

# Infinite Product Representations for Some Surd Numbers and Infinite Sum Representations for Some Logarithm Constants

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*"It is the spirit that quickeneth; the flesh profiteth nothing: the words that I speak unto you, they are spirit, and they are life."* - John 6:63.

ABSTRACT. I derive identities for some surd numbers, involving gamma functions; thence, I have represented them as infinite products.

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## 1. INTRODUCTION

In present paper, I derive the identity below

$$n^{\ell-1} \cdot \sqrt[n]{n} = \frac{\Gamma(\ell + \frac{1}{n})}{\Gamma(\frac{\ell}{n} + \frac{1}{n^2})} \prod_{k=1}^{n-1} \frac{\Gamma(\frac{k}{n})}{\Gamma(\frac{k}{n} + \frac{\ell}{n} + \frac{1}{n^2})},$$

which enabled me to prove the following infinite products

$$\begin{aligned} 2^{\ell-1/2} &= \prod_{k=0}^{\infty} \left( \frac{4k+2\ell+1}{4k+2} \right) \left( \frac{4k+2\ell+3}{4k+4\ell+2} \right), \\ 3^{\ell-2/3} &= \prod_{k=0}^{\infty} \left( \frac{9k+3\ell+1}{9k+3} \right) \left( \frac{9k+3\ell+4}{9k+6} \right) \left( \frac{9k+3\ell+7}{9k+9\ell+3} \right), \\ 4^{\ell-3/4} &= 2^{2\ell-3/2} = \\ &= \prod_{k=0}^{\infty} \left( \frac{16k+4\ell+1}{16k+4} \right) \left( \frac{16k+4\ell+5}{16k+8} \right) \left( \frac{16k+4\ell+9}{16k+12} \right) \left( \frac{16k+4\ell+13}{16k+16\ell+4} \right), \\ 5^{\ell-4/5} &= \prod_{k=0}^{\infty} \left( \frac{25k+5\ell+1}{25k+25\ell+5} \right) \left( \frac{25k+5\ell+6}{25k+5} \right) \left( \frac{25k+5\ell+11}{25k+10} \right) \\ &\quad \left( \frac{25k+5\ell+16}{25k+15} \right) \left( \frac{25k+5\ell+21}{25k+20} \right), \end{aligned}$$

and so on; more specifically, I get

$$\begin{aligned}\sqrt{2} &= \prod_{k=0}^{\infty} \left( \frac{4k+3}{4k+2} \right) \left( \frac{4k+5}{4k+6} \right) \\ &= \frac{3}{2} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{9}{10} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{15}{14} \cdot \frac{17}{18} \cdot \dots\end{aligned}$$

$$\begin{aligned}\sqrt[3]{3} &= \prod_{k=0}^{\infty} \left( \frac{9k+4}{9k+3} \right) \left( \frac{9k+7}{9k+6} \right) \left( \frac{9k+10}{9k+12} \right) \\ &= \frac{4}{3} \cdot \frac{7}{6} \cdot \frac{10}{12} \cdot \frac{13}{12} \cdot \frac{16}{15} \cdot \frac{19}{21} \cdot \frac{22}{21} \cdot \frac{25}{24} \cdot \frac{28}{30} \cdot \dots\end{aligned}$$

$$\begin{aligned}\sqrt[4]{4} = \sqrt{2} &= \prod_{k=0}^{\infty} \left( \frac{16k+5}{16k+4} \right) \left( \frac{16k+9}{16k+8} \right) \left( \frac{16k+13}{16k+12} \right) \left( \frac{16k+17}{16k+20} \right) \\ &= \frac{5}{4} \cdot \frac{9}{8} \cdot \frac{13}{12} \cdot \frac{17}{20} \cdot \frac{21}{20} \cdot \frac{25}{24} \cdot \frac{29}{28} \cdot \frac{33}{36} \cdot \dots\end{aligned}$$

$$\begin{aligned}\sqrt[5]{5} &= \prod_{k=0}^{\infty} \left( \frac{25k+6}{25k+5} \right) \left( \frac{25k+11}{25k+10} \right) \left( \frac{25k+16}{25k+15} \right) \left( \frac{25k+21}{25k+20} \right) \left( \frac{25k+26}{25k+30} \right) \\ &= \frac{6}{5} \cdot \frac{11}{10} \cdot \frac{16}{15} \cdot \frac{21}{20} \cdot \frac{26}{30} \cdot \frac{31}{30} \cdot \frac{36}{35} \cdot \frac{41}{40} \cdot \frac{46}{45} \cdot \frac{51}{55} \cdot \dots\end{aligned}$$

and so on.

On the other hand, I derived the following infinite sum formulas for some logarithm constants

$$\frac{\ln 2}{2} = \sum_{k=0}^{\infty} \frac{4\ell^2 + (8k+4)\ell + 1}{(2k+2\ell+1)(4k+2\ell+1)(4k+2\ell+3)}$$

and

$$\frac{\ln 3}{3} = \sum_{k=0}^{\infty} \frac{81k^2(6\ell-1) + 18k(18\ell^2 + 21\ell - 1) + (6\ell+11)(3\ell+1)^2}{(3k+3\ell+1)(9k+3\ell+1)(9k+3\ell+4)(9k+3\ell+7)}.$$

more specifically, I get

$$\begin{aligned}\frac{\ln 2}{2} &= \sum_{k=0}^{\infty} \frac{8k+9}{(2k+3)(4k+3)(4k+5)} \\ &= \frac{9}{3 \cdot 3 \cdot 5} + \frac{17}{5 \cdot 7 \cdot 9} + \frac{25}{7 \cdot 11 \cdot 13} + \dots\end{aligned}$$

and

$$\begin{aligned}\frac{\ln 3}{3} &= \sum_{k=0}^{\infty} \frac{405k^2 + 684k + 272}{(3k+4)(9k+4)(9k+7)(9k+10)} \\ &= \frac{272}{4 \cdot 4 \cdot 7 \cdot 10} + \frac{1361}{7 \cdot 13 \cdot 16 \cdot 19} + \frac{3260}{10 \cdot 22 \cdot 25 \cdot 28} + \dots\end{aligned}$$

## 2. PRELIMINARIES

I use the following classical formula, [1, Section 12.13; 2], which is a Corollary of the Weierstrass infinite product representation for the gamma function:

**Corollary 2.1.** *If  $k$  is a positive integer and  $a_1 + a_2 + \dots + a_k = b_1 + b_2 + \dots + b_k$ , where the  $a_j$  and  $b_j$  are complex numbers and no  $b_j$  is zero or a negative integer, then*

$$\prod_{\ell=0}^{\infty} \frac{(\ell + a_1) \cdot \dots \cdot (\ell + a_k)}{(\ell + b_1) \cdot \dots \cdot (\ell + b_k)} = \frac{\Gamma(b_1) \cdot \dots \cdot \Gamma(b_k)}{\Gamma(a_1) \cdot \dots \cdot \Gamma(a_k)}. \quad (2.1)$$

**Proof.** See [1, Section 12.13]. □

The Gauss multiplication formula for gamma function assures me that

**Theorem 2.2.** (*Gauss*)

$$\forall z \notin \left\{ -\frac{m}{n} : m \in \mathbb{N} \right\} : \prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n}\right) = (2\pi)^{(n-1)/2} n^{1/2-nz} \Gamma(nz), \quad (2.2)$$

where  $\Gamma(z)$  denotes the gamma function.

**Proof.** See [3]. □

### 3. THE MAIN THEOREM

**3.1. The numbers of form  $n^{\ell-1} \cdot \sqrt[n]{n}$  like the finite product of gamma functions.**

**Theorem 3.1.** *If  $\ell \in \mathbb{N}$  and  $n \in \mathbb{N}_{\geq 2}$ , then*

$$n^{\ell-1} \cdot \sqrt[n]{n} = \frac{\Gamma\left(\ell + \frac{1}{n}\right)}{\Gamma\left(\frac{\ell}{n} + \frac{1}{n^2}\right)} \prod_{k=1}^{n-1} \frac{\Gamma\left(\frac{k}{n}\right)}{\Gamma\left(\frac{k}{n} + \frac{\ell}{n} + \frac{1}{n^2}\right)}. \quad (3.1)$$

**Proof.** From Theorem 2.2, I obtain with a bit of manipulation

$$\begin{aligned} \Gamma(z) \prod_{k=1}^{n-1} \Gamma\left(z + \frac{k}{n}\right) &= (2\pi)^{(n-1)/2} n^{1/2-nz} \Gamma(nz) \\ \Rightarrow \prod_{k=1}^{n-1} \Gamma\left(z + \frac{k}{n}\right) &= (2\pi)^{(n-1)/2} n^{1/2-nz} \frac{\Gamma(nz)}{\Gamma(z)}. \end{aligned} \quad (3.2)$$

Replace  $z$  by  $\left(\frac{\ell}{n} + \frac{1}{n^2}\right)$  in (3.2)

$$\begin{aligned} \prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n} + \frac{\ell}{n} + \frac{1}{n^2}\right) &= (2\pi)^{(n-1)/2} n^{1/2-\ell-1/n} \frac{\Gamma\left(\ell + \frac{1}{n}\right)}{\Gamma\left(\frac{\ell}{n} + \frac{1}{n^2}\right)} \\ \Rightarrow \frac{1}{\prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n} + \frac{\ell}{n} + \frac{1}{n^2}\right)} &= (2\pi)^{-(n-1)/2} n^{-1/2+\ell+1/n} \frac{\Gamma\left(\frac{\ell}{n} + \frac{1}{n^2}\right)}{\Gamma\left(\ell + \frac{1}{n}\right)}. \end{aligned} \quad (3.3)$$

From right hand side of the Theorem 3.1 and the right hand side of (3.3), it follows that

$$\begin{aligned} \frac{\Gamma\left(\ell + \frac{1}{n}\right)}{\Gamma\left(\frac{\ell}{n} + \frac{1}{n^2}\right)} \prod_{k=1}^{n-1} \frac{\Gamma\left(\frac{k}{n}\right)}{\Gamma\left(\frac{k}{n} + \frac{\ell}{n} + \frac{1}{n^2}\right)} &= \frac{\Gamma\left(\ell + \frac{1}{n}\right)}{\Gamma\left(\frac{\ell}{n} + \frac{1}{n^2}\right)} \cdot \frac{\prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right)}{\prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n} + \frac{\ell}{n} + \frac{1}{n^2}\right)} \\ &= \frac{\Gamma\left(\ell + \frac{1}{n}\right)}{\Gamma\left(\frac{\ell}{n} + \frac{1}{n^2}\right)} \cdot \left[ (2\pi)^{-(n-1)/2} n^{-1/2+\ell+1/n} \frac{\Gamma\left(\frac{\ell}{n} + \frac{1}{n^2}\right)}{\Gamma\left(\ell + \frac{1}{n}\right)} \right] \cdot \prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right) \\ &= (2\pi)^{-(n-1)/2} n^{-1/2+\ell+1/n} \prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right). \end{aligned} \quad (3.4)$$

On the other hand, replace  $z$  by  $1/\ell$ , let  $\ell$  tends to infinity in both members of (3.2)

$$\lim_{\ell \rightarrow \infty} \prod_{k=1}^{n-1} \Gamma\left(\frac{1}{\ell} + \frac{k}{n}\right) = \lim_{\ell \rightarrow \infty} (2\pi)^{(n-1)/2} n^{1/2-n/\ell} \frac{\Gamma\left(\frac{n}{\ell}\right)}{\Gamma\left(\frac{1}{\ell}\right)}. \quad (3.5)$$

Note that

$$\lim_{\ell \rightarrow \infty} \Gamma\left(\frac{1}{\ell} + \frac{k}{n}\right) = \Gamma\left(\frac{k}{n}\right), \quad (3.6)$$

$$\lim_{\ell \rightarrow \infty} n^{-n/\ell} = 1 \quad (3.7)$$

and

$$\lim_{\ell \rightarrow \infty} \frac{\Gamma\left(\frac{n}{\ell}\right)}{\Gamma\left(\frac{1}{\ell}\right)} = \frac{1}{n}. \quad (3.8)$$

From (3.5), (3.6), (3.7) and (3.8), I conclude that

$$\prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right) = \frac{(2\pi)^{(n-1)/2} n^{1/2}}{n}. \quad (3.9)$$

Now, from (3.4) and (3.9), it follows that

$$\begin{aligned} \frac{\Gamma\left(\ell + \frac{1}{n}\right)}{\Gamma\left(\frac{\ell}{n} + \frac{1}{n^2}\right)} \prod_{k=1}^{n-1} \frac{\Gamma\left(\frac{k}{n}\right)}{\Gamma\left(\frac{k}{n} + \frac{\ell}{n} + \frac{1}{n^2}\right)} &= (2\pi)^{-(n-1)/2} n^{-1/2+\ell+1/n} \prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right) \\ &= (2\pi)^{-(n-1)/2} n^{-1/2+\ell+1/n} \left[ \frac{(2\pi)^{(n-1)/2} n^{1/2}}{n} \right] \\ &= \frac{n^{\ell+1/n}}{n} = n^{\ell-1} \cdot \sqrt[n]{n}, \end{aligned}$$

which is the desired result of the left hand side of the Theorem above. This completes the proof.  $\square$

#### 4. SOME SURD NUMBERS AND INFINITE PRODUCT REPRESENTATION

##### 4.1. The $\sqrt{2}$ and some multiples of the form $2^{\ell-1/2}$ .

**Corollary 4.1.** *If  $\ell \in \mathbb{Z}^+$ , then*

$$2^{\ell-1/2} = \prod_{k=0}^{\infty} \left( \frac{4k+2\ell+1}{4k+2} \right) \left( \frac{4k+2\ell+3}{4k+4\ell+2} \right). \quad (4.1)$$

**Proof.** Set  $n=2$  in Theorem 3.1 and find

$$\begin{aligned} 2^{\ell-1} \cdot \sqrt{2} &= \frac{\Gamma(\ell + \frac{1}{2})}{\Gamma(\frac{\ell}{2} + \frac{1}{4})} \cdot \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2} + \frac{\ell}{2} + \frac{1}{4})} \\ \Rightarrow 2^{\ell-1/2} &= \frac{\Gamma(\ell + \frac{1}{2})}{\Gamma(\frac{\ell}{2} + \frac{1}{4})} \cdot \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{\ell}{2} + \frac{3}{4})}. \end{aligned} \quad (4.2)$$

Note that

$$\ell + \frac{1}{2} + \frac{1}{2} = \frac{\ell}{2} + \frac{1}{4} + \frac{\ell}{2} + \frac{3}{4},$$

satisfy the condition  $a_1 + a_2 = b_1 + b_2$ ;  $k=2$  is a positive integer; the  $a_j$  and  $b_j$  are complex numbers and no  $b_j$  is zero or a negative integer. From Corollary 2.1 and (4.2), it follows that

$$\begin{aligned} 2^{\ell-1/2} &= \prod_{k=0}^{\infty} \left( \frac{k + \frac{\ell}{2} + \frac{1}{4}}{k + \ell + \frac{1}{2}} \right) \left( \frac{k + \frac{\ell}{2} + \frac{3}{4}}{k + \frac{1}{2}} \right) \\ &= \prod_{k=0}^{\infty} \left( \frac{4k+2\ell+1}{4k+2} \right) \left( \frac{4k+2\ell+3}{4k+4\ell+2} \right), \end{aligned}$$

which is the desired result. □

**Example 4.2.** Let  $\ell=1$  in Corollary 4.1 and encounter

$$\begin{aligned} \sqrt{2} &= \prod_{k=0}^{\infty} \left( \frac{4k+3}{4k+2} \right) \left( \frac{4k+5}{4k+6} \right) \\ &= \frac{3}{2} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{9}{10} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{15}{14} \cdot \frac{17}{18} \cdot \dots \end{aligned}$$

**Example 4.3.** Let  $\ell=2$  in Corollary 4.1 and encounter

$$\begin{aligned} 2\sqrt{2} &= \prod_{k=0}^{\infty} \left( \frac{4k+5}{4k+2} \right) \left( \frac{4k+7}{4k+10} \right) \\ &= \frac{5}{2} \cdot \frac{7}{10} \cdot \frac{9}{6} \cdot \frac{11}{14} \cdot \frac{13}{10} \cdot \frac{15}{18} \cdot \frac{17}{14} \cdot \frac{19}{22} \cdot \dots \end{aligned}$$

**Example 4.4.** Let  $\ell=3$  in Corollary 4.1 and encounter

$$\begin{aligned} 4\sqrt{2} &= \prod_{k=0}^{\infty} \left( \frac{4k+7}{4k+2} \right) \left( \frac{4k+9}{4k+14} \right) \\ &= \frac{7}{2} \cdot \frac{9}{14} \cdot \frac{11}{6} \cdot \frac{13}{18} \cdot \frac{15}{10} \cdot \frac{17}{22} \cdot \frac{19}{14} \cdot \frac{21}{26} \cdot \dots \end{aligned}$$

**4.2.** The  $\sqrt[3]{3}$  and some multiples of the form  $3^{\ell-2/3}$ .

**Corollary 4.5.** *If  $\ell \in \mathbb{Z}^+$ , then*

$$3^{\ell-\frac{2}{3}} = \prod_{k=0}^{\infty} \left( \frac{9k+3\ell+1}{9k+3} \right) \left( \frac{9k+3\ell+4}{9k+6} \right) \left( \frac{9k+3\ell+7}{9k+9\ell+3} \right). \quad (4.3)$$

**Proof.** Set  $n=3$  in Theorem 3.1 and find

$$\begin{aligned} 3^{\ell-1} \cdot \sqrt[3]{3} &= \frac{\Gamma(\ell+\frac{1}{3})}{\Gamma(\frac{\ell}{3}+\frac{1}{9})} \prod_{k=1}^2 \frac{\Gamma(\frac{k}{3})}{\Gamma(\frac{k}{3}+\frac{\ell}{3}+\frac{1}{9})} \\ \Rightarrow 3^{\ell-\frac{2}{3}} &= \frac{\Gamma(\ell+\frac{1}{3})}{\Gamma(\frac{\ell}{3}+\frac{1}{9})} \cdot \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{\ell}{3}+\frac{4}{9})} \cdot \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{\ell}{3}+\frac{7}{9})} \end{aligned} \quad (4.4)$$

Note that

$$\ell + \frac{1}{3} + \frac{1}{3} + \frac{2}{3} = \frac{\ell}{3} + \frac{1}{9} + \frac{\ell}{3} + \frac{4}{9} + \frac{\ell}{3} + \frac{7}{9},$$

satisfy the condition  $a_1 + a_2 + a_3 = b_1 + b_2 + b_3$ ;  $k=3$  is a positive integer; the  $a_j$  and  $b_j$  are complex numbers and no  $b_j$  is zero or a negative integer. From Corollary 2.1 and (4.4), it follows that

$$\begin{aligned} 3^{\ell-\frac{2}{3}} &= \prod_{k=0}^{\infty} \left( \frac{k+\frac{\ell}{3}+\frac{1}{9}}{k+\frac{1}{3}} \right) \left( \frac{k+\frac{\ell}{3}+\frac{4}{9}}{k+\frac{2}{3}} \right) \left( \frac{k+\frac{\ell}{3}+\frac{7}{9}}{k+\ell+\frac{1}{3}} \right) \\ &= \prod_{k=0}^{\infty} \left( \frac{9k+3\ell+1}{9k+3} \right) \left( \frac{9k+3\ell+4}{9k+6} \right) \left( \frac{9k+3\ell+7}{9k+9\ell+3} \right) \end{aligned}$$

which is the desired result. □

**Example 4.6.** Let  $\ell=1$  in Corollary 4.5 and encounter

$$\begin{aligned} \sqrt[3]{3} &= \prod_{k=0}^{\infty} \left( \frac{9k+4}{9k+3} \right) \left( \frac{9k+7}{9k+6} \right) \left( \frac{9k+10}{9k+12} \right) \\ &= \frac{4}{3} \cdot \frac{7}{6} \cdot \frac{10}{12} \cdot \frac{13}{12} \cdot \frac{16}{15} \cdot \frac{19}{21} \cdot \frac{22}{21} \cdot \frac{25}{24} \cdot \frac{28}{30} \cdots \end{aligned}$$

**Example 4.7.** Let  $\ell=2$  in Corollary 4.5 and encounter

$$\begin{aligned} 3\sqrt[3]{3} &= \prod_{k=0}^{\infty} \left( \frac{9k+7}{9k+3} \right) \left( \frac{9k+10}{9k+6} \right) \left( \frac{9k+13}{9k+21} \right) \\ &= \frac{7}{3} \cdot \frac{10}{6} \cdot \frac{13}{21} \cdot \frac{16}{12} \cdot \frac{19}{15} \cdot \frac{22}{30} \cdot \frac{25}{21} \cdot \frac{28}{24} \cdot \frac{31}{39} \cdots \end{aligned}$$

**Example 4.8.** Let  $\ell=3$  in Corollary 4.5 and encounter

$$\begin{aligned} 9\sqrt[3]{3} &= \prod_{k=0}^{\infty} \left( \frac{9k+10}{9k+3} \right) \left( \frac{9k+13}{9k+6} \right) \left( \frac{9k+16}{9k+30} \right) \\ &= \frac{10}{3} \cdot \frac{13}{6} \cdot \frac{16}{30} \cdot \frac{19}{12} \cdot \frac{22}{15} \cdot \frac{25}{39} \cdot \frac{28}{21} \cdot \frac{31}{24} \cdot \frac{34}{48} \cdots \end{aligned}$$

**4.3. The  $\sqrt[4]{4}$  and some multiples of the form  $4^{\ell-3/4}$ .**

**Corollary 4.9.** *If  $\ell \in \mathbb{Z}^+$ , then*

$$4^{\ell-3/4} = 2^{2\ell-3/2} = \prod_{k=0}^{\infty} \left( \frac{16k+4\ell+1}{16k+4} \right) \left( \frac{16k+4\ell+5}{16k+8} \right) \left( \frac{16k+4\ell+9}{16k+12} \right) \left( \frac{16k+4\ell+13}{16k+16\ell+4} \right). \quad (4.5)$$

**Proof.** Set  $n=4$  in Theorem 3.1 and find

$$\begin{aligned} 4^{\ell-1} \cdot \sqrt[4]{4} &= \frac{\Gamma(\ell + \frac{1}{4})}{\Gamma(\frac{\ell}{4} + \frac{1}{16})} \prod_{k=1}^3 \frac{\Gamma(\frac{k}{4})}{\Gamma(\frac{k}{4} + \frac{\ell}{4} + \frac{1}{16})} \\ \Rightarrow 4^{\ell-3/4} &= \frac{\Gamma(\ell + \frac{1}{4})}{\Gamma(\frac{\ell}{4} + \frac{1}{16})} \cdot \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{\ell}{4} + \frac{5}{16})} \cdot \frac{\Gamma(\frac{2}{4})}{\Gamma(\frac{\ell}{4} + \frac{9}{16})} \cdot \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{\ell}{4} + \frac{13}{16})} \end{aligned} \quad (4.6)$$

Note that

$$\ell + \frac{1}{4} + \frac{1}{4} + \frac{2}{4} + \frac{3}{4} = \frac{\ell}{4} + \frac{1}{16} + \frac{\ell}{4} + \frac{5}{16} + \frac{\ell}{4} + \frac{9}{16} + \frac{\ell}{4} + \frac{13}{16},$$

satisfy the condition  $a_1 + a_2 + a_3 + a_4 = b_1 + b_2 + b_3 + b_4$ ;  $k=4$  is a positive integer; the  $a_j$  and  $b_j$  are complex numbers and no  $b_j$  is zero or a negative integer. From Corollary 2.1 and (4.6), it follows that

$$\begin{aligned} 4^{\ell-3/4} &= \prod_{k=0}^{\infty} \left( \frac{k + \frac{\ell}{4} + \frac{5}{16}}{k + \frac{1}{4}} \right) \left( \frac{k + \frac{\ell}{4} + \frac{9}{16}}{k + \frac{5}{4}} \right) \left( \frac{k + \frac{\ell}{4} + \frac{13}{16}}{k + \frac{9}{4}} \right) \left( \frac{k + \frac{\ell}{4} + \frac{1}{16}}{k + \ell + \frac{1}{4}} \right) \\ &\Rightarrow 4^{\ell-3/4} = 2^{2\ell-3/2} = \\ &= \prod_{k=0}^{\infty} \left( \frac{16k+4\ell+1}{16k+4} \right) \left( \frac{16k+4\ell+5}{16k+8} \right) \left( \frac{16k+4\ell+9}{16k+12} \right) \left( \frac{16k+4\ell+13}{16k+16\ell+4} \right), \end{aligned}$$

which is the desired result.  $\square$

**Example 4.10.** Let  $\ell=1$  in Corollary 4.9 and encounter

$$\begin{aligned} \sqrt[4]{4} = \sqrt{2} &= \prod_{k=0}^{\infty} \left( \frac{16k+5}{16k+4} \right) \left( \frac{16k+9}{16k+8} \right) \left( \frac{16k+13}{16k+12} \right) \left( \frac{16k+17}{16k+20} \right) \\ &= \frac{5}{4} \cdot \frac{9}{8} \cdot \frac{13}{12} \cdot \frac{17}{20} \cdot \frac{21}{20} \cdot \frac{25}{24} \cdot \frac{29}{28} \cdot \frac{33}{36} \cdots \end{aligned}$$

**Example 4.11.** Let  $\ell=2$  in Corollary 4.9 and encounter

$$\begin{aligned} \sqrt[4]{4} = 4\sqrt{2} &= \prod_{k=0}^{\infty} \left( \frac{16k+9}{16k+4} \right) \left( \frac{16k+13}{16k+8} \right) \left( \frac{16k+17}{16k+12} \right) \left( \frac{16k+21}{16k+36} \right) \\ &= \frac{9}{4} \cdot \frac{13}{8} \cdot \frac{17}{12} \cdot \frac{21}{36} \cdot \frac{25}{20} \cdot \frac{29}{24} \cdot \frac{33}{28} \cdot \frac{38}{52} \cdots \end{aligned}$$

**Example 4.12.** Let  $\ell=3$  in Corollary 4.9 and encounter

$$\begin{aligned} \sqrt[4]{4} = 16\sqrt{2} &= \prod_{k=0}^{\infty} \left( \frac{16k+13}{16k+4} \right) \left( \frac{16k+17}{16k+8} \right) \left( \frac{16k+21}{16k+12} \right) \left( \frac{16k+25}{16k+52} \right) \\ &= \frac{13}{4} \cdot \frac{17}{8} \cdot \frac{21}{12} \cdot \frac{25}{52} \cdot \frac{29}{20} \cdot \frac{33}{24} \cdot \frac{37}{28} \cdot \frac{41}{68} \cdots \end{aligned}$$

#### 4.4. The $\sqrt[5]{5}$ and some multiples of the form $5^{\ell-4/5}$ .

**Corollary 4.13.** *If  $\ell \in \mathbb{Z}^+$ , then*

$$5^{\ell-4/5} = \prod_{k=0}^{\infty} \left( \frac{25k+5\ell+1}{25k+25\ell+5} \right) \left( \frac{25k+5\ell+6}{25k+5} \right) \left( \frac{25k+5\ell+11}{25k+10} \right) \left( \frac{25k+5\ell+16}{25k+15} \right) \left( \frac{25k+5\ell+21}{25k+20} \right). \quad (4.7)$$

**Proof.** Set  $n=5$  in Theorem 3.1 and find

$$5^{\ell-1} \cdot \sqrt[5]{5} = \frac{\Gamma(\ell + \frac{1}{5})}{\Gamma(\frac{\ell}{5} + \frac{1}{25})} \prod_{k=1}^4 \frac{\Gamma(\frac{k}{5})}{\Gamma(\frac{k}{5} + \frac{\ell}{5} + \frac{1}{25})} \Rightarrow 5^{\ell-4/5} = \frac{\Gamma(\ell + \frac{1}{5})}{\Gamma(\frac{\ell}{5} + \frac{1}{25})} \cdot \frac{\Gamma(\frac{1}{5})}{\Gamma(\frac{\ell}{5} + \frac{6}{25})} \cdot \frac{\Gamma(\frac{2}{5})}{\Gamma(\frac{\ell}{5} + \frac{11}{25})} \cdot \frac{\Gamma(\frac{3}{5})}{\Gamma(\frac{\ell}{5} + \frac{16}{25})} \cdot \frac{\Gamma(\frac{4}{5})}{\Gamma(\frac{\ell}{5} + \frac{21}{25})} \quad (4.8)$$

Note that

$$\ell + \frac{1}{5} + \frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \frac{4}{5} = \frac{\ell}{5} + \frac{1}{25} + \frac{\ell}{5} + \frac{6}{25} + \frac{\ell}{5} + \frac{11}{25} + \frac{\ell}{5} + \frac{16}{25} + \frac{\ell}{5} + \frac{21}{25},$$

satisfy the condition  $a_1 + a_2 + a_3 + a_4 + a_5 = b_1 + b_2 + b_3 + b_4 + b_5$ ;  $k=5$  is a positive integer; the  $a_j$  and  $b_j$  are complex numbers and no  $b_j$  is zero or a negative integer. From Corollary 2.1 and (4.8), it follows that

$$5^{\ell-4/5} = \prod_{k=0}^{\infty} \left( \frac{k + \frac{\ell}{5} + \frac{1}{25}}{k + \ell + \frac{1}{5}} \right) \left( \frac{k + \frac{\ell}{5} + \frac{6}{25}}{k + \frac{1}{5}} \right) \left( \frac{k + \frac{\ell}{5} + \frac{11}{25}}{k + \frac{2}{5}} \right) \left( \frac{k + \frac{\ell}{5} + \frac{16}{25}}{k + \frac{3}{5}} \right) \left( \frac{k + \frac{\ell}{5} + \frac{21}{25}}{k + \frac{4}{5}} \right) = \prod_{k=0}^{\infty} \left( \frac{25k+5\ell+1}{25k+25\ell+5} \right) \left( \frac{25k+5\ell+6}{25k+5} \right) \left( \frac{25k+5\ell+11}{25k+10} \right) \left( \frac{25k+5\ell+16}{25k+15} \right) \left( \frac{25k+5\ell+21}{25k+20} \right),$$

which is the desired result.  $\square$

**Example 4.14.** Let  $\ell=1$  in Corollary 4.13 and encounter

$$\sqrt[5]{5} = \prod_{k=0}^{\infty} \left( \frac{25k+6}{25k+5} \right) \left( \frac{25k+11}{25k+10} \right) \left( \frac{25k+16}{25k+15} \right) \left( \frac{25k+21}{25k+20} \right) \left( \frac{25k+26}{25k+30} \right) = \frac{6}{5} \cdot \frac{11}{10} \cdot \frac{16}{15} \cdot \frac{21}{20} \cdot \frac{26}{30} \cdot \frac{31}{30} \cdot \frac{36}{35} \cdot \frac{41}{40} \cdot \frac{46}{45} \cdot \frac{51}{55} \cdot \dots$$

**Example 4.15.** Let  $\ell=2$  in Corollary 4.13 and encounter

$$5\sqrt[5]{5} = \prod_{k=0}^{\infty} \left( \frac{25k+11}{25k+5} \right) \left( \frac{25k+16}{25k+10} \right) \left( \frac{25k+21}{25k+15} \right) \left( \frac{25k+26}{25k+20} \right) \left( \frac{25k+31}{25k+30} \right) = \frac{11}{5} \cdot \frac{16}{10} \cdot \frac{21}{15} \cdot \frac{26}{20} \cdot \frac{31}{30} \cdot \frac{36}{30} \cdot \frac{41}{35} \cdot \frac{46}{40} \cdot \frac{51}{45} \cdot \frac{56}{80} \cdot \dots$$



**Example 4.16.** Let  $\ell = 3$  in Corollary 4.13 and encounter

$$\begin{aligned} 25^5\sqrt{5} &= \prod_{k=0}^{\infty} \left( \frac{25k+16}{25k+5} \right) \left( \frac{25k+21}{25k+10} \right) \left( \frac{25k+26}{25k+15} \right) \left( \frac{25k+31}{25k+20} \right) \left( \frac{25k+36}{25k+25} \right) \\ &= \frac{16}{5} \cdot \frac{21}{10} \cdot \frac{26}{15} \cdot \frac{31}{20} \cdot \frac{36}{25} \cdot \frac{41}{30} \cdot \frac{46}{35} \cdot \frac{51}{40} \cdot \frac{56}{45} \cdot \frac{61}{50} \cdots \end{aligned}$$

## 5. EXERCISES

**Exercise 5.1.** Using the Theorem 3.1 and Corollary 2.1, prove that

$$\begin{aligned} 6^{\ell-5/6} &= \prod_{k=0}^{\infty} \left( \frac{36k+6\ell+1}{36k+36\ell+6} \right) \left( \frac{36k+6\ell+7}{36k+6} \right) \left( \frac{36k+6\ell+13}{36k+12} \right) \\ &\quad \left( \frac{36k+6\ell+19}{36k+18} \right) \left( \frac{36k+6\ell+25}{36k+24} \right) \left( \frac{36k+6\ell+31}{36k+30} \right), \end{aligned} \quad (5.1)$$

$$\begin{aligned} 7^{\ell-6/7} &= \prod_{k=0}^{\infty} \left( \frac{49k+7\ell+1}{49k+49\ell+7} \right) \left( \frac{49k+7\ell+8}{49k+7} \right) \left( \frac{49k+7\ell+15}{49k+14} \right) \\ &\quad \left( \frac{49k+7\ell+22}{49k+21} \right) \left( \frac{49k+7\ell+29}{49k+28} \right) \left( \frac{49k+7\ell+36}{49k+35} \right) \left( \frac{49k+7\ell+43}{49k+42} \right), \end{aligned} \quad (5.2)$$

$$\begin{aligned} 8^{\ell-7/8} &= 2^{3\ell-21/8} = \prod_{k=0}^{\infty} \left( \frac{64k+8\ell+1}{64k+64\ell+8} \right) \left( \frac{64k+8\ell+9}{64k+8} \right) \left( \frac{64k+8\ell+17}{64k+16} \right) \\ &\quad \left( \frac{64k+8\ell+25}{64k+24} \right) \left( \frac{64k+8\ell+33}{64k+32} \right) \left( \frac{64k+8\ell+41}{64k+40} \right) \\ &\quad \left( \frac{64k+8\ell+49}{64k+48} \right) \left( \frac{64k+8\ell+57}{64k+56} \right), \end{aligned} \quad (5.3)$$

$$\begin{aligned} 9^{\ell-8/9} &= \prod_{k=0}^{\infty} \left( \frac{81k+9\ell+1}{81k+81\ell+9} \right) \left( \frac{81k+9\ell+10}{81k+9} \right) \left( \frac{81k+9\ell+19}{81k+18} \right) \\ &\quad \left( \frac{81k+9\ell+28}{81k+27} \right) \left( \frac{81k+9\ell+37}{81k+36} \right) \left( \frac{81k+9\ell+46}{81k+45} \right) \\ &\quad \left( \frac{81k+9\ell+55}{81k+54} \right) \left( \frac{81k+9\ell+64}{81k+63} \right) \left( \frac{81k+9\ell+73}{81k+72} \right), \end{aligned} \quad (5.4)$$

$$\begin{aligned} 10^{\ell-9/10} &= \prod_{k=0}^{\infty} \left( \frac{100k+10\ell+1}{100k+100\ell+10} \right) \left( \frac{100k+10\ell+11}{100k+10} \right) \left( \frac{100k+10\ell+21}{100k+20} \right) \\ &\quad \left( \frac{100k+10\ell+31}{100k+30} \right) \left( \frac{100k+10\ell+41}{100k+40} \right) \left( \frac{100k+10\ell+51}{100k+50} \right) \left( \frac{100k+10\ell+61}{100k+60} \right) \\ &\quad \left( \frac{100k+10\ell+71}{100k+70} \right) \left( \frac{100k+10\ell+81}{100k+80} \right) \left( \frac{100k+10\ell+91}{100k+90} \right) \end{aligned} \quad (5.5)$$

and

$$\begin{aligned} 11^{\ell-10/11} &= \prod_{k=0}^{\infty} \left( \frac{121k+11\ell+1}{121k+121\ell+11} \right) \left( \frac{121k+11\ell+12}{121k+11} \right) \left( \frac{121k+11\ell+23}{121k+22} \right) \\ &\quad \left( \frac{121k+11\ell+34}{121k+33} \right) \left( \frac{121k+11\ell+45}{121k+44} \right) \left( \frac{121k+11\ell+56}{121k+55} \right) \left( \frac{121k+11\ell+67}{121k+66} \right) \\ &\quad \left( \frac{121k+11\ell+78}{121k+77} \right) \left( \frac{121k+11\ell+89}{121k+88} \right) \left( \frac{121k+11\ell+100}{121k+99} \right) \left( \frac{121k+11\ell+111}{121k+110} \right) \end{aligned} \quad (5.6)$$

**Exercise 5.2.** Prove that

$$\begin{aligned} \sqrt[6]{6} &= \prod_{k=0}^{\infty} \left( \frac{36k+7}{36k+6} \right) \left( \frac{36k+13}{36k+12} \right) \left( \frac{36k+19}{36k+18} \right) \\ &\quad \left( \frac{36k+25}{36k+24} \right) \left( \frac{36k+31}{36k+30} \right) \left( \frac{36k+37}{36k+42} \right) \\ &= \frac{7}{6} \cdot \frac{13}{12} \cdot \frac{19}{18} \cdot \frac{25}{24} \cdot \frac{31}{30} \cdot \frac{37}{42} \cdot \dots, \end{aligned} \quad (5.7)$$

$$\begin{aligned} \sqrt[7]{7} &= \prod_{k=0}^{\infty} \left( \frac{49k+8}{49k+7} \right) \left( \frac{49k+15}{49k+14} \right) \left( \frac{49k+22}{49k+21} \right) \\ &\quad \left( \frac{49k+29}{49k+28} \right) \left( \frac{49k+36}{49k+35} \right) \left( \frac{49k+43}{49k+42} \right) \left( \frac{49k+50}{49k+56} \right) \\ &= \frac{8}{7} \cdot \frac{15}{14} \cdot \frac{22}{21} \cdot \frac{29}{28} \cdot \frac{36}{35} \cdot \frac{43}{42} \cdot \frac{50}{56} \cdot \dots, \end{aligned} \quad (5.8)$$

$$\begin{aligned} \sqrt[8]{8} &= \prod_{k=0}^{\infty} \left( \frac{64k+9}{64k+8} \right) \left( \frac{64k+17}{64k+16} \right) \left( \frac{64k+25}{64k+24} \right) \\ &\quad \left( \frac{64k+33}{64k+32} \right) \left( \frac{64k+41}{64k+40} \right) \left( \frac{64k+49}{64k+48} \right) \\ &\quad \left( \frac{64k+57}{64k+56} \right) \left( \frac{64k+65}{64k+72} \right) \\ &= \frac{9}{8} \cdot \frac{17}{16} \cdot \frac{25}{24} \cdot \frac{33}{32} \cdot \frac{41}{40} \cdot \frac{49}{48} \cdot \frac{57}{56} \cdot \frac{65}{72} \cdot \dots, \end{aligned} \quad (5.9)$$

$$\begin{aligned} \sqrt[9]{9} &= \prod_{k=0}^{\infty} \left( \frac{81k+10}{81k+9} \right) \left( \frac{81k+19}{81k+18} \right) \left( \frac{81k+28}{81k+27} \right) \\ &\quad \left( \frac{81k+37}{81k+36} \right) \left( \frac{81k+46}{81k+45} \right) \left( \frac{81k+55}{81k+54} \right) \\ &\quad \left( \frac{81k+64}{81k+63} \right) \left( \frac{81k+73}{81k+72} \right) \left( \frac{81k+82}{81k+90} \right) \\ &= \frac{10}{9} \cdot \frac{19}{18} \cdot \frac{28}{27} \cdot \frac{37}{36} \cdot \frac{46}{45} \cdot \frac{55}{54} \cdot \frac{64}{63} \cdot \frac{73}{72} \cdot \frac{82}{90} \cdot \dots, \end{aligned} \quad (5.10)$$

$$\begin{aligned} \sqrt[10]{10} &= \prod_{k=0}^{\infty} \left( \frac{100k+11}{100k+10} \right) \left( \frac{100k+21}{100k+20} \right) \left( \frac{100k+31}{100k+30} \right) \\ &\quad \left( \frac{100k+41}{100k+40} \right) \left( \frac{100k+51}{100k+50} \right) \left( \frac{100k+61}{100k+60} \right) \left( \frac{100k+71}{100k+70} \right) \\ &\quad \left( \frac{100k+81}{100k+80} \right) \left( \frac{100k+91}{100k+90} \right) \left( \frac{100k+101}{100k+110} \right) \\ &= \frac{11}{10} \cdot \frac{21}{20} \cdot \frac{31}{30} \cdot \frac{41}{40} \cdot \frac{51}{50} \cdot \frac{61}{60} \cdot \frac{71}{70} \cdot \frac{81}{80} \cdot \frac{91}{90} \cdot \frac{101}{110} \cdot \dots \end{aligned} \quad (5.11)$$

and

$$\begin{aligned} \sqrt[11]{11} &= \prod_{k=0}^{\infty} \left( \frac{121k+12}{121k+11} \right) \left( \frac{121k+23}{121k+22} \right) \left( \frac{121k+34}{121k+33} \right) \\ &\quad \left( \frac{121k+45}{121k+44} \right) \left( \frac{121k+56}{121k+55} \right) \left( \frac{121k+67}{121k+66} \right) \left( \frac{121k+78}{121k+77} \right) \\ &\quad \left( \frac{121k+89}{121k+88} \right) \left( \frac{121k+100}{121k+99} \right) \left( \frac{121k+111}{121k+110} \right) \left( \frac{121k+122}{121k+132} \right) \\ &= \frac{12}{11} \cdot \frac{23}{22} \cdot \frac{34}{33} \cdot \frac{45}{44} \cdot \frac{56}{55} \cdot \frac{67}{66} \cdot \frac{78}{77} \cdot \frac{89}{88} \cdot \frac{100}{99} \cdot \frac{111}{110} \cdot \frac{122}{132} \cdot \dots \end{aligned} \quad (5.12)$$

## 6. SOME LOGARITHM CONSTANTS AND INFINITE SUM REPRESENTATION

6.1. The  $\ln 2/2$ .

**Corollary 6.1.** *If  $\ell \in \mathbb{Z}^+$ , then*

$$\frac{\ln 2}{2} = \sum_{k=0}^{\infty} \frac{4\ell^2 + (8k+4)\ell + 1}{(2k+2\ell+1)(4k+2\ell+1)(4k+2\ell+3)}. \quad (6.1)$$

**Proof.** The logarithmic differentiation of the Corollary 4.1 with respect to  $\ell$  give me the desired result.  $\square$

**Example 6.2.** Let  $\ell = 1$  in Corollary 6.1 and encounter

$$\begin{aligned} \frac{\ln 2}{2} &= \sum_{k=0}^{\infty} \frac{8k+9}{(2k+3)(4k+3)(4k+5)} \\ &= \frac{9}{3 \cdot 3 \cdot 5} + \frac{17}{5 \cdot 7 \cdot 9} + \frac{25}{7 \cdot 11 \cdot 13} + \dots \end{aligned}$$

**Example 6.3.** Let  $\ell = 2$  in Corollary 6.1 and encounter

$$\begin{aligned} \frac{\ln 2}{2} &= \sum_{k=0}^{\infty} \frac{16k+25}{(2k+5)(4k+5)(4k+7)} \\ &= \frac{25}{5 \cdot 5 \cdot 7} + \frac{41}{7 \cdot 9 \cdot 11} + \frac{57}{9 \cdot 13 \cdot 15} + \dots \end{aligned}$$

6.2. The  $\ln 3/3$ .

**Corollary 6.4.** *If  $\ell \in \mathbb{Z}^+$ , then*

$$\frac{\ln 3}{3} = \sum_{k=0}^{\infty} \frac{81k^2(6\ell-1) + 18k(18\ell^2 + 21\ell - 1) + (6\ell+11)(3\ell+1)^2}{(3k+3\ell+1)(9k+3\ell+1)(9k+3\ell+4)(9k+3\ell+7)}. \quad (6.2)$$

**Proof.** The logarithmic differentiation of the Corollary 4.5 with respect to  $\ell$  give me the desired result.  $\square$

**Example 6.5.** Let  $\ell = 1$  in Corollary 6.4 and encounter

$$\begin{aligned} \frac{\ln 3}{3} &= \sum_{k=0}^{\infty} \frac{405k^2 + 684k + 272}{(3k+4)(9k+4)(9k+7)(9k+10)} \\ &= \frac{272}{4 \cdot 4 \cdot 7 \cdot 10} + \frac{1361}{7 \cdot 13 \cdot 16 \cdot 19} + \frac{3260}{10 \cdot 22 \cdot 25 \cdot 28} + \dots \end{aligned}$$

**Example 6.6.** Let  $\ell = 2$  in Corollary 6.4 and encounter

$$\begin{aligned} \frac{\ln 3}{3} &= \sum_{k=0}^{\infty} \frac{891k^2 + 2034k + 1127}{(3k+7)(9k+7)(9k+10)(9k+13)} \\ &= \frac{1127}{7 \cdot 7 \cdot 10 \cdot 13} + \frac{4052}{10 \cdot 16 \cdot 19 \cdot 22} + \frac{8759}{13 \cdot 25 \cdot 28 \cdot 31} + \dots \end{aligned}$$

7. APPLICATION FOR THE DIVISION BETWEEN GAMMA FUNCTIONS

**Theorem 7.1.** *If  $a \in \mathbb{Z}_{\geq 2}$ ,  $b \in \mathbb{Z}_{\geq 3}$ ,  $a < b$  and  $k \in \mathbb{N}$ , then*

$$\left(\frac{a}{b}\right)^k = \frac{a^b \sqrt{b}}{b^a \sqrt{a}} \cdot \frac{\Gamma(k + \frac{1}{a}) \Gamma(\frac{k}{b} + \frac{1}{b^2})}{\Gamma(k + \frac{1}{b}) \Gamma(\frac{k}{a} + \frac{1}{a^2})} \cdot \prod_{\nu=1}^{a-1} \frac{\Gamma(\frac{\nu}{a})}{\Gamma(\frac{\nu}{a} + \frac{k}{a} + \frac{1}{a^2})} \cdot \prod_{\nu=1}^{b-1} \frac{\Gamma(\frac{\nu}{b} + \frac{k}{b} + \frac{1}{b^2})}{\Gamma(\frac{\nu}{b})},$$

where  $\Gamma(z)$  denotes the gamma function.

**Proof.** In Theorem 3.1, replace  $n$  by  $a$ ,  $\ell$  by  $k$  and  $k$  by  $\nu$

$$a^{k-1} \cdot \sqrt[a]{a} = \frac{\Gamma(k + \frac{1}{a})}{\Gamma(\frac{k}{a} + \frac{1}{a^2})} \prod_{\nu=1}^{a-1} \frac{\Gamma(\frac{\nu}{a})}{\Gamma(\frac{\nu}{a} + \frac{k}{a} + \frac{1}{a^2})}. \quad (7.1)$$

In (7.2), replace  $a$  by  $b$

$$b^{k-1} \cdot \sqrt[b]{b} = \frac{\Gamma(k + \frac{1}{b})}{\Gamma(\frac{k}{b} + \frac{1}{b^2})} \prod_{\nu=1}^{b-1} \frac{\Gamma(\frac{\nu}{b})}{\Gamma(\frac{\nu}{b} + \frac{k}{b} + \frac{1}{b^2})}. \quad (7.2)$$

Divide (7.2) by (7.3) and find

$$\begin{aligned} \left(\frac{a}{b}\right)^k \cdot \frac{b^a \sqrt{a}}{a^b \sqrt{b}} &= \frac{\Gamma(k + \frac{1}{a}) \Gamma(\frac{k}{b} + \frac{1}{b^2})}{\Gamma(k + \frac{1}{b}) \Gamma(\frac{k}{a} + \frac{1}{a^2})} \cdot \frac{\prod_{\nu=1}^{a-1} \frac{\Gamma(\frac{\nu}{a})}{\Gamma(\frac{\nu}{a} + \frac{k}{a} + \frac{1}{a^2})}}{\prod_{\nu=1}^{b-1} \frac{\Gamma(\frac{\nu}{b})}{\Gamma(\frac{\nu}{b} + \frac{k}{b} + \frac{1}{b^2})}} \\ &\Rightarrow \left(\frac{a}{b}\right)^k = \frac{a^b \sqrt{b}}{b^a \sqrt{a}} \cdot \frac{\Gamma(k + \frac{1}{a}) \Gamma(\frac{k}{b} + \frac{1}{b^2})}{\Gamma(k + \frac{1}{b}) \Gamma(\frac{k}{a} + \frac{1}{a^2})} \cdot \frac{\prod_{\nu=1}^{a-1} \Gamma(\frac{\nu}{a})}{\prod_{\nu=1}^{b-1} \Gamma(\frac{\nu}{b})} \cdot \frac{\prod_{\nu=1}^{b-1} \Gamma(\frac{\nu}{b} + \frac{k}{b} + \frac{1}{b^2})}{\prod_{\nu=1}^{a-1} \Gamma(\frac{\nu}{a} + \frac{k}{a} + \frac{1}{a^2})} \\ &\Rightarrow \left(\frac{a}{b}\right)^k = \frac{a^b \sqrt{b}}{b^a \sqrt{a}} \cdot \frac{\Gamma(k + \frac{1}{a}) \Gamma(\frac{k}{b} + \frac{1}{b^2})}{\Gamma(k + \frac{1}{b}) \Gamma(\frac{k}{a} + \frac{1}{a^2})} \cdot \frac{\prod_{\nu=1}^{a-1} \Gamma(\frac{\nu}{a}) \prod_{\nu=1}^{b-1} \Gamma(\frac{\nu}{b} + \frac{k}{b} + \frac{1}{b^2})}{\prod_{\nu=1}^{b-1} \Gamma(\frac{\nu}{b}) \prod_{\nu=1}^{a-1} \Gamma(\frac{\nu}{a} + \frac{k}{a} + \frac{1}{a^2})} \\ &\Rightarrow \left(\frac{a}{b}\right)^k = \frac{a^b \sqrt{b}}{b^a \sqrt{a}} \cdot \frac{\Gamma(k + \frac{1}{a}) \Gamma(\frac{k}{b} + \frac{1}{b^2})}{\Gamma(k + \frac{1}{b}) \Gamma(\frac{k}{a} + \frac{1}{a^2})} \cdot \prod_{\nu=1}^{a-1} \frac{\Gamma(\frac{\nu}{a})}{\Gamma(\frac{\nu}{a} + \frac{k}{a} + \frac{1}{a^2})} \cdot \prod_{\nu=1}^{b-1} \frac{\Gamma(\frac{\nu}{b} + \frac{k}{b} + \frac{1}{b^2})}{\Gamma(\frac{\nu}{b})}, \end{aligned}$$

which is the desired result. □

**Corollary 7.2.** *I have*

$$\sqrt[3]{3} = \frac{\Gamma(\frac{1}{9}) \Gamma(\frac{4}{9}) \Gamma(\frac{6}{9}) \Gamma(\frac{7}{9})}{\Gamma(\frac{2}{9}) \Gamma(\frac{3}{9}) \Gamma(\frac{5}{9}) \Gamma(\frac{8}{9})}. \quad (7.3)$$

**Proof.** Set  $a = 2$  and  $b = 3$  in Theorem 7.1

$$\left(\frac{2}{3}\right)^k = \frac{\sqrt[3]{3}}{\sqrt{6\pi}} \cdot \frac{\Gamma(k + \frac{1}{2})\Gamma(\frac{k}{3} + \frac{1}{9})\Gamma(\frac{k}{3} + \frac{4}{9})\Gamma(\frac{k}{3} + \frac{7}{9})}{\Gamma(k + \frac{1}{3})\Gamma(\frac{k}{2} + \frac{1}{4})\Gamma(\frac{k}{2} + \frac{3}{4})}. \quad (7.4)$$

For  $k = 1$  in (7.4), I obtain

$$\frac{2\pi\sqrt{3}}{\sqrt[3]{3}} = \frac{\Gamma(\frac{1}{9})\Gamma(\frac{4}{9})\Gamma(\frac{7}{9})}{\Gamma(\frac{1}{3})}. \quad (7.5)$$

Multiply the numerator and the denominator in the right hand side of (7.5) by  $\Gamma(2/9)\Gamma(5/9)\Gamma(8/9)$  and apply the Euler's reflection formula for gamma function, i. e.,  $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ ,

$$\begin{aligned} \frac{2\pi\sqrt{3}}{\sqrt[3]{3}} &= \frac{\Gamma(\frac{1}{9})\Gamma(\frac{8}{9})\Gamma(\frac{4}{9})\Gamma(\frac{5}{9})\Gamma(\frac{2}{9})\Gamma(\frac{7}{9})}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{9})\Gamma(\frac{5}{9})\Gamma(\frac{8}{9})} \\ &\Rightarrow \Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{3}{9}\right)\Gamma\left(\frac{5}{9}\right)\Gamma\left(\frac{8}{9}\right) = \frac{4\pi^2\sqrt[3]{3}}{3}. \end{aligned} \quad (7.6)$$

Multiply the numerator and the denominator in the right hand side of (7.5) by  $\Gamma(2/3)$  and apply the Euler's reflection formula for gamma function, i. e.,  $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ ,

$$\begin{aligned} \frac{2\pi\sqrt{3}}{\sqrt[3]{3}} &= \frac{\Gamma(\frac{1}{9})\Gamma(\frac{4}{9})\Gamma(\frac{2}{3})\Gamma(\frac{7}{9})}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})} \\ &\Rightarrow \Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{4}{9}\right)\Gamma\left(\frac{6}{9}\right)\Gamma\left(\frac{7}{9}\right) = \frac{4\pi^2}{\sqrt[3]{3}}. \end{aligned} \quad (7.7)$$

Divide (7.7) by (7.6) and encounter the desired result.  $\square$

**Corollary 7.3.** *I have*

$$\begin{aligned} \sqrt[3]{3} &= \prod_{k=0}^{\infty} \frac{(9k+2)(9k+3)(9k+5)(9k+8)}{(9k+1)(9k+4)(9k+6)(9k+7)} \\ &= \frac{2}{1} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{8}{7} \cdot \frac{11}{10} \cdot \frac{12}{13} \cdot \frac{14}{15} \cdot \frac{17}{16} \cdot \dots \end{aligned} \quad (7.8)$$

**Proof.** I use the Corollary 2.1 in the right hand side of the Corollary 7.2.  $\square$

**Corollary 7.4.** *I have*

$$2 = \frac{\Gamma(\frac{1}{16})\Gamma(\frac{5}{16})\Gamma(\frac{9}{16})\Gamma(\frac{12}{16})\Gamma(\frac{13}{16})}{\Gamma(\frac{3}{16})\Gamma(\frac{4}{16})\Gamma(\frac{7}{16})\Gamma(\frac{11}{16})\Gamma(\frac{15}{16})}. \quad (7.9)$$

**Proof.** Set  $a = 3$  and  $b = 4$  in Theorem 7.1

$$\left(\frac{3}{4}\right)^k = \frac{3^4\sqrt{4}}{4^3\sqrt[3]{3}} \cdot \sqrt{\frac{2}{3\pi}} \cdot \frac{\Gamma(k + \frac{1}{3})\Gamma(\frac{k}{4} + \frac{1}{16})\Gamma(\frac{k}{4} + \frac{5}{16})\Gamma(\frac{k}{4} + \frac{9}{16})\Gamma(\frac{k}{4} + \frac{13}{16})}{\Gamma(k + \frac{1}{4})\Gamma(\frac{k}{3} + \frac{1}{9})\Gamma(\frac{k}{3} + \frac{4}{9})\Gamma(\frac{k}{3} + \frac{7}{9})}. \quad (7.10)$$

For  $k = 1$  in (7.10), I obtain

$$\frac{4\sqrt[3]{3}\sqrt{3\pi}}{3^4\sqrt[4]{4}}\sqrt{\frac{3\pi}{2}} = \frac{\Gamma(\frac{1}{3})\Gamma(\frac{1}{16})\Gamma(\frac{5}{16})\Gamma(\frac{9}{16})\Gamma(\frac{13}{16})}{\Gamma(\frac{1}{4})\Gamma(\frac{1}{9})\Gamma(\frac{4}{9})\Gamma(\frac{7}{9})}. \quad (7.11)$$

Multiply the numerator and the denominator in the right hand side of (7.11) by  $\Gamma(2/3)\Gamma(15/16)\Gamma(11/16)\Gamma(7/16)\Gamma(3/16)$  and apply the Euler's reflection formula for gamma function, i. e.,  $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ ,

$$\begin{aligned} \frac{4\sqrt[3]{3}\sqrt{3\pi}}{3^4\sqrt[4]{4}}\sqrt{\frac{3\pi}{2}} &= \frac{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})\Gamma(\frac{1}{16})\Gamma(\frac{15}{16})\Gamma(\frac{5}{16})\Gamma(\frac{11}{16})\Gamma(\frac{7}{16})\Gamma(\frac{9}{16})\Gamma(\frac{3}{16})\Gamma(\frac{13}{16})}{\Gamma(\frac{1}{9})\Gamma(\frac{4}{9})\Gamma(\frac{6}{9})\Gamma(\frac{7}{9})\Gamma(\frac{3}{16})\Gamma(\frac{4}{16})\Gamma(\frac{7}{16})\Gamma(\frac{11}{16})\Gamma(\frac{15}{16})} \\ &\Rightarrow \Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{4}{9}\right)\Gamma\left(\frac{6}{9}\right)\Gamma\left(\frac{7}{9}\right)\Gamma\left(\frac{3}{16}\right)\Gamma\left(\frac{4}{16}\right)\Gamma\left(\frac{7}{16}\right)\Gamma\left(\frac{11}{16}\right)\Gamma\left(\frac{15}{16}\right) \\ &= \frac{8\pi^5\sqrt[4]{4}}{\sqrt{\pi^3\sqrt[3]{3}}}. \end{aligned} \quad (7.12)$$

Multiply the numerator and the denominator in the right hand side of (7.5) by  $\Gamma(2/9)\Gamma(5/9)\Gamma(8/9)\Gamma(3/4)$  and apply the Euler's reflection formula for gamma function, i. e.,  $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ ,

$$\begin{aligned} \frac{4\sqrt[3]{3}\sqrt{3\pi}}{3^4\sqrt[4]{4}}\sqrt{\frac{3\pi}{2}} &= \frac{\Gamma(\frac{2}{9})\Gamma(\frac{3}{9})\Gamma(\frac{5}{9})\Gamma(\frac{8}{9})\Gamma(\frac{1}{16})\Gamma(\frac{5}{16})\Gamma(\frac{9}{16})\Gamma(\frac{12}{16})\Gamma(\frac{13}{16})}{\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})\Gamma(\frac{1}{9})\Gamma(\frac{8}{9})\Gamma(\frac{4}{9})\Gamma(\frac{5}{9})\Gamma(\frac{2}{9})\Gamma(\frac{7}{9})} \\ &\Rightarrow \Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{3}{9}\right)\Gamma\left(\frac{5}{9}\right)\Gamma\left(\frac{8}{9}\right)\Gamma\left(\frac{1}{16}\right)\Gamma\left(\frac{5}{16}\right)\Gamma\left(\frac{9}{16}\right)\Gamma\left(\frac{12}{16}\right)\Gamma\left(\frac{13}{16}\right) \\ &= \frac{32\pi^4\sqrt{\pi^3\sqrt[3]{3}}}{3^4\sqrt[4]{4}}. \end{aligned} \quad (7.13)$$

Divide (7.12) by (7.13) and encounter

$$\frac{\sqrt[3]{3}}{2} = \frac{\Gamma(\frac{1}{9})\Gamma(\frac{4}{9})\Gamma(\frac{6}{9})\Gamma(\frac{7}{9})\Gamma(\frac{3}{16})\Gamma(\frac{4}{16})\Gamma(\frac{7}{16})\Gamma(\frac{11}{16})\Gamma(\frac{15}{16})}{\Gamma(\frac{2}{9})\Gamma(\frac{3}{9})\Gamma(\frac{5}{9})\Gamma(\frac{8}{9})\Gamma(\frac{1}{16})\Gamma(\frac{5}{16})\Gamma(\frac{9}{16})\Gamma(\frac{12}{16})\Gamma(\frac{13}{16})}. \quad (7.14)$$

From Corollary 7.2 and (7.14), it follows that

$$\begin{aligned} \frac{1}{2} &= \frac{\Gamma(\frac{3}{16})\Gamma(\frac{4}{16})\Gamma(\frac{7}{16})\Gamma(\frac{11}{16})\Gamma(\frac{15}{16})}{\Gamma(\frac{1}{16})\Gamma(\frac{5}{16})\Gamma(\frac{9}{16})\Gamma(\frac{12}{16})\Gamma(\frac{13}{16})} \\ &\Rightarrow 2 = \frac{\Gamma(\frac{1}{16})\Gamma(\frac{5}{16})\Gamma(\frac{9}{16})\Gamma(\frac{12}{16})\Gamma(\frac{13}{16})}{\Gamma(\frac{3}{16})\Gamma(\frac{4}{16})\Gamma(\frac{7}{16})\Gamma(\frac{11}{16})\Gamma(\frac{15}{16})}, \end{aligned}$$

which is the desired result.  $\square$

**Corollary 7.5.** *I have*

$$\begin{aligned} 2 &= \prod_{k=0}^{\infty} \frac{(16k+3)(16k+4)(16k+7)(16k+11)(16k+15)}{(16k+1)(16k+5)(16k+9)(16k+12)(16k+13)} \\ &= \frac{3}{1} \cdot \frac{4}{5} \cdot \frac{7}{9} \cdot \frac{11}{12} \cdot \frac{15}{13} \cdot \frac{19}{17} \cdot \frac{20}{21} \cdot \frac{23}{25} \cdot \frac{27}{28} \cdot \frac{31}{29} \cdots \end{aligned} \quad (7.15)$$

**Proof.** I use the Corollary 2.1 in the right hand side of the Corollary 7.4.  $\square$

## 8. EXERCISES

**Exercise 8.1.** Use the Theorem 7.1 and prove that

$$\sqrt[5]{5^3} = \frac{\Gamma(\frac{1}{25})\Gamma(\frac{6}{25})\Gamma(\frac{11}{25})\Gamma(\frac{16}{25})\Gamma(\frac{20}{25})\Gamma(\frac{21}{25})}{\Gamma(\frac{4}{25})\Gamma(\frac{5}{25})\Gamma(\frac{9}{25})\Gamma(\frac{14}{25})\Gamma(\frac{19}{25})\Gamma(\frac{24}{25})}, \quad (8.1)$$

$$\sqrt[6]{6^4} = \sqrt[3]{6^2} = \frac{\Gamma(\frac{1}{36})\Gamma(\frac{7}{36})\Gamma(\frac{13}{36})\Gamma(\frac{19}{36})\Gamma(\frac{25}{36})\Gamma(\frac{30}{36})\Gamma(\frac{31}{36})}{\Gamma(\frac{5}{36})\Gamma(\frac{6}{36})\Gamma(\frac{11}{36})\Gamma(\frac{17}{36})\Gamma(\frac{23}{36})\Gamma(\frac{29}{36})\Gamma(\frac{35}{36})}, \quad (8.2)$$

$$\sqrt[7]{7^5} = \frac{\Gamma(\frac{1}{49})\Gamma(\frac{8}{49})\Gamma(\frac{15}{49})\Gamma(\frac{22}{49})\Gamma(\frac{29}{49})\Gamma(\frac{36}{49})\Gamma(\frac{42}{49})\Gamma(\frac{43}{49})}{\Gamma(\frac{6}{49})\Gamma(\frac{7}{49})\Gamma(\frac{13}{49})\Gamma(\frac{20}{49})\Gamma(\frac{27}{49})\Gamma(\frac{34}{49})\Gamma(\frac{41}{49})\Gamma(\frac{48}{49})}, \quad (8.3)$$

$$\sqrt[8]{8^6} = 4\sqrt[4]{2} = \frac{\Gamma(\frac{1}{64})\Gamma(\frac{9}{64})\Gamma(\frac{17}{64})\Gamma(\frac{25}{64})\Gamma(\frac{33}{64})\Gamma(\frac{41}{64})\Gamma(\frac{49}{64})\Gamma(\frac{56}{64})\Gamma(\frac{57}{64})}{\Gamma(\frac{7}{64})\Gamma(\frac{8}{64})\Gamma(\frac{15}{64})\Gamma(\frac{23}{64})\Gamma(\frac{31}{64})\Gamma(\frac{39}{64})\Gamma(\frac{47}{64})\Gamma(\frac{55}{64})\Gamma(\frac{63}{64})}, \quad (8.4)$$

$$\sqrt[9]{9^7} = \frac{\Gamma(\frac{1}{81})\Gamma(\frac{10}{81})\Gamma(\frac{19}{81})\Gamma(\frac{28}{81})\Gamma(\frac{37}{81})\Gamma(\frac{46}{81})\Gamma(\frac{55}{81})\Gamma(\frac{64}{81})\Gamma(\frac{72}{81})\Gamma(\frac{73}{81})}{\Gamma(\frac{8}{81})\Gamma(\frac{9}{81})\Gamma(\frac{17}{81})\Gamma(\frac{26}{81})\Gamma(\frac{35}{81})\Gamma(\frac{44}{81})\Gamma(\frac{53}{81})\Gamma(\frac{62}{81})\Gamma(\frac{71}{81})\Gamma(\frac{80}{81})}, \quad (8.5)$$

$$\begin{aligned} & \sqrt[10]{10^8} = \\ & = \frac{\Gamma(\frac{1}{100})\Gamma(\frac{11}{100})\Gamma(\frac{21}{100})\Gamma(\frac{31}{100})\Gamma(\frac{41}{100})\Gamma(\frac{51}{100})\Gamma(\frac{61}{100})\Gamma(\frac{71}{100})\Gamma(\frac{81}{100})\Gamma(\frac{90}{100})\Gamma(\frac{91}{100})}{\Gamma(\frac{9}{100})\Gamma(\frac{10}{100})\Gamma(\frac{19}{100})\Gamma(\frac{29}{100})\Gamma(\frac{39}{100})\Gamma(\frac{49}{100})\Gamma(\frac{59}{100})\Gamma(\frac{69}{100})\Gamma(\frac{79}{100})\Gamma(\frac{89}{100})\Gamma(\frac{99}{100})} \end{aligned} \quad (8.6)$$

and so on.

**Exercise 8.2.** Use the Corollary 2.1 and prove that

$$\sqrt[5]{5^3} = \prod_{k=0}^{\infty} \frac{(25k+4)(25k+5)(25k+9)(25k+14)(25k+19)(25k+24)}{(25k+1)(25k+6)(25k+11)(25k+16)(25k+20)(25k+21)}, \quad (8.7)$$

$$\begin{aligned} & \sqrt[6]{6^4} = \sqrt[3]{6^2} = \\ & = \prod_{k=0}^{\infty} \frac{(36k+5)(36k+6)(36k+11)(36k+17)(36k+23)(36k+29)(36k+35)}{(36k+1)(36k+7)(36k+13)(36k+19)(36k+25)(36k+30)(36k+31)} \end{aligned} \quad (8.8)$$

and so on.

## REFERENCES

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