2.2. *If p and q are positive integers, $p \leq q$ and $p/q \neq 1/2$, then*

$$\cos\left(\frac{p\pi}{q}\right) = \prod_{s=1}^q \frac{\Gamma^2\left(\frac{2s-1}{2q}\right)}{\Gamma\left(\frac{2s-1}{2q} + \frac{p}{q^2}\right)\Gamma\left(\frac{2s-1}{2q} - \frac{p}{q^2}\right)}, \quad (2.2)$$

where $\Gamma(z)$ denotes the gamma function and $\cos(z)$ denotes the cosine function.

3. THE MAIN COROLLARY

3.1. New infinite product representation for the cosine function at rational argument.

Corollary 3.1. *If p and q are positive integers, $p \leq q$ and $p/q \neq 1/2$, then*

$$\cos\left(\frac{p\pi}{q}\right) = \prod_{\ell=0}^{\infty} \prod_{s=1}^q \left(1 - \frac{4p^2}{(2\ell q^2 + 2sq - q)^2}\right), \quad (3.1)$$

where $\sin(z)$ denotes the sine function.

Proof. Note that

$$\frac{2s-1}{2q} + \frac{2s-1}{2q} = \frac{2s-1}{2q} + \frac{p}{q^2} + \frac{2s-1}{2q} - \frac{p}{q^2},$$

satisfy the condition $a_1 + a_2 = b_1 + b_2$; $k = 2$ is a positive integer; the a_j and b_j are complex numbers and no b_j is zero or a negative integer. From Corollary 2.1 and the right hand side of the (2.2), it follows that

$$\begin{aligned} \frac{\Gamma^2\left(\frac{2s-1}{2q}\right)}{\Gamma\left(\frac{2s-1}{2q} + \frac{p}{q^2}\right)\Gamma\left(\frac{2s-1}{2q} - \frac{p}{q^2}\right)} &= \prod_{\ell=0}^{\infty} \frac{\left(\ell + \frac{2s-1}{2q} - \frac{p}{q^2}\right)\left(\ell + \frac{2s-1}{2q} + \frac{p}{q^2}\right)}{\left(\ell + \frac{2s-1}{2q}\right)\left(\ell + \frac{2s-1}{2q}\right)} \\ &= \prod_{\ell=0}^{\infty} \left(1 - \frac{4p^2}{(2\ell q^2 + 2sq - q)^2}\right). \end{aligned} \quad (3.2)$$

From Theorem 2.2 and (3.2), I conclude that

$$\cos\left(\frac{p\pi}{q}\right) = \prod_{\ell=0}^{\infty} \prod_{s=1}^q \left(1 - \frac{4p^2}{(2\ell q^2 + 2sq - q)^2}\right),$$

which is the desired result. \square

Example 3.2. Set $p=1$ and $q=3$ in Corollary 3.1

$$\begin{aligned} \frac{1}{2} &= \prod_{\ell=0}^{\infty} \left(1 - \frac{4}{(18\ell+3)^2}\right) \left(1 - \frac{4}{(18\ell+9)^2}\right) \left(1 - \frac{4}{(18\ell+15)^2}\right) \\ &= \left(1 - \frac{4}{3^2}\right) \left(1 - \frac{4}{9^2}\right) \left(1 - \frac{4}{15^2}\right) \left(1 - \frac{4}{21^2}\right) \left(1 - \frac{4}{27^2}\right) \left(1 - \frac{4}{33^2}\right) \cdots \\ &= \frac{5}{9} \cdot \frac{77}{81} \cdot \frac{221}{225} \cdot \frac{437}{441} \cdot \frac{725}{729} \cdot \frac{1085}{1089} \cdots \end{aligned}$$

Example 3.3. Set $p=1$ and $q=4$ in Corollary 3.1

$$\begin{aligned} \frac{\sqrt{2}}{2} &= \prod_{\ell=0}^{\infty} \left(1 - \frac{4}{(32\ell+4)^2}\right) \left(1 - \frac{4}{(32\ell+12)^2}\right) \left(1 - \frac{4}{(32\ell+20)^2}\right) \left(1 - \frac{4}{(32\ell+28)^2}\right) \\ &= \left(1 - \frac{4}{4^2}\right) \left(1 - \frac{4}{12^2}\right) \left(1 - \frac{4}{20^2}\right) \left(1 - \frac{4}{28^2}\right) \\ &\quad \left(1 - \frac{4}{36^2}\right) \left(1 - \frac{4}{44^2}\right) \left(1 - \frac{4}{52^2}\right) \left(1 - \frac{4}{60^2}\right) \cdots \\ &= \frac{12}{16} \cdot \frac{140}{144} \cdot \frac{396}{400} \cdot \frac{780}{784} \cdot \frac{1292}{1296} \cdot \frac{1932}{1936} \cdot \frac{2700}{2704} \cdot \frac{3596}{3600} \cdots \end{aligned}$$

Example 3.4. Set $p=1$ and $q=5$ in Corollary 3.1

$$\begin{aligned} \frac{1+\sqrt{5}}{4} &= \prod_{\ell=0}^{\infty} \left(1 - \frac{4}{(50\ell+5)^2}\right) \left(1 - \frac{4}{(50\ell+15)^2}\right) \\ &\quad \left(1 - \frac{4}{(50\ell+25)^2}\right) \left(1 - \frac{4}{(50\ell+35)^2}\right) \left(1 - \frac{4}{(50\ell+45)^2}\right) \\ &= \left(1 - \frac{4}{5^2}\right) \left(1 - \frac{4}{15^2}\right) \left(1 - \frac{4}{25^2}\right) \left(1 - \frac{4}{35^2}\right) \left(1 - \frac{4}{45^2}\right) \\ &\quad \left(1 - \frac{4}{55^2}\right) \left(1 - \frac{4}{65^2}\right) \left(1 - \frac{4}{75^2}\right) \left(1 - \frac{4}{85^2}\right) \left(1 - \frac{4}{95^2}\right) \cdots \\ &= \frac{21}{25} \cdot \frac{221}{225} \cdot \frac{621}{625} \cdot \frac{1221}{1225} \cdot \frac{2021}{2025} \cdot \frac{3021}{3025} \cdot \frac{4221}{4225} \cdot \frac{5621}{5625} \cdot \frac{7221}{7225} \cdot \frac{9021}{9025} \cdots \end{aligned}$$

- [1] Guedes, Edigles, *Infinite Product Representations for Binomial Coefficient, Pochhammer's Symbol, Newton's Binomial and Exponential Function*, June 27, 2016, [viXra:1611.0049](#).
- [2] Whittaker, E. T. and Watson, G. N., *A Course of Modern Analysis*, Cambridge University Press, Cambridge, 1978.
- [3] Sondow, Jonathan and Yi, Huang, *New Wallis- and Catalan-Type Infinite Products for π , e and $\sqrt{2 + \sqrt{2}}$* , [arXiv:1005.2712](#).