A Geometrical Model of Gravity

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June 2018†

Abstract

A mathematical model for interpreting Newtonian gravity by means of elastic deformation of space is given.

Key Words: Gravity, Elastostatics, Particle Physics, Topology.

1 Introduction

In this paper we show that Newtonian gravity may be interpreted in a framework of equilibrium in elastic material.

2 An Elastic Model for Gravity

In this section we will show how it is possible to derive a non relativistic model for gravity based on the theory of elasticity.

2.1 Shearless Displacements in Elastic Material

We want to study the equilibrium in an elastic material in presence of a solution with vanishing curl for the displacement field (i.e a solution with shearless stress tensor). The vanishing curl may be due to several reasons but in particular we are interested in the following one:

- **Spherical symmetry:** A solution to a problem with spherical symmetry has clearly displacements with spherical symmetry and vanishing curl. If we consider a solution given by the superposition of several spherical symmetric solutions, given the linearity of the curls, this will also have have a vanishing curl.

To find our solution we start from the Navier-Lame equation for the equilibrium in elastic materials:

$$p + \mu_L \nabla^2 \mathbf{u} + (\lambda_L + \mu_L)\nabla\nabla \cdot \mathbf{u} = 0 \tag{1}$$

where p is the distributed force in the material, μ_L and λ_L are Lame's parameters end $\mathbf{u}$ is a field of displacements that solves our problem. In absence of distributed force ($p = 0$) and using the following identity:

$$\nabla^2 \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u}) \tag{2}$$

we can write Eq. (1) as folows:

$$\mu_L \nabla^2 \mathbf{u} + (\lambda_L + \mu_L)[\nabla^2 \mathbf{u} + \nabla \times (\nabla \times \mathbf{u})] = 0 \tag{3}$$

since we assume that $\nabla \times \mathbf{u} = 0$, we have immediately that the above equation simplifies as follows:

$$\nabla^2 \mathbf{u} = 0 \tag{4}$$

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†Posted at: http://vixra.org/abs/1806.0251 - Current version: v4 - Aug. 2021

The above vectorial equation together with the boundary conditions having the appropriate symmetry, can be used to find the field of displacements for problems where we know that the solution has shearless displacements. (i.e. problems with vanishing curl).

Finally, we are interested in strains. By definition, strains are represented by a dimensionless tensor $\boldsymbol{\epsilon}$ defined by:

$$\epsilon_{ii} = \frac{\partial u_i}{\partial x^i} \;;\; \epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i}\right) \tag{5}$$

Since in our case there are no shears, meaning that $\epsilon_{ij} = 0$ for $i \neq j$, strains are just a vector $\boldsymbol{\epsilon} = (\epsilon_{xx},\, \epsilon_{yy},\, \epsilon_{zz})$ which is the diagonal of a rank 2 tensor with two covariant indices.

2.2 Strains for the Space Deficiency Problem

Let us suppose we have a three-dimensional space composed of a material which is uniform, isotropic and elastic and that we remove some material as described by a space deficiency function $\lambda(x)$ (which will be better defined further on). The material will readjust creating displacements and strains and we want to find an equation to evaluate them. We will call this the Elastostatic Space Deficiency Problem.

We start from a simple case where we remove a discrete quantity of material from the origin of the axis described by the density deficiency function $\lambda_0^2\delta(x)$. This corresponds to removing from the space a ball of material having radius λ_0 and identifying the boundary of the ball to a point. The reason why we choose the coefficient of the delta to have dimensions of surfaces (and not volume or length) is that this is needed for the final equation to match for a dimensional point of view.

Using Eq. (4) we find that the solution to our spherical space deficiency problem is the following field of displacements and strains (see Appendix A.2):

$$\mathbf{u}(r,\theta,\phi) = -\frac{\lambda_0^2}{r}\hat{\mathbf{i}}_r \tag{6}$$

$$\boldsymbol{\epsilon}(r,\theta,\phi) = \frac{\lambda_0^2}{r^2}\hat{\mathbf{i}}_r \tag{7}$$

both defined for $r > \lambda_0$.

Now define a field $\boldsymbol{\mu}$ as follows:

$$\boldsymbol{\mu} = -Y\boldsymbol{\epsilon} \qquad \left[\frac{N}{m^2} \;\Rightarrow\; \frac{kg}{m\ s^2}\right] \tag{8}$$

where Y is the Young's modulus of the elastic material and $\boldsymbol{\mu}$ is basically a field equal to the usual field $\boldsymbol{\sigma}$ of stress but with opposite sign. Dimension of $\boldsymbol{\mu}$, being stress, is force per unit surface (i.e. pressure). Moreover, if we have a distributed space deficiency in our material obtained by removing tiny balls of material of external area s_0 and radius λ_0 with $\lambda_0^2 = \frac{s_0}{4\pi}$, we can define a space deficiency density function:

$$\lambda(x) = n(x)s_0 \;\;\Rightarrow\;\; n(x)\lambda_0^2 = \frac{\lambda(x)}{4\pi} \tag{9}$$

where $n(x)$ is the density of removed balls per unit area.

We note that given Eq. (7), if Σ is a sphere of equation $r = R$ in spherical coordinates, oriented outward and having interior V, we have:

$$\int_\Sigma (-\boldsymbol{\epsilon})\cdot \mathbf{dS} = \int_\Sigma \frac{\lambda_0^2}{R^2}R^2 dS = 4\pi\lambda_0^2 = 4\pi\int_V \lambda_0^2\delta(x) \tag{10}$$

Now, given any volume V having a closed surface Σ as its boundary with Σ oriented outward, for the Gauss theorem we have:

$$\int_\Sigma \boldsymbol{\mu}(x)\cdot \mathbf{dS} = 4\pi Y\int_V n(x)\lambda_0^2 dV = 4\pi Y\int_V \frac{\lambda(x)}{4\pi}dV \;\;\Rightarrow\;\; \nabla\cdot\boldsymbol{\mu} = Y\lambda \tag{11}$$

Since the field $\boldsymbol{\mu}$ is conservative (and has vanishing curl), it can be expressed in terms of a scalar potential ϕ,

$$\boldsymbol{\mu} = -\nabla\phi \tag{12}$$

from which we have:

$$\nabla \cdot (-\nabla\phi) = -Y\lambda \tag{13}$$

and eventually the Poisson's equation for the space deficiency problem:

$$\nabla^2\phi = Y\lambda \tag{14}$$

This was highly expected. Deficiency of space has dimensions of $[m^2]$ and therefore it is proportional to the external surfaces of the removed balls rather then their volume or radius.

Note that by solving our problem using Eq. (14), we assign to each point of the solution the original coordinate r and not the displaced one $r - \frac{\lambda_0^2}{r^2}$. In other words we give curvature to space by using the metric underlying the system of coordinates with the notation $\mathcal{O}(r, \theta, \phi)$ in Appendix A.1. Moreover, by doing so, we will have a region of space ($r > \lambda_0$) where we have a mathematical solution which is the extension of above metric, where we know there is no space at all.

2.3 Analogy between Elastostatics and Gravity

Equation (14) is also the equation of gravitational field in empty space. For the above reason, there is a perfect analogy between the field $\boldsymbol{\mu} = -Y\boldsymbol{\epsilon}$ and the Newtonian gravitational field $\mathbf{g}$. In this analogy we interpret stress (with opposite sign) due to the displacements field to be the equivalent to gravitational field and the function ϕ present in (14) to be the equivalent to gravitational potential.

It is possible to show that two space deficiencies experience an attractive force to each other as two masses would do and, if they are free to move, they would fall into each other. This is because the field $\boldsymbol{\mu}$ generated by each deficiency goes outward as $\frac{1}{r^2}$ exactly as for gravitational field and if we reduce the distance between the two deficiencies the energy stored in the sum of the two filed would be given by:

$$E = \frac{1}{2}Y\int_U |\boldsymbol{\epsilon}|^2 dV = \frac{1}{2}\frac{1}{Y}\int_U |\boldsymbol{\mu}|^2 dV \tag{15}$$

where V_0 is the whole space, V_1 and V_2 are the removed spaces and $U = V_0 - V_1 - V_2$.

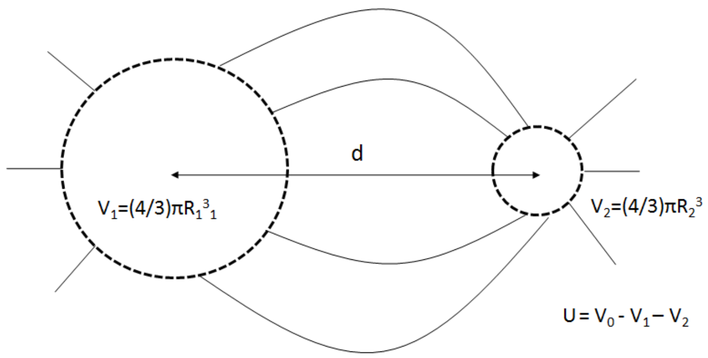

Figure 1: Field Generated by 2 space deficiencies

Now, if we reduce the distance d between the two deficiencies by a virtual displacement δd, the total energy stored in the field $\boldsymbol{\mu}$ decreases and the force experienced by the two deficiencies is equal to:

$$F = -\frac{\delta E}{\delta d} = \frac{Y}{4\pi}\frac{s_1^2 s_2^2}{d^2} \tag{16}$$

where s_1 and s_2 are the areas of the external surfaces of the two spherical decencies. This calculation is not difficult and it could be done explicitly. To see how it works refer to [1] where the calculation is carried out explicitly for the gravitational field but it could be done in exactly the same way for our deficiency problem.

In our analogy, since we identify μ with gravitational field, we can say that they are proportional by a constant Ω as follows:

$$\Omega = \frac{g}{\mu} \quad \left[\frac{m^2}{kg}\right] \tag{17}$$

Comparing the two expressions for the energy of gravity and space deficiency (see Eq. (20e)) we find:

$$\Omega^2 = 4\pi \frac{G}{Y} \tag{18}$$

Finally, using the definition (17) above and comparing the two expressions for the fields of gravity and space deficiency (see the table below) we find:

$$m = \frac{Y\Omega}{4\pi G} s = \frac{\Omega}{\Omega^2} s = \frac{s}{\Omega} \quad \Rightarrow \quad \Omega = \frac{s}{m} \tag{19}$$

The analogy is summarised in the following table:

		Gravitation	Space Deficency
a)	Poisson's eq.	$\nabla^2 \phi(x) = 4\pi G \rho(x)$	$\nabla^2 \phi(x) = Y \lambda(x) = n(x)\, s_0$
b)	Potential	$\phi(r) = -G \frac{m}{r} \hat{\mathbf{i}}_r$ for $m\delta(x)$	$\phi(r) = -\frac{Y}{4\pi} \frac{s}{r} \hat{\mathbf{i}}_r$ for $s\delta(x)$
c)	Gradient	$\mathbf{g} = -\nabla \phi$	$\boldsymbol{\mu} = -\nabla \phi = -Y\epsilon = \frac{\mathbf{g}}{\Omega}$
d)	Field	$\mathbf{g}(r) = -G \frac{m}{r^2} \hat{\mathbf{i}}_r$ for $m\delta(x)$	$\boldsymbol{\mu}(r) = -\frac{Y}{4\pi} \frac{s}{r^2} \hat{\mathbf{i}}_r$ for $s\delta(x)$
e)	Energy	$E = \frac{1}{2} \left(\frac{1}{4\pi G}\right) \int_U \lvert\mathbf{g}\rvert^2 dV$	$E = \frac{1}{2} \frac{1}{Y} \int_U \lvert\boldsymbol{\mu}\rvert^2 dV$
f)	Force	$F = G \frac{m_1^2 m_2^2}{d^2}$	$F = \frac{Y}{4\pi} \frac{s_1^2 s_2^2}{d^2}$
g)	Mass	m	$s = \Omega m = 4\pi \lambda_0^2$

(20)

For what we have said so far, we can conclude this paragraph with our personal equivalence principle:

Equivalence principle. *Space deficiency is equivalent to mass and there is no classical mechanics experiment that can tell an observer if the measured gravitational field is due to space deficiency or mass. Moreover a spherical space deficiency having area of its external surface equal to* s *is equivalent to a mass* $m = \frac{s}{\Omega}$.

2.4 The Newtonian Event Horizon

In our model space is curved and curvature is expressed by the metric given by Eq. (32). An observer at distance $r > R$ would see a sphere or radius R in the origin with noting in it because space has been removed. We call this a Newtonian event horizon and we define it to be the radius of the mass generating the metric.

2.5 Space Deficiency vs Relativity

Our theory introduces curvature in space as the theory of General Relativity does. We want to compare the amount of curvature introduced by the two theories by comparing the amount of length contraction. We start from the General relativity. A mass M curves spaces as described by the metric given by the Schwarzschild solution:

$$ds^2 = -\left(1 - \frac{2MG}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2MG}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta + r^2 \sin\theta d\phi \tag{21}$$

for $dt = 0$, $d\theta =$ and $d\phi = 0$ we have the length contraction of a rood parallel to the radial coordinate given by:

$$\sqrt{\tilde{g}_{rr}} = \sqrt{\left(1 - \frac{2MG}{c^2 r}\right)^{(-1)}} \tag{22}$$

For $r \to \infty$ we have:

$$\sqrt{\tilde{g}_{rr}} = 1 + \frac{MG}{c^2 r} + 3\left(\frac{MG}{c^2 r}\right)^2 + o\left(\frac{1}{r}\right)^3 \tag{23}$$

For the space deficiency, using Eq.(32), we can evaluate the metric as:

$$\sqrt{g_{rr}} = 1 + \frac{R^2}{r^2} \tag{24}$$

If we compare $\sqrt{\tilde{g_{rr}}}$ with $\sqrt{g_{rr}}$ we see that the former goes like $\frac{1}{r}$ while the latter goes like $\frac{1}{r^2}$. This is expected since Eq. (20a) is not Lorentz invariant and therefore our theory is not relativistic. Although both metrics above represent space curvature, the former is a relativistic relation while the latter is Newtonian.

3 A Topological Particle Physics Theory

So far we have shown simple interesting mathematical facts from which it is clear that it is possible to have a full analogy between an elastic space deficiency theory and a Newtonian theory.

In Quantum Mechanics, all successful theory have been developed starting from an analogous classical theory which is then quantised. For Example in QFT, the starting point for the theory is a classical theory of fields with its Lagrangians.

Regardless the likelihood of space to be an elastic material, we think that the space deficiency model (maybe in a Lorentz invariant version) may be used as a starting point for making a quantum theory of particle.

In such a theory, we suggest to suppose space to be an elastic material. Energy can be interpreted as elastic energy of the material. The field to be quantised would be the space deficiency field which can be treated as a quantum field and at the same time would generate curvature in space where other field are present allowing a link between gravity and the other forces.

Moreover, supposing space is conserved, when a particle is created this may be interpreted as a change in the configuration of the material where space is reorganised in a final configuration equal to a compact 3D manifold attached to the space by means of a connected sum. The removed space would then be present in the attached manifold giving the space conservation. Space around the manifold would pull it due to its elastic property and the manifold would shrink to a tiny region (i.e. our particle) where the energy due to the bending of the space in the attached manifold would be in equilibrium with the energy due to the space which is pulled. In a few words, space, rather then disappear, would turn in a manifold attached to space itself and pulling it.

In this theory, particles are simply 3D manifolds attached to space and their mass is simply the deficiency of space caused by the fact that space assumes a different topological configuration (not locally homeomorphic to $\mathbb{R}^3$). Each particle would be associated to a specific (in the topological sense) manifold attached to the space and physical property of particles may then be explained by the topological properties of the relevant manifolds associated with them.

Appendix

A.1 Systems of Coordinates

In this paper, when dealing with the spherical space deficiency of radius R, we use spherical coordinates (r, θ, ϕ) that will always refer to coordinates where θ is the polar angle and ϕ is the azimuthal angle:

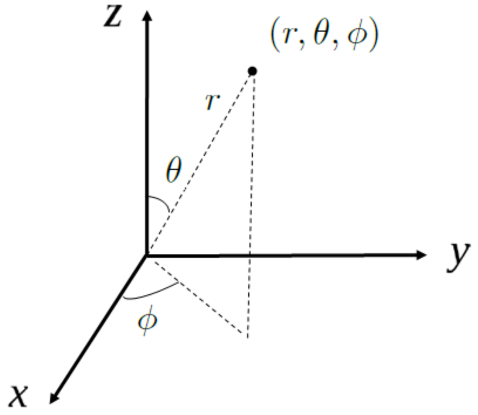

Figure 2: Spherical Coordinate System

Moreover, for our Spherical coordinate systems we use different notations and by that we imply a different metric underling them. The notation we use are the following:

System	Defined for	Definition	Note
$O(r, \theta, \phi)$	$r > R$	Spherical	flat
$\tilde{\mathcal{O}}(r, \theta, \phi)$	$r > R$	$\bar{r} = r - \frac{R^2}{r}$	flat
$\mathcal{O}(r, \theta, \phi)$	$r > 0$	$g_{rr} = \left(1 + \frac{R^2}{r^2}\right)^2$	curved

(25)

System $O(r, \theta, \phi)$:

With this notation we imply the standard metric of the spherical coordinates systems which is::

$$g_{rr} = 1; \qquad g_{\theta\theta} = r^2; \qquad g_{\phi\phi} = r^2 \sin^2 \theta \tag{26}$$

all other components vanish. The line element is:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \tag{27}$$

System $\tilde{\mathcal{O}}(r, \theta, \phi)$:

With this notation we imply the metric of the coordinate transformation given by the displacements $-\frac{R^2}{r}$ of a spherical space deficiency of radius R (see Eq. 6)). These displacements are applied to the coordinate r leading eventually to the following transformations:

$$x^{\bar{r}} : \bar{r} = r - \frac{R^2}{r}; \qquad x^{\bar{\theta}} : \bar{\theta} = \theta; \qquad x^{\bar{\phi}} : \bar{\phi} = \phi \tag{28}$$

Now, the displaced coordinate $\bar{r}$ is a natural coordinate sitting in Euclidean space with $\bar{r} = 0$ in the origin of the axis and therefore the underlying metric is the standard metric of the spherical coordinates system. For example $g_{\bar{r}\bar{r}} = 1$. If we go to the original r coordinates we have:

$$g_{rr} = \frac{\partial x^{\bar{r}}}{\partial x^r} \frac{\partial x^{\bar{r}}}{\partial x^r} g_{\bar{r}\bar{r}} = \left(1 + \frac{R^2}{r^2}\right)^2 \tag{29}$$

and in the same way:

$$g_{\theta\theta} = \frac{\partial x^{\bar{\theta}}}{\partial x^\theta} \frac{\partial x^{\bar{\theta}}}{\partial x^\theta} g_{\bar{\theta}\bar{\theta}} = \bar{r}^2 = \left(r - \frac{R^2}{r}\right)^2 \tag{30}$$

and:

$$g_{\phi\phi} = \frac{\partial x^{\bar{\phi}}}{\partial x^{\phi}} \frac{\partial x^{\bar{\phi}}}{\partial x^{\phi}} g_{\bar{\phi}\bar{\phi}} = \bar{r}^2 \sin^2\theta = \left(r - \frac{R^2}{r}\right)^2 \sin^2\theta \tag{31}$$

This metric describes a flat space because the $\bar{r}$ sits in flat space.

System $\mathcal{O}(r, \theta, \phi)$:

With the notation $\tilde{\mathcal{O}}(r, \theta, \phi)$ we imply a metric where we take into account that each point of space is displaced when we associate the metric (29) to it. In notation $\mathcal{O}(r, \theta, \phi)$ instead, we associate the very same metric to the undisplaced points and we extend this metric to points for $r < R$. $\mathcal{O}(r, \theta, \phi)$ is a good approximation of $\tilde{\mathcal{O}}(r, \theta, \phi)$ when $r >> R$ and it is the system we use when we solve the deficiency problem given by Eq. (20a). This approximation gives curvature to space.

This system of coordinates is simply defined by the following metric:

$$g_{rr} = \left(1 + \frac{R^2}{r^2}\right)^2; \quad g_{\theta\theta} = r^2; \quad g_{\phi\phi} = r^2 \sin^2\theta \tag{32}$$

The above metric is the metric of a 3-dimensional space $\mathcal{S}(\bar{r}, \bar{\theta}, \bar{\phi})$, with $\bar{r} > -\infty$, embedded into a 4-dimensional Euclidean space $\hat{\mathcal{S}}(h, r, \theta, \phi)$ with $r > 0$ (i.e. $\mathbb{R} \times \mathbb{R}^3$ where $\mathbb{R}^3$ is given in spherical coordinates). Although when we do intrinsic geometry we should never think about ambient spaces, in this case we will make explicit this embedding just for fun.

The embedding goes like this:

$$\theta = \bar{\theta}; \quad \phi = \bar{\phi} \tag{33}$$

moreover, given the metric (32), we add an additional h coordinate to the existing (r, θ, ϕ) coordinates. For each r we give a value to h so that a curve moving radially on the manifold from a distance r_1 to a distance r_2 and with θ and ϕ constant, has a length equal to the space obtained integrating the metric between r_1 and r_2.

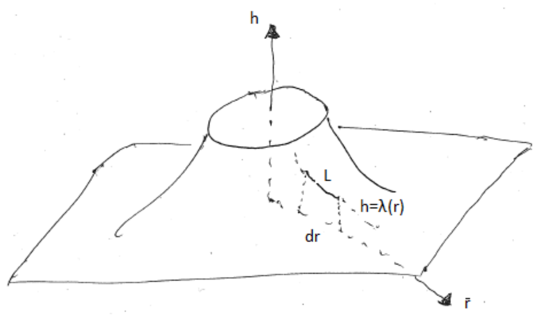

Figure 3: Embedding in $\mathbb{R}^4$

We are basically looking for a function $h = \lambda(r)$ which arc length L between two values of r is equal to the space distance of them in the ambient Euclidean space $\hat{\mathcal{S}}$ (see Fig. 3). The classical formula for evaluating the arc length of a curve is:

$$L = \int_{r_1}^{r_2} \sqrt{1 + (\lambda')^2}\, dr \tag{34}$$

we define the function $L(r)$ to be:

$$L(r) = \int_1^r \sqrt{g_{rr}(v)} dv = \int_1^r \left(1 + \frac{R^2}{v^2}\right) dv \tag{35}$$

clearly we have that $L = L(r_2) - L(r_1)$. Using the function $L(r)$ into Eq. (34) we have:

$$\int_1^r \left(1 + \frac{R^2}{v^2}\right) dv = \int_1^r \sqrt{1 + (\lambda(v)')^2}\, dv \tag{36}$$

Taking the derivative of both sides we have:

$$\left(1+\frac{R^2}{r^2}\right)=\sqrt{1+(\lambda')^2} \tag{37}$$

from which, by choosing the negative sign for the square root (we like the funnel shape of the manifold to point up and not down) we get easily:

$$\lambda'(r)=-\sqrt{\frac{R^4}{r^4}+\frac{2R^2}{r^2}} \tag{38}$$

integrating and taking into account that $h=\lambda(r)$:

$$h=\frac{R\sqrt{2r^2+R^2}}{r}-\sqrt{2}R\,\mathrm{arsinh}\left(\frac{\sqrt{2}r}{R}\right)+constant \tag{39}$$

where we can set the constant to 0. Note that:

$$\lim_{r\to 0}h(r)=+\infty;\quad \lim_{r\to\infty}h(r)=-\infty; \tag{40}$$

Eventually we get the embedding:

$$\bar{r}=r;\quad h=\frac{R\sqrt{2r^2+R^2}}{r}-\sqrt{2}R\,\mathrm{arsinh}\left(\frac{\sqrt{2}r}{R}\right);\quad \bar{\theta}=\theta;\quad \bar{\phi}=\phi \tag{41}$$

Compare the above result with [2].

A.2 Solution of EQ. (4) for Discrete Space Deficiency

We want to solve Eq.(4) for a discrete space deficiency in the origin $s_0\delta(x)$. Since s_0 is the external area of the removed material, this correspond to a ball of radius $\lambda_0=\frac{s_0}{4\pi}$. Due to symmetry of the problem we know that $\mathbf{u}(x)$ has non vanishing components only in the radial direction and vanishing derivative with respect of θ and ϕ for all components. Given the above, the radial component of the vector Laplacian in spherical coordinates can be written as:

$$(\nabla^2 u)_r=\frac{1}{r}\frac{\partial^2(ru_r)}{\partial r^2}-\frac{2u_r}{r^2} \tag{42}$$

Given Eq. (4) we have:

$$r(ru_r)''-2u_r=0 \tag{43}$$

Which is:

$$r^2u_r''+ru_r'-u_r=0 \tag{44}$$

This is a Cauchy-Euler Differential Equation and can be solved through trial solution. By setting $u_r=r^m$ and substituting we get:

$$m^2-1=0 \tag{45}$$

From which we have:

$$u_r(r)=c_1\frac{1}{r}+c_2r \tag{46}$$

From the boundary condition at infinity we get $c_2=0$ and from the boundary condition $r(\lambda_0)=-\lambda_0$ we get $c_1=-\lambda_0^2$ from which eventually:

$$\mathbf{u}(r,\theta,\phi)=-\frac{\lambda_0^2}{r}\hat{\mathbf{i}}_r \tag{47}$$

defined for $r>\lambda_0$.

As far as the strains are concerned, they also have radial components only which are given by the directional derivative of $\mathbf{u}$ in spherical coordinates along r:

$$\boldsymbol{\epsilon}(r,\theta,\phi)=\frac{\lambda_0^2}{r^2}\hat{\mathbf{i}}_r \tag{48}$$

defined for $r>\lambda_0$.

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