Liénard's generalisation: a direct derivation

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Abstract

In this pedagogical article, we elucidate on direct derivation of total power emitted by an accelerating charged particle, known as Liénard's generalisation.

INTRODUCTION

Larmor's formula [1] for total power radiated by an accelerating non-relativistic charged particle is a regular theme in M.Sc. electrodynamics course. Generalisation of Larmor's formula for arbitrary velocity, via putting it in covariant form[2] is known as Liénard's generalisation and is also a regular feature in any electrodynamics text-book. What is missing is straightforward derivation of total power for arbitrary velocity of charge, starting from electric and magnetic fields of a radiating charge through Poynting vector through angular emission of power to total power on integration over 4π solid angle. One may look in [3] for similar derivation. David J. Griffiths[4] poses it as a problem and writes "It's not a picnic". Here, in this pedagogical article, we elucidate on one straightforward derivation.

In the far-field region for an acclerating point charge q, electric field, is given by ([2], [5],[4]),

$$\vec{E} = \frac{Z_0}{4\pi}\frac{q}{R}\frac{\hat{n}\times((\hat{n}-\vec{\beta})\times\dot{\vec{\beta}})}{(1-\hat{n}.\vec{\beta})^3}.$$

where, $\vec{\beta} = \frac{1}{c}\vec{v}$, $\vec{v}$ being the velocity of the charged particle. This electric field is called "accelerating field". There is another part, called, "velocity field", which falls off with distance as $\frac{1}{R^2}$ and hence negligible compared to the "accelerating field" in the far-field domain. Magnetic field in that region is given by

$$\vec{B} = \frac{1}{c}\hat{n}\times\vec{E}.$$

where, $\hat{n}$ is a unit vector from the charge to an observer. Z_0 is vacuum impedance, $\mu_0 c$, with value 378 ohm. $\frac{Z_0}{4\pi}$ goes over to $\frac{1}{c}$ in CGS unit.

Total power emitted

Poynting vector, $\vec{S}$, given by

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$$

is a measure of energy passing out per unit time(observer's time) per unit area about an observer in the far-field region along the direction of radiated electromagnetic wave. As a result power received per unit solid angle about the charged particle, at a time, t, by the observer is given by $\frac{dP(t)}{d\Omega} = \vec{S}.\hat{n}R^2$. Consequently, total power emitted by a charged particle, in its time t_r, is given by

$$P(t_r) = \int d\Omega \vec{S}.\hat{n}R^2 \frac{dt}{dt_r}.$$

But $\frac{dt}{dt_r} = 1 - \hat{n}.\vec{\beta}$ ([2],[4]). On the top of it, $\hat{n}$ and $\vec{E}$ are perpendicular to each other. Expression for the total power reduces to

$$P(t_r) = \int d\Omega \frac{1}{Z_0} E^2 R^2 \frac{dt}{dt_r}.$$

implying

$$P(t_r) = \frac{Z_0}{(4\pi)^2} q^2 \int d\Omega (\frac{\hat{n} \times ((\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}})}{(1 - \hat{n}.\vec{\beta})^3})^2 (1 - \hat{n}.\vec{\beta})$$

or,

$$P(t_r) = \frac{Z_0}{(4\pi)^2} q^2 \int d\Omega \frac{(\hat{n} \times ((\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}))^2}{(1 - \hat{n}.\vec{\beta})^5}$$

Using the vector identity $\vec{a} \times (\vec{b} \times \vec{c}) = -\vec{c}(\vec{a}.\vec{b}) + \vec{b}(\vec{a}.\vec{c})$,

$$P(t_r) = \frac{Z_0}{(4\pi)^2} q^2 [-\int d\Omega \frac{1}{\gamma^2} \frac{(\hat{n}.\dot{\vec{\beta}})^2}{(1 - \hat{n}.\vec{\beta})^5} + \int d\Omega \frac{(\dot{\beta})^2}{(1 - \hat{n}.\vec{\beta})^3} + 2(\vec{\beta}.\dot{\vec{\beta}}) \int d\Omega \frac{\hat{n}.\dot{\vec{\beta}}}{(1 - \hat{n}.\vec{\beta})^4}]$$

From now onwards, for the sake of simplicity, we shall drop the subscript r in t and denote the retarded time as t.

Evaluation of the three integrals

Second integral :

$$\int d\Omega \frac{1}{(1 - \hat{n}.\vec{\beta})^3} = 4\pi\gamma^4$$

(see appendix).
To evaluate the third integral, we observe,

$$\frac{d}{dt}\frac{1}{(1-\hat{n}.\vec{\beta})^3} = 3\frac{1}{(1-\hat{n}.\vec{\beta})^4}\hat{n}.\dot{\vec{\beta}}$$

Hence, the third integral:

$$\begin{aligned}
2(\vec{\beta}.\dot{\vec{\beta}})\int d\Omega\frac{\hat{n}.\dot{\vec{\beta}}}{(1-\hat{n}.\vec{\beta})^4} &= \frac{2}{3}(\vec{\beta}.\dot{\vec{\beta}})\int d\Omega\frac{d}{dt}\frac{1}{(1-\hat{n}.\vec{\beta})^3}\\
&= \frac{2}{3}(\vec{\beta}.\dot{\vec{\beta}})\frac{d}{dt}\int d\Omega\frac{1}{(1-\hat{n}.\vec{\beta})^3}\\
&= \frac{2}{3}(\vec{\beta}.\dot{\vec{\beta}})\frac{d}{dt}4\pi\gamma^4\\
&= \frac{32\pi}{3}\gamma^6(\vec{\beta}.\dot{\vec{\beta}})^2
\end{aligned}$$

To compute the first integral, we use

$$\frac{d^2}{dt^2}\frac{1}{(1-\hat{n}.\vec{\beta})^3} = 3\frac{1}{(1-\hat{n}.\vec{\beta})^4}\hat{n}.\ddot{\vec{\beta}} + 12(\hat{n}.\dot{\vec{\beta}})^2\frac{1}{(1-\hat{n}.\vec{\beta})^5}$$

which implies

$$\frac{1}{12\gamma^2}\frac{d^2}{dt^2}\frac{1}{(1-\hat{n}.\vec{\beta})^3} = \frac{1}{4\gamma^2}\frac{1}{(1-\hat{n}.\vec{\beta})^4}\hat{n}.\ddot{\vec{\beta}} + \frac{1}{\gamma^2}(\hat{n}.\dot{\vec{\beta}})^2\frac{1}{(1-\hat{n}.\vec{\beta})^5}$$

which further implies

$$\frac{1}{12\gamma^2}\int d\Omega\frac{d^2}{dt^2}\frac{1}{(1-\hat{n}.\vec{\beta})^3} = \frac{1}{4\gamma^2}\int d\Omega\frac{1}{(1-\hat{n}.\vec{\beta})^4}\hat{n}.\ddot{\vec{\beta}} + \frac{1}{\gamma^2}\int d\Omega(\hat{n}.\dot{\vec{\beta}})^2\frac{1}{(1-\hat{n}.\vec{\beta})^5}$$

which yields the first integral:

$$\begin{aligned}
\frac{1}{\gamma^2}\int d\Omega(\hat{n}.\dot{\vec{\beta}})^2\frac{1}{(1-\hat{n}.\vec{\beta})^5} &= \frac{1}{12\gamma^2}\int d\Omega\frac{d^2}{dt^2}\frac{1}{(1-\hat{n}.\vec{\beta})^3} - \frac{1}{4\gamma^2}\int d\Omega\frac{1}{(1-\hat{n}.\vec{\beta})^4}\hat{n}.\ddot{\vec{\beta}}\\
&= \frac{1}{12\gamma^2}\frac{d^2}{dt^2}(4\pi\gamma^4) - \frac{1}{4\gamma^2}\frac{16\pi}{3}\gamma^6(\ddot{\vec{\beta}}.\vec{\beta})\\
&= \frac{\pi}{3\gamma^2}[24\gamma^8(\vec{\beta}.\dot{\vec{\beta}})^2 + 4\gamma^6(\ddot{\vec{\beta}}.\vec{\beta} + \dot{\beta}^2)] - \frac{4\pi}{3}\gamma^4(\ddot{\vec{\beta}}.\vec{\beta})\\
&= \frac{4\pi}{3}\gamma^4[\dot{\beta}^2 + 6\gamma^2(\vec{\beta}.\dot{\vec{\beta}})^2]
\end{aligned}$$

Hence, putting the three integration results in, we get

$$\begin{aligned}
P(t) &= \frac{Z_0}{(4\pi)^2}q^2[-\frac{4\pi}{3}\gamma^4\dot{\beta}^2 - \frac{24\pi}{3}\gamma^6(\vec{\beta}.\dot{\vec{\beta}})^2 + 4\pi\gamma^4\dot{\beta}^2 + \frac{32\pi}{3}\gamma^6(\vec{\beta}.\dot{\vec{\beta}})^2]\\
&= \frac{2}{3}\frac{Z_0}{4\pi}q^2\gamma^6[(\vec{\beta}.\dot{\vec{\beta}})^2 + \gamma^{-2}\dot{\beta}^2]\\
&= \frac{2}{3}\frac{Z_0}{4\pi}q^2\gamma^6[\dot{\beta}^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2].
\end{aligned}$$

which is Liénard's generalisation of Larmor's formula,

CONCLUSION

In this pedagogical article, we have elucidated on a straightforward derivation of total power emitted by an accelerating point charge moving with any velocity upto velocity of light. The expression for total power is known as Liénard's generalisation of Larmor's formula.

Appendix

$$\int d\Omega \frac{1}{(1-\hat{n}.\vec{\beta})^3} = 2\pi \int_{-1}^{1} d(cos\theta) \frac{1}{(1-\beta cos\theta)^3} = -\frac{2\pi}{\beta}\frac{1}{2}[\frac{1}{(1+\beta)^2} - \frac{1}{(1-\beta)^2}]$$
$$= 4\pi \frac{1}{(1-\beta^2)^2} = 4\pi\gamma^4.$$

$$\int d\Omega \frac{\hat{n}.\ddot{\vec{\beta}}}{(1-\hat{n}.\vec{\beta})^4} = \int d\Omega \frac{\ddot{\beta}_z cos\theta + \ddot{\beta}_x sin\theta cos\phi + \ddot{\beta}_y sin\theta sin\phi}{(1-\beta cos\theta)^4}$$
$$= 2\pi\ddot{\beta}_z \int_{-1}^{1} d(cos\theta) \frac{cos\theta}{(1-\beta cos\theta)^4} = \frac{16\pi}{3}\ddot{\beta}_z \beta\gamma^6 = \frac{16\pi}{3}(\ddot{\vec{\beta}}.\vec{\beta})\gamma^6.$$

where, $\vec{\beta}$ is taken as z-direction for spherical polar coordinate system.

$$\frac{d}{dt}\gamma^4 = \frac{d}{dt}\gamma^2\gamma^2 = 2\gamma^2\frac{d}{dt}(1-\beta^2)^{-1} = 2\gamma^2 2(\dot{\vec{\beta}}.\vec{\beta})\gamma^4 = 4(\dot{\vec{\beta}}.\vec{\beta})\gamma^6$$

$$\frac{d^2}{dt^2}\gamma^4 = \frac{d}{dt}(4(\dot{\vec{\beta}}.\vec{\beta})\gamma^6) = 4\frac{d}{dt}(\gamma^2)^3(\dot{\vec{\beta}}.\vec{\beta}) + 4\gamma^6(\dot{\beta}^2 + \ddot{\vec{\beta}}.\vec{\beta}) = 24\gamma^8(\dot{\vec{\beta}}.\vec{\beta})^2 + 4\gamma^6(\dot{\beta}^2 + \ddot{\vec{\beta}}.\vec{\beta})$$

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References

[1] J. Larmor, *On the theory of the magnetic influence on spectra; and on the radiation from moving ions*, Philos. Mag. Vol.44, pp.503–512, 1897.

[2] John David Jackson, *Classical Electrodynamics*, 3rd ed., p.666, Wiley India Pvt. Limited, New Delhi, 1999.

[3] Leigh Page, Norman IIsley Adams, *Electrodynamics*, ch.7, Chapman and Hall, 1940.

[4] David J. Griffiths, *Introduction to Electrodynamics*, 3rd ed., p.463, Prentice-Hall Inc., New Jersy, 1999.

[5] Ashok K. Singal, *A first principles derivation of the electromagnetic fields of a point charge in arbitrary motion*, AJP 79, pp.1036–1041, 2011.