Bimetric Theory of Gravitational-Inertial Field in Riemannian and in Finsler-Lagrange Approximation.

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Abstract

In present article the original proposition is a generalization of the Einstein's world tensor g_{ij} by the introduction of pure inertial field tensor $\breve{g}_{ij}$ such that $R^{\alpha}_{\mu\nu\lambda}(\breve{g}_{ij}) \neq 0$. Bimetric theory of gravitational-inertial field is considered for the case when the gravitational-Inertial field is governed by either a perfect magnetic fluid.

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I.Introduction

I.1.GTR in Riemannian Approximation.

General theory of Relativity (GTR) in Riemannian approximation to proceed from assumptions :

- (**I**) One-metric geometric structures of the space-time continuum on the standard assumption of Lorentzianian geometry

$$ds^2 = g_{ik}dx^i dx^k, g_{ik} = g_{ki}, \det\|g_{ik}\| \neq 0; \qquad (1.1.1)$$

- (**II**) Equivalence of gravitational field and space-time metric tenzor g_{ik}.

I.2.GTR in Finsler-Lagrange Approximation (GTRFL).

- In contemporary literature pure formal a Finslerian-Lagrange extension of general relativity was many developed [20]-[26]. Any extension of GTR such that mentioned above based on a Finsler–Lagrange geometry [27]-[28].
  Any Finsler geometry defined by a fundamental Finsler function $F(x,y)$, being homogeneous of type $F(x,\lambda y) = |\lambda|F(x,y)$, for nonzero $\lambda \in \mathbb{R}$, may be considered as a particular case of Lagrange space when $L(x,y) = F^2(x,y)$.
  A differentiable Lagrangian $L(x,y)$, i.e. a fundamental Lagrange function, is defined by a map $L : (x,y) \in TM \to L(x,y) \in \mathbb{R}$ of class C^∞ on $\widetilde{TM} = TM\setminus\{0\}$. A regular Lagrangian has non-degenerate Hessian

$$^{L}g_{ik}(x,y) = \frac{\partial^2 L(x,y)}{\partial x^i \partial y^k}, \qquad (1.1.2)$$

rank$|{}^{L}g_{ik}| = n, \det\|{}^{L}g_{ik}\| \neq 0.$

A Lagrange space is a pair $L^n = [M, L(x,y)]$ with $^{L}g_{ij}$ being of fixed signature over $V = \widetilde{TM}$. The Euler–Lagrange equations

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i} = 0 \qquad (1.1.3)$$

where $y^i \equiv \dfrac{dx^i(\tau)}{d\tau}$ are equivalent to the “nonlinear” geodetic equations

$$\frac{d^2x^a}{d\tau^2} - 2G^a\left(x^k, \frac{dx^b(\tau)}{d\tau}\right) = 0 \qquad (1.1.4)$$

defining paths of a canonical semispray

$$S = y^i \frac{\partial}{\partial x^i} - 2G^a(x,y)\frac{\partial}{\partial y^a} \qquad (1.1.5)$$

where

$$G^i(x,y) = \frac{1}{2}({}^Lg_{ij})\left(\frac{\partial^2 L(x,y)}{\partial x^k \partial y^i} y^k - \frac{\partial L}{\partial x^i}\right) \qquad (1.1.6)$$

There exists on $V \simeq \widetilde{TM}$ a canonical N–connection

$$ {}^{L}N^{a}_{j} = \frac{\partial G^{a}(x,y)}{\partial y^{i}} \qquad (1.1.7) $$

defined by the fundamental Lagrange function $L(x,y)$, which prescribes nonholonomic frame structures [26]: ${}^{L}\mathbf{e}_{\nu} = ({}^{L}\mathbf{e}_{i}, e_{a})$ and ${}^{L}\mathbf{e}^{\mu} = (e^{i}, {}^{L}\mathbf{e}_{a})$. One obtain the canonical metric structure

$$ {}^{L}\mathbf{g} = [{}^{L}g_{ij}(x,y)](e^{i} \otimes e^{j}) + [{}^{L}g_{ij}(x,y)][({}^{L}e^{i}) \otimes ({}^{L}e^{j})] \qquad (1.1.8) $$

constructed as a Sasaki type lift from M for ${}^{L}g_{ij}(x,y)$. There is a unique canonical $\mathbf{d}$-connection ${}^{L}\mathbf{D} = (h\,{}^{L}\widehat{D}, v\,{}^{L}\widehat{D})$ with the coefficients ${}^{L}\widehat{\Gamma}^{\alpha}{}_{\beta\gamma} = ({}^{L}\widehat{L}^{i}{}_{jk}, {}^{L}\widehat{C}^{a}_{bc})$ computed by formulas

$$ \widehat{L}^{i}{}_{jk} = \frac{1}{2} g^{ih}(\mathbf{e}_{k}g_{jh} + \mathbf{e}_{j}g_{kh} - \mathbf{e}_{h}g_{jk}), $$
$$ (1.1.9) $$
$$ \widehat{C}^{a}{}_{bc} = \frac{1}{2} g^{ae}(e_{b}g_{ec} + e_{c}g_{eb} - e_{e}g_{bc}), $$

for the $\mathbf{d}$–metric (1.1.8) with respect to ${}^{L}\mathbf{e}_{\nu}$ and ${}^{L}\mathbf{e}^{\mu}$. All such geometric objects, including the corresponding to ${}^{L}\widehat{\Gamma}^{\alpha}{}_{\beta\gamma}$, ${}^{L}\mathbf{g}$ and ${}^{L}N^{a}_{j}$ $\mathbf{d}$–curvatures

$$ {}^{L}\widehat{\mathbf{R}}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta\gamma\delta}} = \left({}^{L}\widehat{R}^{i}{}_{hjk}, {}^{L}\widehat{P}^{i}{}_{jka}, {}^{L}\widehat{S}^{a}{}_{bcd} \right), \qquad (1.1.9') $$

where

$$\widehat{R}^{i}_{\ hjk} = \mathbf{e}_k \widehat{L}^{i}_{\ hj} - \mathbf{e}_j \widehat{L}^{i}_{\ hk} + \widehat{L}^{m}_{\ hj} \widehat{L}^{i}_{\ mk} - \widehat{L}^{m}_{\ hk} \widehat{L}^{i}_{\ mj} - \widehat{C}^{i}_{\ ha} \Omega^{a}_{\ kj},$$

$$\widehat{P}^{i}_{\ jka} = e_a \widehat{L}^{i}_{\ jk} - \widehat{\mathbf{D}}_k \widehat{C}^{i}_{\ ja},\ \widehat{S}^{a}_{\ bcd} = e_d \widehat{C}^{a}_{\ bc} - e_c \widehat{C}^{a}_{\ bd} + \widehat{C}^{e}_{\ bc} \widehat{C}^{a}_{\ ed} - \widehat{C}^{e}_{\ bd} \widehat{C}^{a}_{\ ec}, \tag{1.1.10}$$

where all indices $a,b,\ldots,i,j,\ldots$ run the same values and, for instance, $C^{e}_{\ bc} \to C^{i}_{\ jk},\ldots$ Thus any $\mathbf{d}$–curvatures ${}^{L}\widehat{\mathbf{R}}^{\boldsymbol{\alpha}}_{\ \boldsymbol{\beta\gamma\delta}}$ are completely defined by a Lagrange fundamental function $L(x,y)$ for a nondegerate ${}^{L}g_{ij}$. Note that such locally anisotropic configurations are not integrable if $\Omega^{a}_{\ kj} \neq 0,$ even the $\mathbf{d}$–torsion components $\widehat{T}^{i}_{\ jk} = 0$ and $\widehat{T}^{a}_{\ bc} = 0.$
General theory of Relativity (GTR) in Finsler-Lagrange approximation to proceed from assumptions :

- (**I**) One-metric Finsler-Lagrange geometric structures ${}^{L}\mathbf{g}$ of the space-time

  continuum on the standard assumption of Finsler-Lagrange geometry given by Eqs.(1.1.8)-(1.1.10).

- (**II**) Equivalence of gravitational field and space-time structures:

$${}^{L}\mathbf{g} = [{}^{L}g_{ij}(x,y)](e^{i} \otimes e^{j}) + [{}^{L}g_{ij}(x,y)][({}^{L}e^{i}) \otimes ({}^{L}e^{j})],$$

$$\widehat{R}^{i}_{\ hjk} = \mathbf{e}_k \widehat{L}^{i}_{\ hj} - \mathbf{e}_j \widehat{L}^{i}_{\ hk} + \widehat{L}^{m}_{\ hj} \widehat{L}^{i}_{\ mk} - \widehat{L}^{m}_{\ hk} \widehat{L}^{i}_{\ mj} - \widehat{C}^{i}_{\ ha} \Omega^{a}_{\ kj}, \tag{1.1.11}$$

$$\widehat{P}^{i}_{\ jka} = e_a \widehat{L}^{i}_{\ jk} - \widehat{\mathbf{D}}_k \widehat{C}^{i}_{\ ja},\ \widehat{S}^{a}_{\ bcd} = e_d \widehat{C}^{a}_{\ bc} - e_c \widehat{C}^{a}_{\ bd} + \widehat{C}^{e}_{\ bc} \widehat{C}^{a}_{\ ed} - \widehat{C}^{e}_{\ bd} \widehat{C}^{a}_{\ ec}.$$

I.3.Theory of Gravitational-Inertional field with necessity admit Einstein's "Strong Equivalence Principle" (SEP)

When dealing with relativistic theories of gravity one is confronted with three types of equivalence principles [36]:

- **1**.The Weak Equivalence Principle (**WEP**),

- **2**.The Einstein Equivalence Principle (**EEP**), and
- **3**.The Strong Equivalence Principle (**SEP**).

  WEP: In a pure geometrical view the WEP states that all test masses move along geodesics in space-time $\mathcal{L}_{\mathbf{Gr.}} = (M, g_{ik}^{\mathbf{Gr.}})$. Test masses are understood to be bodies with negligible self-energy and therefore with negligible contribution to space-time curvature $R_{ik}^{l}(g_{ik}^{\mathbf{Gr.}})$.

EEP:The EEP demands, besides the validity of the WEP, that in local Lorentz frames the non-gravitational laws of physics are those of special relativity. The EEP implies that space-time has to be curved,i.e. $R_{ik}^{l}(g_{ik}^{\mathbf{Gr.}}) \neq 0$ and thus is the basic ingredient of any metric theory of gravity.

SEP: The SEP states, besides the validity of the EEP, the "universality of free fall for self-gravitating bodies".

Note that one has to be careful with the notion of a freely falling self-gravitating bodies in an external gravitational field. There is no rigorous definition for the SEP in relativistic theories of gravity. Because of non-linearity the split of the metric field into an external and a local part can only be approximate. For a discussion of the SEP within a slow-motion weak-field approximation see [39],[40]. For metric theories of gravity, other than general relativity, it has been found that they typically introduce auxiliary gravitational fields (e.g. scalar fields) and thus predict a violation of the SEP see [36],[37].

Definition *2.3.1."Strong Equivalence Principle" (SEP) asserts that any gravitational field* $g_{ik}^{\mathbf{Gr.}}$ *cannot be distinguished from a suitably chosen accelerated reference frame* $\mathcal{F}(g_{ik}^{\mathbf{ac.}})$ *- essentially because we cannot distinguish between the reciprocal cases of spacetime* $\mathcal{L}_{\mathbf{Gr.}} = (M, g_{ik}^{\mathbf{Gr.}})$ *accelerating through us (gravity), or our own acceleration through spacetime [4].*

- Hence SEP in fact asserts that the gravitational curvature cannot be distinguished from a suitably chosen curved accelerated reference frame (as curved Bravais frame or Hollands frame) - essentially because we cannot distinguish between the reciprocal cases of curved space-time $\mathcal{L}_{\mathbf{Gr.}} = (M, g_{ik}^{\mathbf{Gr.}}), R_{ik}^{l}(g_{ik}^{\mathbf{Gr.}}) \neq 0$ accelerating through us (gravity), or our own comoving curved space-time $\mathcal{L}_{\mathbf{ac.}} = (M, g_{ik}^{\mathbf{ac.}}), R_{ik}^{l}(g_{ik}^{\mathbf{ac.}}) \neq 0$ as curved Hollands frame or curved relativistic Bravais frame.

However as shown by Fock [4] in fact SEP dos not was used by Einstein in GTR, but only Einstein’s "Weak Equivalence Principle" (**WEP**) was used by Einstein in GTR. Recall the **WEP**: all objects are observed to accelerate at the same rate in a given gravitational field.Therefore, the inertial and gravitational masses must be the same for any object.This has been verified experimentally, with fractional difference in masses $< 10^{-11}$.As a consequence, the effects of gravity and of inertial forces (fictitious forces associated with accelerated frames) cannot locally be distinguished.

- **Remark** *2.3.1.Recall the accelerational fields in canonical GTR (in contrast with tensor gravitational fields $g_{ik}^{\mathbf{Gr.}}$) was introduced not as objective physical field but only as fictive tensor fields $g_{ik}^{\mathbf{ac.}}$ which may be created by means of an arbitrary choice of coordinates [1],[4],[35].*

But as shown by Fock [1],[4] the equivalence of gravitational fields $g_{ik}^{\mathbf{Gr.}}$ and accelerational fields $g_{ik}^{\mathbf{ac.}}$ *such that mentioned above* is limited not only to sufficiently small domaines of space and sufficiently short intervals of time, but generally to weak and homogeneous fields and slow motions.

- **Remark** *2.3.2.Thus one can pointed-out with Fock [4] that SEP is inconsistent with canonical GTR.*

By the way, here one should not confuse the law of equality of inertial and gravitational masses with the mentioned principle of equivalence. The mathematical expression of this principle is the possibility of introducing the locally geodetic coordinate system such that

$$g_{ik,l} = 0. \qquad (1.1.12)$$

However from this statement a not quite correct conclusion is drawn by Fock namely since the possibility of introducing of locally-geodetic system is contained in Riemannian geometry, therefore the pointed-out principle does not constituate a separate physical hypothesis. Actually the availability of such a possibility in the Riemannian space-time is not nessessary at all [1].

Definition *2.3.2.Every gravitational field theory which contained standard assumptions of* ***GTR****: (**I**) and (**II**) and which is consistent with* ***SEP*** *given by Def.(2.3.1), refers as* ***Theory of Gravitational-Inertional field in Riemannian Approximation (GIFTR)****.*

Notation *Thus every GIFTR in contrast with GTR to proceed with necessity from additional assumption related to SEP:*

$$R^{l}_{ikm}(g^{\mathbf{ac.}}_{ik}) \neq 0. \qquad (1.1.13)$$

Definition *2.3.3.Every gravitational field theory which contained standard assumptions of **GTRFL** : (**I**) and (**II**) and which is consistent with **SEP**, refers as **Theory of Gravitational-Inertional field in Finsler-Lagrange Approximation (GIFTFL)**.*

For the first time nontrivial gravitational-inertional field theory was proposed by Davtyan
[1]-[3]. In contrast to GTR, in Davtyan's theory [1] of the gravitational-inertial field tensor $\mathbf{g}^{\mathbf{GI}}_{ik}$ is not related to the pure gravitational field $\mathbf{g}^{\mathbf{Gr}}_{ik}$. In Davtyan's GIFT the real space-time metric tensor g_{ik} of the gravitational-inertial field is the metric tensor of the real world (Universe), which formed by pure gravitational $g^{\mathbf{Gr}}_{ik}$ and $g^{\mathbf{ac.}}_{ik}$ pure inertial metric tensors. From the field of this general metric tensor $g^{\mathbf{U}}_{ik}$ the Riemannian space is also formed.
Davtyan's field equations in general looks [1]:

$$g^{lp}g^{mn}\nabla_k\left(\sqrt{-g}\,g^{ik}g_{np;i}\right) - \nabla_i\left(\sqrt{-g}\,g^{ik}g_{np;k}\right) =$$

$$= 8\pi\varkappa k\sqrt{-g}\,g^{mn}\left(T^{l}_{n} - \frac{1}{2}\delta^{l}_{n}T\right), \qquad (1.1.14)$$

$$k = \frac{2}{c^4},$$

where

$$\nabla_l g_{ik} = g_{ik;l} = g_{ik,l} - \breve{\boldsymbol{\Gamma}}^{m}_{il}g_{mk} - \breve{\boldsymbol{\Gamma}}^{m}_{kl}g_{im} = 0 \qquad (1.1.15)$$

and

$$\breve{\Gamma}^{i}_{kl} = \frac{1}{2}\breve{\mathbf{g}}^{im}(\breve{\mathbf{g}}_{mk,l} + \breve{\mathbf{g}}_{ml,k} - \breve{\mathbf{g}}_{kl,m}),$$

$$\breve{\mathbf{g}}_{mk} \neq g_{mk}. \tag{1.1.16}$$

Conclusion *2.3.1.Thus in papers [1]-[3] the author proposes that Inertia, like Gravitation, could be a curved spacetime phenomenon caused by accelerating motion of matter in the full consent with SEP and in contrast with canonical GTR.*

Conclusion *2.3.2.However one can pointed-out that:*

- **1**.The new theory proposed in [1] does not looks as true GIFT but looks only as some kind of modified Einstein gravity with non metrical connection $\breve{\Gamma}^{m}_{il}$ and completely hidden of the pure inertial field $g^{\mathbf{ac.}}_{ik}$ sector.
- **2**.The new theory proposed in [1] explains the origin of the pure inertial field $g^{\mathbf{ac.}}_{ik}$ as being a curved space-time phenomenon, with the implication that accelerating matter might influence the metrical tensor g_{ik} of the real space-time by using of the canonical energy-momentum tensor of matter and this influence completely depend only from a small coupling constant $8\pi\varkappa k$ by using the manner of the canonical GTR.

- **3**. The gravitational-inertial field equations proposed in paper [1] in a weak gravitational- inertional field limit admits only equation of Newtonian gravity and weak pure gravitational wave equations:

(a) in a weak stationary field limit corresponds exactly to Newton's equation of gravitation for continuous distribtion of masses i.e.

$$\Delta\Phi^{\mathbf{Gr}}(x) = 4\pi\varkappa\rho(x),$$

$$\Phi^{\mathbf{Gr}} = -\varkappa\int\frac{\rho dV}{r} = -\frac{\varkappa m}{r}; \tag{1.1.17}$$

(b) in a weak nonstationary field limit corresponds exactly to wave equation, an equation of propagation of weak pure gravitational waves $h_{lm}^{\mathbf{Gr}} \simeq 0$:

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)h_{lm}^{\mathbf{Gr}} = 0; \qquad (1.1.18)$$

- **4**. In GIFT proposed in paper [1] for a weak gravitational-inertial field $g_{ik} \simeq 0$ space-time interval equals

$$ds^2 = \left(1 + \frac{2\Phi^{\mathbf{Gr}}}{c^2}\right)dt^2 - dx_1^2 - dx_2^2 - dx_3^2 = \qquad (1.1.19)$$
$$\left(1 - \frac{2\varkappa m}{c^2}\right)dt^2 - dx_1^2 - dx_2^2 - dx_3^2.$$

Conclusion *2.3.3.Thus one pointed-out that in a weak gravitational-inertial field limit $g_{ik} \simeq 0$ GIFT proposed in [1] does non admit any pure inertial field phenomenon which corresponds directly with hidden pure inertial field sector $g_{ik}^{\mathbf{ac.}}$.*

Definition *2.3.4. (1)* $\Re(g_{ik}) \triangleq \sum_{l,i,k,m}|R_{ikm}^{l}(g_{ik})|$, *(2)* $\Re^{\mathbf{Gr}}(g_{ik}^{\mathbf{Gr}}) \triangleq \sum_{l,i,k,m}|R_{ikm}^{l}(g_{ik}^{\mathbf{Gr}})|$

(3) $\Re^{\mathbf{ac}}(g_{ik}^{\mathbf{ac.}}) \triangleq \sum_{l,i,k,m}|R_{ikm}^{l}(g_{ik}^{\mathbf{ac.}})|$.

Claim *One can pointed-out that the correct gravitational-inertial field equations (GIFE) in a weak gravitational-inertional field limit with necessity admits:*

- **1**. In a weak pure gravitational field limit: $\Re(g_{ik}) \simeq \Re^{\mathbf{Gr}}(g_{ik}^{\mathbf{Gr}}), g_{ik,l}^{\mathbf{Gr}} \simeq 0,$ GIFE admits equation of Newtonian gravity and pure weak gravitational wave equations.

- **2**.In a weak pure inertial field limit: $\Re(g_{ik}) \simeq \Re^{\mathbf{ac}}(g_{ik}^{\mathbf{ac.}}), g_{ik,l}^{\mathbf{ac.}} \simeq 0,$ GIFE admits equation of Newtonian inertia, which describe the Newtonian scalar potential of the Newtonian inertial forces. If $\vec{F}$ denotes pure non gravitational force acting on an sufficiently small object O (particle), $\vec{r}$ denotes its position vector in an inertial frame, on can obtain the Newtonian scalar potential $\Phi^{\mathbf{ac.}}(x)$ directly from Newton's law of motion.
- Let's consider Newtonian inertial forces $\vec{F}_{\mathbf{ac}}$ related by canonical manner to the some electric force $\vec{F}$ experienced by the charged particle in the external stationary electric field $\vec{\mathbf{E}}$. Thus

$$\mathrm{div}\vec{\mathbf{E}} = 4\pi\rho(x),$$

$$\mathrm{rot}\vec{\mathbf{E}} = 0, \qquad (1.1.20)$$

$$\rho(x) = q \bullet \delta(x).$$

Substitution $\vec{\mathbf{E}} = -\mathrm{grad}\Phi^{\mathbf{ac.}}(x)$ gives equation for the corresponding Newtonian scalar potential

$$\Delta\Phi^{\mathbf{ac}}(x) = 4\pi q\rho(x),$$

$$\Phi^{\mathbf{ac}} = -q\int \frac{\rho dV}{r}; \qquad (1.1.21)$$

- **3**. For any GIFTR in a weak pure inertial field limit: $\Re(g_{ik}) \simeq \Re^{\mathbf{ac}}(g_{ik}^{\mathbf{ac.}}), g_{ik}^{\mathbf{ac.}} \simeq 0,$ for the case of the external stationary electric field $\vec{\mathbf{E}}$ given by Eq.(1.1.20), space-time interval with necessity equals:

$$ds^2 = \left(1 + \frac{2\Phi^{\mathbf{ac}}}{c^2}\right)dt^2 - dx_1^2 - dx_2^2 - dx_3^2. \quad (1.1.22)$$

- **4**.For any GIFT in a weak gravitational-inertial field limit: $g_{ik}^{\mathbf{Gr}} \simeq 0, g_{ik}^{\mathbf{ac.}} \simeq 0,$ for the case of the external stationary electric field $\vec{\mathbf{E}}$ given by Eq.(1.1.20), space-time interval with necessity equals:

$$ds^2 = \left(1 + \frac{2\Phi^{\mathbf{ac}}}{c^2} + \frac{2\Phi^{\mathbf{Gr}}}{c^2}\right)dt^2 - dx_1^2 - dx_2^2 - dx_3^2. \quad (1.1.23)$$

- **5**.In a weak pure inertial field limit: $\Re(g_{ik}) \simeq \Re^{\mathbf{ac}}(g_{ik}^{\mathbf{ac.}}), g_{ik,l}^{\mathbf{ac}} \simeq 0,$ any GIFT (in a Finsler-Lagrange approximation) with necessity admits equation of Post-Newtonian inertia, which describe a sufficiently small relativistic inertial forces related to external nonstationary electro-magnetic field $\left(\vec{\mathbf{E}}, \vec{\mathbf{H}}\right)$.

I.4. Bimetric Theory of Gravitational-Inertial Field in Riemannian Approximation.

Note that one can construct the Bimetric Theory of Gravitational-Inertial Field in Riemannian Approximation by using Rosen type bimetric formalism.
Recall the Rosen bimetric gravitational field theory. Rosen [5]-[7] proposed the bimetric gravitational field theory only with the purpose to remove some of the unsatisfactory features of the Einstein gravity, in which there exist two metric tensors at each point of bimetric Lorentzian space-time $\Re = \Re(M, g_{ij}, \gamma_{ij})$ viz a physical metric tensor g_{ij}, which described gravitation and the background flat metric γ_{ij} which does not interact directly with matter fields and describes the inertial forces associated with the acceleration of the frame of reference. Note that in Rosen's theory of gravitation for the derivation gravitational field equations one varies the quantities $g_{\mu\nu}$, not the quantities $\gamma_{\mu\nu}$, in the variational principle.
The gravitational field equations of Rozen reads

$$K_{\alpha\beta} - \frac{1}{2}Kg_{\alpha\beta} = -8\pi k T_{\alpha\beta},$$

$$k = \varkappa\sqrt{\frac{g}{\gamma}}, \qquad (1.1.24)$$

$$K_{\alpha\beta} = K_{\alpha}^{\beta} = \frac{1}{2}\breve{g}^{\mu\nu}(g^{h\alpha}g_{h\beta|\mu})_{|\nu}.$$

Where vertical bar | stands for covariant differentiation with respect to $\gamma_{\mu\nu}$.

Let's consider a bimetric geometry $\Re_2 = \Re(M, g_{ij}, \breve{g}_{ij})$ with metrics g and $\breve{g}$ of Lorentzian signature that define two different ways of measuring angles, distances and volumes on a manifold M. In present article the original proposition is a generalization of the real world tensor g_{ij} by the introduction of a *non flat* inertial field tensor $\breve{g}_{ij}$ such that $K = K(g_{ij}, \breve{g}_{ij}) \neq 0$, $\nabla_g \breve{g}_{ij} \neq 0$ and $\breve{R}^{\alpha}_{\mu\nu\lambda} = R^{\alpha}_{\mu\nu\lambda}(\breve{g}_{ij}) \neq 0$. The first metric tensor g_{ij} in **GIFTR**, refers to the curved Lorentzian space-time $\mathcal{L}_g = \mathcal{L}(M, g_{\alpha\beta})$ and describes Gravitational-Inertial Field. The second metric tensor $\breve{g}_{ij}$ in **GIFTR**, refers to the curved Lorentzian space-time $\mathcal{L}_{\breve{g}} = \mathcal{L}(\breve{g}_{\alpha\beta})$ and describes pure inertial forces. The Rosen's tipe curvature tenzor $K_{\mu\nu}(g_{\alpha\beta}, \breve{g}_{\mu\nu})$ refers to the curved Lorentzian space-time $\mathcal{L}(M, g_{\alpha\beta})$ and describes pure gravitational field.
This demands to use as a Action of the gravitational-inertial field the expression

$$\mathbf{S}(K, \breve{R}) = [\mathbf{S}_1(K, \breve{R})]_{\breve{g}} + [\mathbf{S}_2(\breve{R})]_{\breve{g}} =$$
$$(1.1.25)$$
$$\int \mathcal{L}_1(K(g_{\alpha\beta}, \breve{g}_{\mu\nu}))\sqrt{-\breve{g}}\,d^4x + \int \mathcal{L}_2(\breve{R}(\breve{g}_{\mu\nu}))\sqrt{-\breve{g}}\,d^4x.$$

On the basis of variational principle a system of more general Rozen's tipe covariant equations of the gravitational-inertial field is obtained:

$$\frac{\delta \mathbf{S}(K,\breve{R})}{\delta g_{\alpha\beta}} = K_{\alpha\beta} - \frac{1}{2}Kg_{\alpha\beta} = -8\pi\varkappa\sqrt{\frac{g}{\breve{g}}}\,[T_{\alpha\beta}]_{g},$$

$$K_{\alpha\beta} = K_{\alpha}^{\beta} = \frac{1}{2}\breve{g}^{\mu\nu}(g^{h\alpha}g_{h\beta\|\mu})_{\|\nu}$$

$$\frac{\delta \mathbf{S}_1(K,\breve{R})}{\delta \breve{g}_{\mu\nu}} + \frac{\delta \mathbf{S}_2(K,\breve{R})}{\delta \breve{g}_{\mu\nu}} = \breve{\Theta}_{\mu\nu} + \breve{E}_{\mu\nu} = k_1[\breve{T}^{\mu\nu}]_{\breve{g}}, \tag{1.1.26}$$

$$\breve{\Theta}_{\mu\nu} = \frac{\delta \mathbf{S}(K,\breve{R})}{\delta \breve{g}_{\mu\nu}},$$

$$\breve{E}_{\mu\nu} = \breve{R}_{\mu\nu} - \frac{1}{2}\breve{R}g_{\mu\nu},$$

$$[\breve{T}^{\mu\nu}_{;\nu}]_{\breve{g}} = \breve{F}^{\mu}.$$

Here, a subscripts $g, \breve{g}$ stands for specifying that the labelled quantity is defined by curved space-time metrics $ds_1^2 = g_{\alpha\beta}dx^{\alpha}dx^{\alpha}$ and $ds_2^2 = \breve{g}_{\mu\nu}dx^{\mu}dx^{\nu}$ respectively and $\breve{F}^{\mu}$ denote 4-vector of a pure nongravitational force and vertical duble-bar $\|$ stands for covariant differentiation with respect to $\breve{g}_{\mu\nu}$.

In the Rosen approximation ($\breve{R}^{\alpha}_{\mu\nu\lambda} \approx 0,\ k_1[\breve{T}^{\mu\nu}]_{\breve{g}} \approx 0$) $\Theta^{\alpha\beta} \approx K_{\mu\nu}$ and these equations reduce to the field equations of Rosen: $K_{\mu\nu} - \frac{1}{2}Kg_{\mu\nu} \approx -8\pi kT_{\mu\nu}$.
In the general theory of relativity by means of the new equations gives the same results as the solution by means of Rosen's equations only in the Rosen approximation.
However, application of these equations to the standard astrophysical and cosmologic models coupled with a sufficiently strong electromagnetic field or another nongravitational fields, gives a result different from that obtained by Einstein's or Rosen's equations. In particular, the solution gives Kantowski-Sachs model [8],[9] with source cosmic cloud strings coupled with strong electromagnetic field in contrast with corresponding solution gives in Rosen's bimetric theory [5]-[7].
In this paper we also propose an nontrivial extension of General Relativity with noninertial frames $\mathcal{F}[g]$ that experience space-time to have a metric g different from

usual metric of noninertial frames given in canonical General Relativity.

II.1.Brief review of Rosen's Bimetric Theory

Rosen [1] proposed some simplest type the bimetric gravitational field theory such that at each point of Lorentzian space-time $(\mathcal{L}, g_{ij})$ a flat Lorentzian metric tensor γ_{ij} in addition to the curved Lorentzian metric tensor g_{ij}. Thus at each point of Rosen's space-time $\Re = \Re(g_{ij}, \gamma_{ij})$ there are two metrics:

$$ds_1^2 = g_{ij}dx^i dx^j, \qquad (2.1.1)$$

and

$$ds_2^2 = \gamma_{ij}dx^i dx^j. \qquad (2.1.2)$$

The first metric tensor g_{ij} in Rosen's theory, refers to the curved space-time and thus the gravitational field. The second metric tensor γ_{ij} in Rosen's theory, refers to the flat space-time or space-time of constant curvature, for example such that: Eq.(2.1.3)

$$ds_2^2 = \left(1 - \frac{r^2}{a^2}\right)dt^2 - \frac{dr^2}{1 - r^2/a^2} - r^2(d\theta^2 + \sin\theta d\phi^2). \qquad (2.1.3)$$

Remark *2.1.1. Note that in Rosen's theory element (2.1.3) corresponds only to a background space-time of constant curvature, to which the physical metric reduces in the absence of any kind of energy.*

The Christoffel symbols formed from g_{ij} and γ_{ij} are denoted by $\left\{ \begin{matrix} i \\ jk \end{matrix} \right\}$ and Γ^i_{jk} respectively. The quantities Δ^i_{jk} are defined via formulae

$$\Delta^i_{jk} = \left\{ {i \atop jk} \right\} - \Gamma^i_{jk}. \qquad (2.1.4)$$

Remark *2.1.2. Let $R^{\alpha}_{\mu\nu\lambda}$ and $S^{\alpha}_{\mu\nu\lambda}$ be the curvature tensors calculated from $g_{\mu\nu}$ and $\gamma_{\mu\nu}$ respectively. Note that In the Rosen's approach as $ds_2^2 = \gamma_{\mu\nu} dx^i dx^j$ is the flat metric, the curvature tensor is zero $S^{\alpha}_{\mu\nu\lambda} = 0$.*

Now there arise two kinds of covariant differentiation:
(1) g-differentiation based on $g_{\mu\nu}$ (denoted by semicolon $(;)$)

$$A_{\mu\nu;\lambda} = \left(A_{\mu\nu,\lambda} - \left\{ {\alpha \atop \mu\,\lambda} \right\} A_{\alpha\nu} - \left\{ {\alpha \atop \nu\,\lambda} \right\} A_{\alpha\mu} \right) \qquad (2.1.5)$$

(2) differentiation based on γ_{ij} (denoted by a slash $(|)$)

$$A_{\mu\nu|\lambda} = \left(A_{\mu\nu,\lambda} - \Gamma^{\alpha}_{\mu\,\lambda} A_{\alpha\nu} - \Gamma^{\alpha}_{\nu\,\lambda} A_{\alpha\mu} \right), \qquad (2.1.6)$$

where ordinary partial derivatives are denoted by comma $(,)$.
The straightforward calculations gives

$$R^{\alpha}_{\mu\nu\lambda} = -\Delta^{\alpha}_{\mu\nu|\lambda} + \Delta^{\alpha}_{\mu\lambda|\nu} - \Delta^{\alpha}_{\beta\nu}\Delta^{\beta}_{\mu\lambda} - \Delta^{\alpha}_{\beta\lambda}\Delta^{\beta}_{\mu\nu}. \qquad (2.1.7)$$

Hence

$$R_{\mu\nu} = -\Delta^{\alpha}_{\mu\nu|\alpha} + \Delta^{\alpha}_{\alpha\mu|\nu} - \Delta^{\alpha}_{\alpha\beta}\Delta^{\beta}_{\mu\nu} - \Delta^{\alpha}_{\beta\mu}\Delta^{\beta}_{\alpha\nu}. \qquad (2.1.8)$$

This is the curvature tensor $R_{\mu\nu}$ associated with the curvature effects of pure gravitation acting in the spacetime.

The geodesic equation in Rosen's bimetric relativity takes the form

$$\begin{gathered} \frac{d^2x^i}{ds} + \Delta^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = \\ \frac{d^2x^i}{ds} + \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} \frac{dx^j}{ds}\frac{dx^k}{ds} - \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0. \end{gathered} \qquad (2.1.9)$$

It is seen from Eqs. (2.1.4) and (2.1.9) that Γ^i_{jk} can be regarded as describing the flat inertial field because it vanishes by a suitable coordinate transformation.

II.2.Brief review of Davtyan's One-Metric Theory of Gravitational-Inertial Field.

Einstein theory of General Relativity (GTR) and Einstein Gravitational field theory to proceed from assumptions:

- (**I**) One-metric geometric structures of the space-time continuum on the standard assumption of Lorentzianian geometry

$$ds^2 = g_{ik}dx^i dx^k, g_{ik} = g_{ki}, \det\|g_{ik}\| \neq 0; \qquad (2.2.1)$$

- (**II**) From equivalence of gravitational field and spacc-tirne metric tenzor $g_{ik}(M) = g_{ik}(x_1, x_2, x_3, , x_4)$;
- (**III**) From equivalence of accelerational field γ_{ik} and flat space-time metric tenzor $\gamma_{ik}(M) = \gamma_{ik}(x_1, x_2, x_3, , x_4)$, i.e. $R^i_{klm}[\gamma_{ik}] = 0$.

Axiom *2.2.1.**GTR is based on the following postulates**:*

- (**1**) In nonrelativistic approximation and very far from the localized masses the metric tensor describes a flat space-time, i.e. $R^i_{klm} = 0$.
- (**2**) A sufficiently small domain of Lorentzian space-time is flat, i.e. $R^i_{klm} \approx 0$.
- (**3**) Homogenious gravitational field and acceleratíonal field are equivalent.
- (**4**) Every acceleratíonal field $g^{\mathbf{ac.}}_{ik}$ is flat, i.e. $R^i_{klm}(g^{\mathbf{ac.}}_{ik}) = 0$.

Let's consider this postulates in more detail [1].

1.The exsistence of remote masses of this Universe, including the field masses, will undoubtedly influence metric of the real world and create a general metrical background in the Universe different from Galileo's. Therefore everywhere in the Universe in difference to the ideas of GTR $g_{ik,l} \neq 0$. This metrical background will be called henceforth an inertial field.In order to find the way out of this situation the authors of the well-known scalar theory Brans and Dicke [36]-[37] proceeding from Mach's principle suggested an idea within that theory according to which tlere exists a scalar field, besides the usual tensor field, with a long-range radius of action and caused by universal mass - the "mass of fixed stars".

2. A small domain of mathematical Riemannian or Lorentzian space cannot be flat, it should be approximately similar to the flat space, to be more exact : for each of Rieniannian space a tangent flat space may be constructed. Hence it follows that though locally R^i_{klm} are sufficiently small quantities $R^i_{klm} \approx 0$ of higher order, nevertheless $R^i_{klm} \neq 0$.

3. From the mentioned principle of equivalence it follows that the theory automatically permits arbitrary *holonomic transformations of coordinates*, under which at linear conditions the gravitational field vanishes or at other conditions new physical fields originate.This is evidently not correct, because the true gravitational field, which is equivalent to the geometric structure of Riemannian spacc-time, cannot be eliminated by means of choosing coordinates. On the other hand no objectivc physical field (in contrast to fictive fields) may be created by means of an arbitrary choice of coordinates. Moreover it is well known that, the principle of equivalencc in thc mentioned sense has only a local and approximate character.

By the way, here one should not confuse the law of equality of inertial and gravitational masses with the nientioned principle of equivalence. The mathematical expression of this principle is the possibility of introducing the locally geodetic coordinatc system such that $g_{ik,l} = 0$.

However from this statement a not quite correct conclusion is drawn by Fock [4], namely since the possibility of introducting of locally-geodetic system is contained in Riemannian geometry, therefore the pointed-out principle does not constitute a separate physical hypothesis. Actually the availability of such a possibility in the Riemannian space-time is not nessessary at all. On the contrary, as we have seen, everywhere in this space $g_{ik,l} \neq 0$.

Therefore the principle of equivalence of Einstein may be exprcssed mathematically in terms of any abstract tensor $\breve{\mathbf{g}}_{ik}$: $g_{ik,l} \simeq \breve{\mathbf{g}}_{ik,l} = 0$ in locally-geodetic coordinate system or,

$$g_{ik,l} \simeq \breve{\mathbf{g}}_{ik,l} = 0 \qquad (2.2.2)$$

in all coordinate systems. Without this assumption one cannot construct the EGTR. Thus since in the locally-geodetic system $g_{ik;l}$ may be equal to zero only approximately: $g_{ik,l} \simeq 0$ the approximate character of the principle of equivalence follows.

Actually one more very important fact (usually unnoticed) follows from the principle of equivalencc, namely that the geodetic line is identified with a trajectory of motion of a free material particle. Indeed it is well known that the notion of affine connection $\breve{\mathbf{\Gamma}}^{i}_{kl}$. On the basis of parallel transfer the whole tensor analysis may be Constructed, the expression for the tensor of curvature R^{i}_{klm} obtained, and geodetic lines be constructed, i.e., the curves of parallel transfer of vector or tensor with their equations, without introducing the notion of metrical tensor g_{ik}.

Indeed let an arbitrary scalar parameter t be taken as a parameter changing along the curve of parallel transfer of vector u^{i}, i.e. the curve is parametrically defined by the equation $x_i = x_i(t)$ and $u^i = dx_i/dt$ is a unit tangent vector to the curve. Then the variations of vector components as a result of parallel transfcr from point $\mathbf{M}$ to point $\mathbf{M}'$ along the curve will be equal to

$$\frac{dx_i(\mathbf{M}') - dx_i(\mathbf{M})}{dt} =$$

$$\frac{dx_i(\mathbf{M}')}{dt} - \frac{dx_i(\mathbf{M})}{dt} = \qquad (2.2.3)$$

$$= -\breve{\mathbf{\Gamma}}^i_{kl} dx_k dt \times \Delta x_l.$$

Dividing these equations by the value of transfer Δt from $\mathbf{M}'$ to $\mathbf{M}'$ and taking the limit when $\Delta t \to 0$ we obtain the equations of the geodetic line

$$\frac{d^2 x_i}{dt^2} + \breve{\mathbf{\Gamma}}^i_{kl} \frac{dx_k}{dt} \frac{dx_l}{dt} = 0. \qquad (2.2.4)$$

The affine coefficients $\breve{\mathbf{\Gamma}}^i_{kl}$ and the scalar parameter t in these equations are not related to the metric of space-time. However $\breve{\mathbf{\Gamma}}^i_{kl}$ may be expressed in terms of any abstract tensor $\breve{\mathbf{g}}_{ik}$ (and its first derivatives) satisfying $\breve{\mathbf{g}}_{ik;l} = 0$ on the basis of expressions for covariant derivatives :

$$\breve{\mathbf{g}}_{ik;l} = \breve{\mathbf{g}}_{ik,l} - \breve{\mathbf{\Gamma}}^m_{il} \breve{\mathbf{g}}_{mk} - \breve{\mathbf{\Gamma}}^m_{kl} \breve{\mathbf{g}}_{im} = 0 \qquad (2.2.5)$$

and consequently one obtain

$$\breve{\mathbf{\Gamma}}^i_{kl} = \frac{1}{2}\breve{\mathbf{g}}^{im}(\breve{\mathbf{g}}_{mk,l} + \breve{\mathbf{g}}_{ml,k} - \breve{\mathbf{g}}_{kl,m}) \qquad (2.2.6)$$

Thus it follows that in the EGTR the real space-time metric tensor g_{ik} is identified with an
abstract tensor $\breve{\mathbf{g}}_{ik.}$.

Meanwhile the trajectory of motion of a free particle, in contrast to thc geodetic line, can be obtained only on the basis of real space-time metric tensor g_{ik} from the principle of least action

$$\delta S = -mc\delta \int ds = 0, ds^2 = g_{ik}dx^i dx^k \qquad (2.2.7)$$

and consequently from Eq.(2.2.7) one obtain

$$\frac{d^2x_i}{dt^2} + \Gamma^i_{kl}\frac{dx_k}{dt}\frac{dx_l}{dt} = 0,$$

$$(2.2.8)$$

$$\Gamma^i_{kl} = \frac{1}{2}g^{im}(g_{mk,l} + g_{ml,k} - g_{kl,m}).$$

The curvature tensor R^i_{klm} is also defined through parallel transfer and one can expressed curvature tensor by the affine coefficients $\breve{\mathbf{\Gamma}}^i_{kl}$ and their derivatives. In particular

$$\breve{\mathbf{R}}_{ik} = \breve{\mathbf{\Gamma}}^{l}_{i,kl} - \breve{\mathbf{\Gamma}}^{l}_{il,k} + \breve{\mathbf{\Gamma}}^{l}_{ik}\breve{\mathbf{\Gamma}}^{m}_{lm} - \breve{\mathbf{\Gamma}}^{m}_{il}\breve{\mathbf{\Gamma}}^{l}_{km},$$

$$\breve{\mathbf{R}} = \breve{\mathbf{g}}^{ik}\breve{\mathbf{R}}_{ik}. \qquad (2.2.9)$$

Hence Einstein's field equations reads

$$\breve{\mathbf{R}}_{ik} - \frac{1}{2}\breve{\mathbf{g}}_{ik}\breve{\mathbf{R}} = \frac{8\pi\varkappa}{c^4}T_{ik}. \qquad (2.2.10)$$

II.2.1.Postulates of Davtyan's One-Metric Theory of Gravitational-Inertial Field.

The fundamental principle of Davtyan's gravitational-inertial theory [1], like that of EGTR, is the assumption of equivalence of the gravitational-inertial field with the geometric structure of space-time on the basis of statement and conditions (2.2.1) of Rienimanian geometry and its "extension" all over the Universe.The essence of such an "extension" lies in the point that because of the existence of the inertial field (besides the gravitational fields) far from the masses and also locally the metric tensor of the world is everywhere distinct from Calileio's metric, i.e.,

$$g_{ik,l} \neq 0 \qquad (2.2.11)$$

The space is permanently related to a weak metrical background $h^{lm}_{(J)}$. The field formed by this tensor background $h^{lm}_{(J)}(\mathbf{M})$, as already noted, will be called inertial field. The cause of formation of such tensor background, as we shall see in the later development of the theory, is the world energy-momentum tensor $T^{lm}_{(J)}$ related to a field mass of gravitational electromagnetic radiation surrounding the metagalaxy. Further, as already shown, there are no theoretical are experimental reasons to considering g_{ik} that represents a physical field as identical with $\breve{\mathbf{g}}_{ik}$ entering into the coefficients of affine connection $\breve{\mathbf{\Gamma}}^{l}_{km}$, which is being abstractly constructed for operations in tensor analysis. Independently of gravitational fields, in the inertial frame of reference and in the locally geodetic coordinate system

$$\breve{\mathbf{g}}_{ik,l} = 0. \qquad (2.2.12)$$

Remark ***2.2.1.1***. *It should be recalled that according to GTR the gravitational field is defined by the tensor $\breve{\mathbf{g}}_{ik}$. Therefore in the presence of a gravitational field in the locally-geodctic coordinate system though $\breve{\mathbf{g}}_{ik,l} = 0$ nevertheless:*

$$\breve{\mathbf{g}}_{ik,lm} = \frac{\partial^2 \breve{\mathbf{g}}_{ik}}{\partial x_l \partial x_m} \neq 0. \qquad (2.2.13)$$

In contrast to this, in Davtyan's theory [1] of the gravitational inertial field $\breve{\mathbf{g}}_{ik}$ is not related to the gravitational field, so that even at the presence of such a field we have in the locally-geodetic coordinate system

$$\breve{\mathbf{g}}_{ik,lm} = \frac{\partial^2 \breve{\mathbf{g}}_{ik}}{\partial x_l \partial x_m} = 0. \qquad (2.2.14)$$

Thus on the basis of listed propositions one may conclude that the universal metric tensor g_{ik} of the gravitational-inertial field is the metric tensor of the real world, formed by gravitational $g^{\mathbf{Gr}}_{ik}$ and $g^{\mathbf{In}}_{ik}$ inertial metric tensors. From the field of this general metric

tensor $g_{ik}(\mathbf{M})$ the Riemannian space is also formed.
On the basis of original proposition given by Eq. (2.2.11) the following quite obvious theorem may be proved [1]: If first derivatives of the metric tensor gir are nonzero in the locally geodetic coordinate system then also nonzero will be its covariant derivatives in all coordinate systems i.e.

$$g_{ik;l} = g_{ik,l} - \breve{\mathbf{\Gamma}}^{m}_{il} g_{mk} - \breve{\mathbf{\Gamma}}^{m}_{ki} g_{im} \neq 0, \qquad (2.2.15)$$

where

$$\breve{\mathbf{\Gamma}}^{i}_{kl} = \frac{1}{2} \breve{\mathbf{g}}^{im} (\breve{\mathbf{g}}_{mk,l} + \breve{\mathbf{g}}_{ml,k} - \breve{\mathbf{g}}_{kl,m}). \qquad (2.2.16)$$

Since, according to Eq.(2.2.12) in the locally-geodetic coordinate system $\breve{g}_{ik,l} = 0$, then in Eq.(2.2.15) $\breve{\mathbf{\Gamma}}^{m}_{il}$ and $\breve{\mathbf{\Gamma}}^{m}_{ki}$ should be equal to zero and according to Eq. (2.2.11) actually $\breve{\mathbf{g}}_{ik;l} \neq 0$. But since the quantity $\breve{\mathbf{g}}_{ik;l}$ is a tensor, it will be nonzero in all other coordinate systems if it is nonzero in one of them.

- **Remark** *2.2.1.2. Note that the expression given by Eqs.(2.2.15)-(2.2.16) is another important original proposition of Davtyan's gravitational-inertial field theory of the Universe. This proposition actually means that the gravitational-inertia1 field g_{ik} and Riemannian space $\mathcal{L}(M, g_{ik})$ formed by a general metric tensor of real world represents a truely physical field, that may not be eliminated by means of transformation of coordinates.*

Meanwhile the affine field $\breve{\mathbf{\Gamma}}^{m}_{il}$ containing the tensor $\breve{\mathbf{g}}_{ik}$ may be eliminated by the choice of a special coordinate system, $\breve{\mathbf{\Gamma}}^{i}_{kl} = 0$. Further if the world is considered as pseudoeuclidean, i.e. as a world without gravitational and inertial fields, then in curvilinear coordinates or generally in non-inertial frames of reference $\breve{\mathbf{\Gamma}}^{i}_{kl} \neq 0$. Thus whereas the real nietric tensor g_{ik} forms a truely physical field $g_{ik}(\mathbf{M})$, the tensor $\breve{\mathbf{g}}_{ik}$ causes various kineinntical-dynamical effects due to origination of fictitious fields in noninertial frames of reference.

Remark *2.2.1.3. From the introduced propositions of Davtyan's theory it follows apart from the Riemannian space tensor $g_{ik}(\mathbf{M})$ representing the gravitationel-inertial field (of real world) with the quadratic form of space-time interval element*

$$ds^2 = g_{ik}dx^i dx^k \qquad (2.2.17)$$

no other physical space or new tensor or scalar is being introduced, as done in bimetrical and scalar theories.

Remark *2.2.1.4. Though it is formally supposed in Davtyan's theory that g_{ik} may be represented sum of a gravitational tensor $g_{ik}^{\mathbf{Gr}}$ and the metrical tensor background $h_{ik}^{(J)}$, this assumption is not used in the Davtyan's field equations (see [1] section 3.).*

Remark *2.2.1.5.The potentials of the gravitational-inertial field obtained in [1] represent themselves only the components of the universal metric tensor g_{ik}, entering in (2.2.17). Further, as we have seen, in the locally-geodetic coordinate system*

$$g_{ik,l} = g_{ik;l} \simeq 0. \qquad (2.2.18)$$

On the same basis the scalar quantity

$$g^{ik} g_{ik} \simeq 1 \qquad (2.2.19)$$

in all coordinate systems.

II.2.2.Lagrangian Density and Field Equations in Davtyan's Theory of Gravitational-Inertial Field.

The expression (2.2.15) allows us to use the following variational action for obtaining field equations [5]:

$$\mathbf{S} = \mathbf{S}_g + \mathbf{S}_m = \frac{1}{c} \int (\Lambda_g + \Lambda_m) \sqrt{-g}\, d^4 x, \qquad (2.2.20)$$

where

$$\Lambda_g = \frac{1}{8\pi\chi} g_{lm;i} \bullet g^{lm}_{;k} g^{ik}. \qquad (2.2.21)$$

This corresponds to the fact, that in Euclidean space the Lagrange density is defined as a square of gradient of the potential Φ :

$$\Lambda_g = \frac{1}{8\pi\chi} [\mathbf{grad}(\Phi)]^2 \qquad (2.2.22)$$

In Riemann space this quantity should be generalized to an inner product of covariant derivatives of the metric tensor as in Eq.(2.2.21).
By analogy with Eq.(2.2.21) the Lagrangian density for matter will be defined as

$$\Lambda_m = k \cdot f(g^{ik}, g_i^{lm})\left(\sqrt{-g}\right)^{-1} \qquad (2.2.23)$$

and the action will be equal to

$$\mathbf{S}_m = \frac{1}{c}\int \Lambda_m \sqrt{-g}\, d^4x = \frac{k}{c}\int f(g^{ik}, g_i^{lm}) d^4x \qquad (2.2.24)$$

where k is a constant. As was shown by Davtyan's in [1] that

$$k = \frac{2}{c^4} \qquad (2.2.25)$$

or $k = \frac{4}{c^4}$.
According to Eqs.(2.2.20)-(2.2.24) by using variational principle

$$\delta \mathbf{S} = \delta(\mathbf{S}_g + \mathbf{S}_m) = 0 \qquad (2.2.26)$$

one obtain [1]:

$$\delta\mathbf{S} = \delta(\mathbf{S}_g + \mathbf{S}_m) =$$

$$\int\Big\{\frac{1}{8\pi\varkappa}\Big[g^{lp}g^{mn}\nabla_k\Big(\sqrt{-g}\,g^{ik}g_{np;i}\Big)$$

$$-\nabla_i\Big(\sqrt{-g}\,g^{ik}g_{np;k}\Big)\Big] - kg^{lp}g^{mn}\sqrt{-g}\,T_{np}\Big\}, \qquad (2.2.27)$$

$$\sqrt{-g}\,T_{lm} = \frac{\partial\Lambda_m\sqrt{-g}}{\partial g^{lm}} - \frac{\partial}{\partial x_i}\left(\frac{\partial\Lambda_m\sqrt{-g}}{\partial g^{lm}_{,i}}\right).$$

Finally Davtyan's Gravitational-Inertial field equations sees [5]:

$$g^{lp}g^{mn}\nabla_k\Big(\sqrt{-g}\,g^{ik}g_{np;i}\Big) - \nabla_i\Big(\sqrt{-g}\,g^{ik}g_{np;k}\Big) =$$

$$= 8\pi\varkappa k\sqrt{-g}\,g^{lp}g^{mn}T_{np} = \qquad (2.2.28)$$

$$8\pi\varkappa k\sqrt{-g}\,g^{mn}T^l_n.$$

In the cquations (2.2.28) the components of g_{ik} represent the potentials of the gravitational-inertial field. The components of tensor $\breve{g}_{ik}$ entering only into Christoffel symbols $\Gamma^i_{kl}(\breve{g}_{ik}, \breve{g}_{ik,l})$ may be eliminated. The possibility of eliminating $\breve{g}_{ik}$ may be explained in virtue of the fact that, was pointed out, the space of affine connection $\breve{\mathbf{\Gamma}}^i_{kl}(\mathbf{M})$ auxiliary, abstract mathematical space while $g_{ik}(\mathbf{M})$) defines a Riemannian world and the gravitational-inertia1 field. Therefore $\breve{\mathbf{\Gamma}}^i_{kl}(\mathbf{M})$ may be chosen arbitrarily [1]. Thus the $\breve{\mathbf{\Gamma}}^i_{kl}$ may be defined in such a way that $\breve{g}_{ik,l} = 0$ and hence $\breve{\mathbf{\Gamma}}^i_{kl} = 0.$ It sllould be noted that the elimination of Christoffel symbols in (2.2.28) actually means the elimination of various fictive fields. In order that no misunderstanding of these items will arise (due to existing traditional habits) we consider it necessary to specify the essence of one of the major differences between the GTR and present theory. As already pointed out, in GTR it is assumed that $g_{ik} \equiv \breve{g}_{ik}$. Therefore if g_{ik} is taken in some coordinate system, then $\breve{\Gamma}^i_{kl}$ should be taken in the same coordinate system. In contrast to this in the present theory [1] the Riemann space (M, g_{ik}) is everywere and always curved, therefore g_{ik} is always taken in curvilinear coordinates. Since g_{ik} is not physically related to $\breve{g}_{ik}$ the coordinate system for $\breve{g}_{ik}$ may be chosen independently of that for g_{ik}.

For example, in the locally geodetic coordinate system $\breve{g}_{ik,l} = 0$ while $g_{ik,l} \neq 0$. This means that $\breve{g}_{ik}$ is taken in Carhian coordinates, while g_{ik} is still related to the curved space and may be taken in arbitrary curvilinear coordinates. This also means that, as mentioned above, the parallel transfer operation and its curve do not depend on the curvature of space-time and hence on the trajectory of motion of a free particle. Thus according to conditions (2.2.12)-(2.2.14) for $\breve{\Gamma}^i_{kl} = 0$ in (2.2.28).Hence all Christoffel symbols are being eliminated and (2.2.28) are transformed into the equations

$$\begin{gathered} \nabla_k g^{lm} = g^{lm}_{,k}, \\ \nabla_i \nabla_k g^{lm} = g^{lm}_{,ki}. \end{gathered} \qquad (2.2.29)$$

Hence all Christoffe1 symbols are being eliminated and (2.2.28) are transformed into the equations [1]:

$$\begin{gathered} g^{lp} g^{mn} \frac{\partial}{\partial x_k}\left(\sqrt{-g}\, g^{ik} g_{np;i}\right) - \frac{\partial}{\partial x_i}\left(\sqrt{-g}\, g^{ik} g_{np;k}\right) = \\ = 8\pi\varkappa k \sqrt{-g}\, g^{mn}\left(T^l_n - \frac{1}{2}\delta^l_n T\right). \end{gathered} \qquad (2.2.30)$$

These equations are very attractive not only because Christoffel symbols are absent in them (and hence they are extremely simple), but mainly because the solution of fundamental problems of GTR by means of these equations gives the same results as the solution by means of the Einstein equations [1].

II.2.3. Davtyan's Field Equations in Einstein approximation.

It is necessary to observe that the original variational equations make it possible to obtain another version of field equations, somewhat different from Eq.(2.2.28) Indeed in the proccss of variation of Lagrangian Λ_m, for the matter (2.2.23) a tensor density is obtained in the form (2.2.27). This expression with some coefficient may also be considered as the tensor density

$$\sqrt{-g}\,\breve{T}_{lm} = \sqrt{-g}\left(T_{lm} - \frac{1}{2}g_{lm}T\right) \qquad (2.2.31)$$

Anyway one may always (with equal hasis) proceed from the fact. That for the gravitational- inertial field the following formula holds

$$\delta\mathbf{S}_m = \frac{k}{c}\int \breve{T}\sqrt{-g}\,\delta g^{lm}d^4x = \qquad (2.2.32)$$
$$\frac{k}{c}\int(T_{lm} - \tfrac{1}{2}g_{lm}T)\,\sqrt{-g}\,g^{lm}d^4x.$$

Therefore the choice of tensor T_{lm}, is mathematically equivalent to the choice of $\breve{T}_{lm}$ However the analysis of equations (2.2.28) shows that in the Einstein approximation, i.e. in the limit $g_{ik} \simeq \breve{g}_{ik}$ they reduce to

$$R^{lm} = 4\pi k \varkappa T^{lm}. \qquad (2.2.33)$$

Where R^{lm} is the Ricci tensor. Therefore, in order to satisfy the continuity law as well as the Bianchi identity the choice of the second tensor version-the source of gravitational- inertional field as

$$\breve{T}^{lm} = T^{lm} - \frac{1}{2} g^{lm} T \qquad (2.2.34)$$

is more expedient. Then the gravitional-inertial field equations take the following form

$$\begin{aligned} & g^{lp} g^{mn} \nabla_k \left(\sqrt{-g}\, g^{ik} g_{np;i} \right) - \nabla_i \left(\sqrt{-g}\, g^{ik} g_{np;k} \right) = \\ & = 8\pi \varkappa k \sqrt{-g}\, g^{mn} \left(T^l_n - \frac{1}{2} \delta^l_n T \right). \end{aligned} \qquad (2.2.35)$$

In the above mentioned limit one get

$$\begin{aligned} & R^{lm} = 4\pi k \varkappa \left(T^{lm} - \frac{1}{2} g^{lm} T \right) = \\ & \frac{8\pi \varkappa}{c^4} \left(T^{lm} - \frac{1}{2} g^{lm} T \right). \end{aligned} \qquad (2.2.36)$$

Thus in the Einstein approximation the gravitational-inertial field equations reduce to the usual Einstein equations of the gravitational field.

II.2.4. Weak Field limit in Davtyan's Theory of Gravitational-Inertial Field.

Under these conditions the metric of space-time is close to the Galileo metric:

$$g^0_{11} = g^0_{22} = g^0_{33} = -1, g^0_{44} = 1,$$
$$g^0_{ik} = 0, i \neq k. \qquad (2.2.37)$$

with a certain background of an inertial field $h^{jk}_{(J)}$. A weak perturbation, caused by gratvitational field (plus the inertial field $h^{il}_{(J)}$)) may be represented by a tensor h_{ik}, which is a first order small quantity:

$$g_{lm} = g^0_{lm} + h_{lm}. \qquad (2.2.38)$$

With the same accuracy one obtain the expression for the determinant of the inetricsl tensor :

$$g = \det\|g_{lm}\| = -(1 + g^{lm}h_{lm}). \qquad (2.2.39)$$

On the basis of these simplifications one may proceed to the solution of the gravitational field equations (2.2.30). According to Eq.(2.2.38) one obtain

$$(g^{lp(0)} - h^{lp})(g^{nm(0)} - h^{mn})\frac{\partial}{\partial x_k}\left[(g^{ik(0)} - h^{ik})\frac{\partial}{\partial x_i}\left(g^{(0)}_{np} + h_{np}\right)\right] -$$

$$-\frac{\partial}{\partial x_i}\left[(g^{ik(0)} - h^{ik})\frac{\partial}{\partial x_k}(g^{lm(0)} - h^{lm})\right] = \qquad (2.2.40)$$

$$= 8\pi k\varkappa g^{mn(0)}\left(T^i_n - \frac{1}{2}\delta^i_n T\right).$$

From Eq.(2.2.40) one obtain

$$\Box h_{ml} = 4\pi k\varkappa g^{nm(0)}\left(T^i_n - \frac{1}{2}\delta^i_n T\right) \qquad (2.2.41)$$

The solution of this equation is

$$h_{ml} = -k\varkappa \int \frac{\left(g^{nm(0)}\left(T^i_n - \frac{1}{2}\delta^i_n T\right)\right)}{r} d^3x. \qquad (2.2.42)$$

The energy-momentum tensor may be taken in the form

$$T_l^n = \rho v_l v^n,$$

$$v_l = \frac{dx_l}{dt}, v^n = \frac{dx^n}{dt}, \qquad (2.2.43)$$

$$l, n = 1, 2, 3, 4$$

where ρ is the density of mass.

Since according to condition the field is weak the components of thrce-dimensional velocity should be very small with respect to the fudamental velocity c. Therefore all space components of velocity in Eq.(2.2.43) inay he neglected. Hence only the time coinponent c^2 remains. Hence $T_4^4 = c^2\rho$ and system (2.2.41) consisting of 10 equations turns into a single equation

$$\Box h_{44} = 2\pi c^2 k\varkappa\rho \qquad (2.2.44)$$

If the field is stationary we have $\frac{\partial h_{44}}{\partial x_4}$ and hence from Eq.(2.2.44) one obtain

$$\Delta h_{44} = 2\pi c^2 k\varkappa\rho. \qquad (2.2.45)$$

Thc solution of this equation is

$$h_{44} = \frac{k}{2}c^2\varkappa\int\frac{\rho}{r}d^3x. \qquad (2.2.46)$$

Thus

$$h^{44} = g_{44}^{(0)} - \frac{k}{2}c^2\varkappa\int\frac{\rho}{r}d^3x,$$

$$h_{44} = \frac{k}{2}c^2\Phi, g_{44} = g_{44}^{(0)} + \frac{k}{2}c^2\Phi, \qquad (2.2.47)$$

$$\Phi = -\varkappa\int\frac{\rho}{r}d^3x$$

and according to Eq.(2.2.45)

$$\Delta\Phi = 4\pi\varkappa\rho. \qquad (2.2.48)$$

This expression corresponds exactly to Newton's equation of gravitation for continuous
distribution of masses.

III. Variational Action Principles in Rozen's Bimetric Theory.

General variational Action Principle [38] was introduced, in the Rosen's Theory of Gravitation, in view of deriving field equations, motion equations,canonical energy tensor. and conservative principles. Using the constraint of metric invariance during the variational process along the trajectory, a certain relationship between the canonical tensor and the motion equations is established as a test for selfconsistency. As was pointed out in the paper [38] the Equivalence Principle between gravitational mass and inertial mass does hold in a weak version, i.e. equality of masses but not also of their space distributions.

III.1.Simple Variational Action Principle in Rozen's Bimetric Theory. Field Equations.

The field and motion equations, as well as the canonical energy tensor, may by derived, in the case of Rosen's theory of gravitation [5]-[7], from a certain Action Principle adopting the perfect magnetic fluid scheme for the matter tensor and specifying the field part of the Action as depending on Minkowskian quantities of definite variance not exceeding the first order derivatives, the Action integral takes the form

$$\mathbf{S} = \frac{1}{64\pi\chi}\int \mathcal{L}\left(g_{\mu\nu},\boldsymbol{\gamma}_{\mu\nu}\right)\sqrt{-\boldsymbol{\gamma}}\, d^4x + \mathbf{S}_m \qquad (3.1.1)$$

where

$$\mathcal{L}\left(g_{\mu\nu},\boldsymbol{\gamma}_{\mu\nu}\right) = \boldsymbol{\gamma}^{\mu\nu} g^{\alpha\beta} g^{\gamma\delta}\left(g_{\alpha\gamma|\mu} g_{\alpha\delta|\nu} - \frac{1}{2} g_{\alpha\beta|\mu} g_{\gamma\delta|\nu}\right), \qquad (3.1.2)$$

where the bar ("|") denotes covariant derivative with respect to $\boldsymbol{\gamma}_{\mu\nu}$. The corresponding field equations may be written in the form:

$$\Box_{\gamma} g_{\mu\nu} - g^{\alpha\beta}\boldsymbol{\gamma}^{\gamma\delta} = -16\pi\varkappa\sqrt{g/\boldsymbol{\gamma}}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \qquad (3.1.3)$$

or in the form

$$\gamma^{\alpha\beta} g_{\mu\nu|\alpha\beta} - g^{\alpha\beta}\boldsymbol{\gamma}^{\gamma\delta} = -16\pi\varkappa\sqrt{g/\boldsymbol{\gamma}}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \qquad (3.1.4)$$

Remark *3.1.1. Note that in Rosen's theory of gravitation for the derivetion gravitational field equations by simple variational principle one varies the quantities $g_{\mu\nu}$, not the quantities $\gamma_{\mu\nu}$, in the simple variational principle.*

Suppose that the second metric tensor γ_{ij} in Rosen's theory, refers to space-time of constant curvature, given by Eq.(2.1.3).The gravitational field equations reads

$$K_{\mu\nu} - \frac{1}{2}Kg_{\mu\nu} = -8\pi kT_{\mu\nu} \qquad (3.1.5)$$

or

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \frac{3}{a^2}\left(\gamma_{\mu\nu} - \frac{1}{2} g^{\alpha\underline{\alpha}} g_{\mu\nu}\right) = -8\pi k T_{\mu\nu},$$

$$k = \frac{\varkappa}{c^4}. \tag{3.1.6}$$

One sees that (3.1.6) differs from the Einstein field equations by an additional term on the left hand side.

Suppose that the background space-time metric $\gamma_{\mu\nu}$ corresponding to the metric $g_{\mu\nu}$ is

$$ds_2^2 = dt^2 - dx^2 - dy^2 - dz^2. \tag{3.1.7}$$

The energy momentum tensor for perfect fluid is given by

$$T^{FL}_{\alpha\beta} = (P + \rho) u_\alpha u_\beta - g_{\alpha\beta} P. \tag{3.1.8}$$

The Rozen-Maxwell equations for ideal MHD flow are

$$K_{\alpha\beta} - \frac{1}{2} K g_{\alpha\beta} = -8\pi k (T^{FL}_{\alpha\beta} + T^{EM}_{\alpha\beta}),$$

$$k = \varkappa \sqrt{\frac{g}{\gamma}}, \tag{3.1.9}$$

$$K_{\alpha\beta} = K^{\beta}_{\alpha} = \frac{1}{2} \breve{g}^{\mu\nu} (g^{h\alpha} g_{h\beta|\mu})_{|\nu},$$

where the energy momentum tensor of the electromagnetic field denoted by $T^{EM}_{\alpha\beta}$.

The electromagnetic field tensor for the MHD fluid is given by the covariant expression

$$F_{\alpha\beta} = u_\alpha E_\beta - u_\beta E_\alpha + \eta_{\alpha\beta\gamma\delta} u^\gamma B^\delta \qquad (3.1.10)$$

and similarly in a contravariant form

$$F^{\alpha\beta} = u^\alpha E^\beta - u^\beta E^\beta + \eta^{\alpha\beta\gamma\delta} u_\gamma B_\delta, \qquad (3.1.11)$$

where the four vectors E_α and B_α, denoting the electric and magnetic field components in the four dimensional spacetime, are orthogonal to the velocity four vector u^α. Here the volume element 4-form of V_4 namely $\eta_{\alpha\beta\gamma\delta}$ and its dual $\eta^{\alpha\beta\gamma\delta}$ is defined

$$\begin{aligned} \eta_{\alpha\beta\gamma\delta} &= \sqrt{-g}\,\epsilon_{\alpha\beta\gamma\delta}, \\ \eta^{\alpha\beta\gamma\delta} &= \frac{1}{\sqrt{-g}}\epsilon^{\alpha\beta\gamma\delta}, \end{aligned} \qquad (3.1.13)$$

where g represents the determinant of the metric tensor $g_{\alpha\beta}$ and $\epsilon_{\alpha\beta\gamma\delta}$ is the Levi -Civita symbol, which is $+1, -1,$ and 0 for a cyclic, anti-cyclic, and noncyclic permutation of $\alpha\beta\gamma\delta$ respectively. It should be noticed that the choice of spacetime V_4 is quite general here, namely of a four dimensional vector space, or more generally of a differentiable manifold of four dimensions. Also the signature of an axially symmetric metric defined on this manifold must be either $(+---)$ or $(-+++)$.

III.2. General Variational Action Principle in Rozen's Bimetric Theory.

Let's consider bimetric Rosen's space-time $\Re(M, g_{\mu\nu}, \breve{g}_{\mu\nu})$ with $R(g_{\mu\nu}) \neq 0,\ R(\breve{g}_{\mu\nu}) = 0.$ Action integral takes the form

$$\mathbf{S} = \frac{1}{c}\int \mathcal{L}\sqrt{-\breve{g}}\, d^4x,$$
$$\mathcal{L}(g_{\mu\nu}, \breve{g}_{\mu\nu}) = \mathcal{L}_m(g_{\mu\nu}, \breve{g}_{\mu\nu}) + \mathcal{L}_f\left(\chi^{\mu\nu}; g_{\mu\nu|\lambda}, \breve{h}^{\mu\nu}\right), \tag{3.2.1}$$

where

$$\mathcal{L}_m = [(c^2 + \Xi)\rho - p]_g \cdot K, K = \sqrt{\frac{-g}{-\breve{g}}},$$

$$\mathcal{L}_f = \frac{c^4}{16\pi G} f\left(\chi^{\alpha\beta}, g_{\alpha\beta|\lambda}; \breve{h}^{\alpha\beta}\right).$$

$$g_{\mu\lambda}\chi^{\lambda\nu} = \delta^\nu_\mu;\ \breve{g}_{\mu\lambda}\breve{h}^{\lambda\nu} = \delta^\nu_\mu,$$

$$g^{\alpha\beta} = g_{\mu\lambda}\breve{h}^{\mu\alpha}\breve{h}^{\nu\beta} \neq \chi^{\alpha\beta},$$

$$(3.2.2)$$

$$\chi_{\alpha\beta} = \chi^{\mu\nu}\breve{g}_{\mu\alpha}\breve{g}_{\nu\beta} \neq g_{\alpha\beta}.$$

$$\Xi = \int_0^{p(\rho)} \frac{dp}{p(\rho)},$$

$$p = p(\rho), \rho = \rho(p),$$

$$g = \det\|g_{\mu\lambda}\| < 0, \breve{g} = \det\|\breve{g}_{\mu\lambda}\| < 0.$$

Here, a subscript g stands for specifying that the labelled quantity is defined by curved space-time metric $ds_1^2 = g_{\alpha\beta}dx^\alpha dx^\alpha$. Besid this, a flat space-time metric is $ds_2^2 = \breve{g}_{\mu\nu}dx^\mu dx^\nu$.

The quantity K in (3.2.2)) is a Minkowskian scalar, while the quantities ρ (mass density) p (pressure) and Ξ (Helmoltz potential) are scalar in both M_4 and $\mathcal{L}(M, g_{\alpha\beta})$.

For ensuring coherence of the whole variational process, not only the coordinates x^α and the signatures (time-like) should be the same for the two metrics, but also the metric tensor $g_{\alpha\beta}$ of the curved Lorentzian space-time $\mathcal{L}_g = \mathcal{L}(M, g_{\alpha\beta})$ must be considered as a tensor in M_4 (Minkowskian space time). Arbitrary coordinates in M_4 are adopted (necessary for carrying out the variational calculations) and $x^0 = ct$ is taken as the time coordinate (with physical dimension of a length). So, in Rosen's theory the role of the two metrics is strongly dissymmetrized, $g_{\alpha\beta}$ are some quantities preserving only the meaning of gravitational potentials, and the metric $ds_1^2 = g_{\alpha\beta}dx^\alpha dx^\alpha$ turns out to be a simple mathematical artifact necessary to formulate the specific

coupling of gravitational field to its sources This bimetric philosophy (which restricts the main role of Lorentzian metric ds_1^2 to the motion equations) entitles us to treat the quantities $g_{\mu\nu}$, as true gravitational potentials, distinct from the metric functions $\breve{g}_{\mu\nu}$. Now, performing the variations against $g_{\mu\nu}$, in the action integral, we come to field equations; variation against $\breve{g}_{\mu\nu}$ delivers a canonical energy tensor, while the variation against the coordinates of a fluid particle delivers motion equations.
The Minkowskian covariant derivatives are denoted by a vertical bar (|) followed by a certain (Greek) subscript, or (equivalently) by a derivative symbol (D_α) followed by the same subscript. For example

$$g_{\alpha\beta|\lambda} = \breve{D}_\lambda g_{\alpha\beta} = g_{\alpha\beta,\lambda} - \breve{G}^{\ \sigma}_{\alpha\lambda} g_{\alpha\beta} - \breve{G}^{\ \sigma}_{\beta\lambda} g_{\sigma\alpha},$$

$$\breve{G}^{\lambda}_{\mu\nu} = \frac{1}{2}\breve{h}^{\lambda\sigma}(\breve{g}_{\mu\sigma,\eta} + \breve{g}_{\nu\sigma,\rho} - \breve{g}_{\mu\nu,\sigma}). \tag{3.2.3}$$

For obtaining the field equations and the energy canonical tensor one obtain to the following identity

$$\frac{1}{\sqrt{-\breve{g}}}\delta\left(\sqrt{-\breve{g}}\,L\right) = \breve{D}_\lambda q^\lambda -$$

$$\frac{1}{2}K\left([T_{\alpha\beta}]_g + \frac{c^4}{8\pi G}E_{\alpha\beta}\right)\delta\chi^{\alpha\beta} - \frac{1}{2}\Im_{\alpha\beta}\delta\breve{h}^{\alpha\beta}, \tag{3.2.4}$$

where

$$q^{\lambda} = \mathbf{P}^{\alpha\beta\|\lambda}\delta g_{\alpha\beta} + 2\gamma_{\sigma\nu}\mathbf{P}^{\mu\nu\|\eta}\Omega_{\nu\mu\|\alpha\beta}\delta h^{\alpha\beta},$$

$$\mathbf{P}^{\alpha\beta\|\lambda} = \frac{\partial L_f}{\partial g_{\alpha\beta|\lambda}},$$

$$\breve{\Omega}^{\lambda\sigma}_{\nu\mu\|\alpha\beta} = \frac{1}{4}(\delta^{\lambda}_{\beta}\delta^{\sigma}_{\nu}\breve{g}_{\mu\alpha} + \delta^{\alpha}_{\lambda}\delta^{\sigma}_{\nu}\breve{g}_{\mu\beta} + \delta^{\lambda}_{\beta}\delta^{\sigma}_{\mu}\breve{g}_{\nu\alpha} +$$

$$+\delta^{\lambda}_{\alpha}\delta^{\sigma}_{\mu}\breve{g}_{\nu\beta} - \breve{h}^{\lambda\sigma}\breve{g}_{\mu\alpha}\breve{g}_{\nu\beta} - \breve{h}^{\lambda\sigma}\breve{g}_{\mu\beta}\breve{g}_{\nu\alpha}),$$

$$[T_{\alpha\beta}]_g = \{(c^2 + \Xi)\rho U_\alpha U_\beta - p g_{\alpha\beta}\}_g,$$

$$E_{\alpha\beta} = \widetilde{\breve{R}}_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\left(\chi^{\mu\nu}\widetilde{\breve{R}}_{\mu\nu}\right) \qquad (3.2.5)$$

$$\widetilde{\breve{R}}_{\alpha\beta} = -\frac{1}{K}\left\{\left(\frac{\partial f}{\partial\chi_{\alpha\beta}} - \frac{1}{2}g_{\alpha\beta}\chi^{\mu\nu}\frac{\partial f}{\partial\chi^{\mu\nu}}\right) +\right.$$

$$\left.\left(g_{\mu\alpha}g_{\nu\beta} - \frac{1}{2}g_{\alpha\beta}g_{\mu\nu}\right)D_\lambda p^{\mu\nu\|\lambda}\right\},$$

$$p^{\alpha\beta\|\lambda} = \frac{16\pi G}{c^4}\mathbf{P}^{\alpha\beta\|\lambda},$$

$$\mathfrak{I}_{\alpha\beta} = \breve{D}_\sigma Q^{\sigma}_{\alpha\beta} - 2\left(\frac{\partial L_f}{\partial\breve{h}^{\alpha\beta}} - \frac{1}{2}\breve{g}_{\alpha\beta}L_f\right)$$

$$Q^{\sigma}_{\alpha\beta} = 4g_{\lambda\zeta}\mathbf{P}^{\mu\zeta\|\nu}\breve{\Omega}^{\lambda\sigma}_{\mu\nu\|\alpha\beta}.$$

Putting the variation of the action integral against $\chi^{\mu\nu}$ to vanish, one obtains the field equations

$$\breve{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}\left(\chi^{\mu\nu}\breve{R}_{\mu\nu}\right) = -\frac{8\pi G}{c^4}\left[T_{\alpha\beta}\right]_g \qquad (3.2.6)$$

Rosen's gravitation field equations may be obtained out of the general bimetric theory so far presented by specifying the function f

$$f = -\frac{1}{8}\breve{h}^{\alpha\beta} W^{\lambda\sigma\rho\tau}(g_{\lambda\sigma|\alpha} \bullet g_{\rho\tau|\beta}),$$
$$(3.2.7)$$
$$W^{\lambda\sigma\rho\tau} = \chi^{\lambda\rho}\chi^{\sigma\tau} + \chi^{\sigma\rho}\chi^{\lambda\tau} - \chi^{\lambda\sigma}\chi^{\rho\tau}.$$

Hence

$$\breve{R}_{\mu\nu} = \frac{1}{2K}\left(\breve{h}^{\alpha\beta}\breve{D}_\alpha\breve{D}_\beta g_{\mu\nu} - \chi^{\lambda\sigma} g_{\mu\lambda|\alpha} g_{\nu\sigma}^{|\alpha}\right), \qquad (3.2.8)$$

and

$$\Im_{\alpha\beta} = \frac{c^4}{32\pi G}\left\{\chi^{\mu\lambda}\chi^{\nu\sigma}\left(g_{\mu\nu|\alpha} g_{\lambda\sigma|\beta} - \frac{1}{2}\breve{g}_{\alpha\beta} g_{\mu\nu|\eta} g_{\lambda\sigma}^{|\eta}\right) - \right.$$
$$(3.2.9)$$
$$2\left[(\ln K)_{,\alpha}(\ln K)_{,\beta} - \frac{1}{2}\breve{g}_{\alpha\beta}\breve{h}^{\mu\nu}(\ln K)_{,\mu}(\ln K)_{,\nu}\right] - \mathbf{P}_{\alpha\beta},$$

where

$$\mathbf{P}_{\alpha\beta} = (g_{\mu\beta|\alpha}\chi^{\mu\nu})_{|\nu} + (g_{\mu\alpha|\beta}\chi^{\mu\nu})_{|\nu} + \left(g_{\mu\beta}^{|\nu}\chi_{\alpha}^{\mu}\right)_{|\nu} +$$

$$\left(g_{\mu\alpha}^{|\nu}\chi_{\beta}^{\mu}\right)_{|\nu} - \left(g_{\mu|\alpha}^{\nu}\chi_{\beta}^{\mu}\right)_{|\nu} - \left(g_{\mu|\beta}^{\nu}\chi_{\alpha}^{\mu}\right)_{|\nu} - 2\breve{g}_{\alpha\beta}\breve{h}^{\alpha\beta}\breve{D}_{\alpha}\breve{D}_{\beta}\ln K, \qquad (3.2.10)$$

where the covariant derevative $(\ldots^{|\alpha})$ is a lift via $\breve{h}^{\alpha\beta}$.

IV.Bimetric Theory of Gravitational-Inertial Field in Riemannian approximation.

Bimetric Theory of Gravitational-Inertial Field in Riemannian approximation to proceed from assumptions:

- **I**. Bimetric geometrical structures $\Re_2(M, g_{ij}, \breve{g}_{ij})$ of the space-time continuum on the standard assumption of Bimetric Lorentzianian geometry:

$$ds_1^2 = g_{ik}dx^i dx^k, g_{ik} = g_{ki}, \det\|g_{ik}\| \neq 0;$$

$$ds_2^2 = \breve{\mathbf{g}}_{ik}dx^i dx^k, \breve{\mathbf{g}}_{ik} = \breve{\mathbf{g}}_{ki}, \det\|\breve{g}_{ik}\| \neq 0; \qquad (4.1)$$

- **II**. Equivalence of gravitational-inertial field and spacc-time metric tenzor

$g_{ik}(M) = g_{ik}(x_1, x_2, x_3, , x_4);$

● **III**. Equivalence of pure accelerational field $\breve{\mathbf{g}}_{ik}$ and space-time metric tenzor

$\breve{\mathbf{g}}_{ik}(M) = \breve{\mathbf{g}}_{ik}(x_1,x_2,x_3,,x_4)$.

Axiom *Bimetric Theory of Gravitational-Inertial Field in Riemannian approximation is based on the following postulates:*

1.In nonrelativistic approximation, i.e. if reference frame (body) is accelerate by a sufficiently small external nongravitational force and very far from the localized masses, then the metric tensors g_{ik} and $\breve{\mathbf{g}}_{ik}$ describes a flat space-time, i.e. $R^i_{klm}(g_{ik}) = 0$ and $\breve{\mathbf{R}}^i_{klm}(\breve{\mathbf{g}}_{ik}) = 0$.
2. A sufficiently small domain of bimetric space-time $\Re_2\left(M,g_{ij},\breve{\mathbf{g}}_{ij}\right)$ is flat, i.e. $R^i_{klm} \approx 0$ and $\breve{\mathbf{R}}^i_{klm} \approx 0$.
3. If reference frame (body) is accelerate by an arbitrary external nongravitational force but very far avay from the localized masses, then $g_{ik\|l} \simeq 0$.
4. SEP is satisfied. In particular: there is no experiment observers can perform to distinguish whether an acceleration arises because of a gravitational force or because their reference frame is accelerating by an external nonravitational force.

IV.1.Simple Variational Action Principle in Bimetric Theory of Gravitational-Inertial Field.

Let's consider Gravitational-Inertial field theory (GIFTR) such that at each point of Lorentzian space-time $(\mathcal{L},g_{ij})$ a curved Lorentzian metric tensor $\breve{g}_{ij}$ in addition to the curved Lorentzian metric tensor g_{ij}. Thus at each point of generalized Rosen's space-time $\Re_2 = \Re_2(M,g_{ij},\breve{g}_{ij})$ there are two curved metrics:

$$\begin{aligned} ds_1^2 &= g_{ij}dx^i dx^j, \\ & \qquad\qquad\qquad (4.1.1) \\ ds_2^2 &= \breve{g}_{ij}dx^i dx^j. \end{aligned}$$

Notation *The first metric tensor g_{ij} in GIFTR, refers to the curved space-time and describes Gravitational-Inertial Field. The second metric tensor $\breve{g}_{ij} \triangleq g^{\mathbf{ac.}}_{ij}$ in*

GIFTR, space-time and describes pure inertial forces. The Christoffel symbols formed from g_{ij} and $\breve{g}_{ij}$ are denoted by Γ^i_{jk} and $\breve{\Gamma}^i_{jk}$ respectively.

The quantities $\tilde{\Delta}^i_{jk}$ are defined via formulae

$$\tilde{\Delta}^i_{jk} = \Gamma^i_{jk} - \breve{\Gamma}^i_{jk}. \qquad (4.1.2)$$

Remark *4.1.1. Let $R^{\alpha}_{\mu\nu\lambda}$ and $\breve{R}^{\alpha}_{\mu\nu\lambda}$ be the curvature tensors calculated from $g_{\mu\nu}$ and $\breve{g}_{\mu\nu}$ respectively. We set $R^{\alpha}_{\mu\nu\lambda} \neq 0, \breve{R}^{\alpha}_{\mu\nu\lambda} \neq 0$.*

Now there arise two kinds of covariant differentiation:

- (**1**) g-differentiation based on $g_{\mu\nu}$ (denoted by a semicolon $(;)$)

$$A_{\mu\nu;\lambda} = \left(A_{\mu\nu,\lambda} - \Gamma^{\alpha}_{\mu\ \lambda} A_{\alpha\nu} - \Gamma^{\alpha}_{\nu\ \lambda} A_{\alpha\mu}\right) \qquad (4.1.3)$$

- (**2**) $\breve{g}$-differentiation based on $\breve{g}_{\mu\nu}$ (denoted by a bislash $(\|)$)

$$A_{\mu\nu\|\lambda} = \left(A_{\mu\nu,\lambda} - \breve{\Gamma}^{\alpha}_{\mu\ \lambda} A_{\alpha\nu} - \breve{\Gamma}^{\alpha}_{\nu\ \lambda} A_{\alpha\mu}\right), \qquad (4.1.4)$$

where ordinary partial derivatives are denoted by comma $(,)$.
The straightforward calculations gives

$$\tilde{R}^{\alpha}_{\mu\nu\lambda} = -\tilde{\Delta}^{\alpha}_{\mu\nu\|\lambda} + \tilde{\Delta}^{\alpha}_{\mu\lambda\|\nu} - \tilde{\Delta}^{\alpha}_{\beta\nu}\tilde{\Delta}^{\beta}_{\mu\lambda} - \tilde{\Delta}^{\alpha}_{\beta\lambda}\tilde{\Delta}^{\beta}_{\mu\nu}. \qquad (4.1.5)$$

Hence

$$\tilde{R}_{\mu\nu} = -\tilde{\Delta}^{\alpha}_{\mu\nu\|\alpha} + \tilde{\Delta}^{\alpha}_{\alpha\mu\|\nu} - \tilde{\Delta}^{\alpha}_{\alpha\beta}\Delta^{\beta}_{\mu\nu} - \tilde{\Delta}^{\alpha}_{\beta\mu}\tilde{\Delta}^{\beta}_{\alpha\nu}. \qquad (4.1.6)$$

This is the curvature tensor $\tilde{R}_{\mu\nu}$ associated with the curvature effects of pure gravitation acting in the bimetric spacetime $\Re_2 = \Re_2(M, g_{ij}, \breve{g}_{ij})$.
The geodesic equation in bimetric spacetime $\Re_2$ takes the form:

$$\frac{d^2x^i}{ds_1} + \tilde{\Delta}^i_{jk}\frac{dx^j}{ds_1}\frac{dx^k}{ds_1} =$$

$$(4.1.7)$$

$$\frac{d^2x^i}{ds_1} + \Gamma^i_{jk}\frac{dx^j}{ds_1}\frac{dx^k}{ds_1} - \breve{\Gamma}^i_{jk}\frac{dx^j}{ds_1}\frac{dx^k}{ds_1} = 0.$$

Action integral takes the form

$$\mathbf{S} = [\mathbf{S}_1(K, \breve{R})]_{\breve{g}} + [\mathbf{S}_2(\breve{R})]_{\breve{g}} + \mathbf{S}_m =$$

$$(4.1.8)$$

$$\frac{1}{64\pi\varkappa}\int \mathcal{L}_1\left(g_{\mu\nu}, \breve{\mathbf{g}}_{\mu\nu}\right)\sqrt{-\breve{\mathbf{g}}}\, d^4x + \frac{1}{8\pi\varkappa}\int \mathcal{L}_2\left(\breve{\mathbf{g}}_{\mu\nu}\right)\sqrt{-\breve{\mathbf{g}}}\, d^4x + \mathbf{S}_m,$$

where

$$\mathcal{L}_1\left(g_{\mu\nu}, \breve{\mathbf{g}}_{\mu\nu}\right) = \breve{\mathbf{g}}^{\mu\nu} g^{\alpha\beta} g^{\gamma\delta}\left(g_{\alpha\gamma\|\mu} g_{\alpha\delta\|\nu} - \frac{1}{2} g_{\alpha\beta\|\mu} g_{\gamma\delta\|\nu}\right), \tag{4.1.9}$$

where the duble bar ("$\|$") denotes covariant derivative with respect to $\breve{\mathbf{g}}_{\mu\nu}$. The corresponding field equations may be written in the form:

$$\begin{gathered}
\Box_{\breve{\mathbf{g}}} g_{\mu\nu} - g^{\alpha\beta}\breve{\mathbf{g}}^{\gamma\delta} = -16\pi\varkappa\sqrt{g/\breve{\mathbf{g}}}\left([T_{\mu\nu}]_g - \frac{1}{2} g_{\mu\nu}[T]_g\right), \\
\breve{\Theta}_{\mu\nu} + \breve{E}_{\mu\nu} = k_1[\breve{T}^{\mu\nu}]_{\breve{g}}, \\
\breve{\Theta}_{\mu\nu} = \frac{\delta \mathbf{S}(R, \breve{R})}{\delta \breve{g}_{\mu\nu}}, \\
\breve{E}_{\mu\nu} = \breve{R}_{\mu\nu} - \frac{1}{2}\breve{R} g_{\mu\nu}, \\
[\breve{T}^{\mu\nu}_{;\nu}]_{\breve{g}} = \breve{F}^{\mu}.
\end{gathered} \tag{4.1.10}$$

or in the form

$$\breve{\mathbf{g}}^{\alpha\beta}g_{\mu\nu\|\alpha\beta} - g^{\alpha\beta}\breve{\mathbf{g}}^{\gamma\delta} = -16\pi\varkappa\sqrt{g/\breve{\mathbf{g}}}\left([T_{\mu\nu}]_g - \frac{1}{2}g_{\mu\nu}[T]_g\right),$$

$$\breve{\Theta}_{\mu\nu} + \breve{E}_{\mu\nu} = k_1[\breve{T}^{\mu\nu}]_{\breve{g}}. \tag{4.1.11}$$

Here, a subscripts $g, \breve{g}$ stands for specifying that the labelled quantity is defined by curved space-time metrics $ds_1^2 = g_{\alpha\beta}dx^\alpha dx^\alpha$ and $ds_2^2 = \breve{g}_{\mu\nu}dx^\mu dx^\nu$ respectively and $\breve{F}^\mu$ denote 4-vector of a pure nongravitational force and vertical duble bar || stands for covariant differentiation with respect to $\breve{g}_{\mu\nu}$.

IV.2.The Weak Field Limit of the Bimetric Theory of Gravitational-Inertial Field in Riemannian Approximation.

IV.2.1.

Let us recall the weak field limit procedure of the GTR.
Using Cartesian space coordinates, the metric determined from the radar method is,

$$g_{ab} = g^{ab} = \begin{Vmatrix} k^{-2} & 0 & 0 & 0 \\ 0 & -k^2 & 0 & 0 \\ 0 & 0 & -k^2 & 0 \\ 0 & 0 & 0 & -k^2 \end{Vmatrix} \qquad (4.2.1.1)$$

The partial derivatives of the metric are

$$g_{ab,c} = \begin{Vmatrix} -2k^{-3}k_{,c} & 0 & 0 & 0 \\ 0 & -2kk_{,c} & 0 & 0 \\ 0 & 0 & -2kk_{,c} & 0 \\ 0 & 0 & 0 & -2kk_{,c} \end{Vmatrix} \qquad (4.2.1.2)$$

The geodesic equation is $\dot{x}^a = -\Gamma^a_{bc}\dot{x}^b\dot{x}^c$ where the Christoffel symbols are $\Gamma_{abc} = (g_{ab,c} + g_{ac,b} - g_{cb,a})/2.$ For the Christoffel symbols to be non-zero, two indices must be the same, and, in a constant field, the other must not be zero. For non-relativistic velocities, terms in the order of velocity squared can be ignored, and we have $\dot{x}^0 \approx 1$. Then, 3-acceleration is given by, for $a \neq 0, \ddot{x}^a \approx -\Gamma^a_{00} = -g^{ab}\Gamma_{b00} = -k(-k^2)(-2kk_{,c}/2) = -k^{-1}k_{,c}$. Writing $k = 1 + \varphi$, where φ is small, we have that acceleration is minus the gradient of φ,i.e. $\ddot{x}^a \approx -\varphi_{,a}$. So, gravitational redshift can be identified with the scalar potential in Newtonian gravity. Hence the time component of the metric is $g_{00} = k^{-2} \approx 1 + 2\varphi(x,y,z)$.
This is called the *weak gravitational field limit.* $m\varphi$ has units of kinetic energy, $\frac{1}{2}mv^2$. To convert to conventional units, we must divide by c^2, i.e.

$$\begin{aligned} g_{00} &= k^{-2} \approx 1 + \frac{2\varphi(x,y,z)}{c^2}, \\ k &= 1 - \frac{\varphi(x,y,z)}{c^2}. \end{aligned} \qquad (4.2.1.3)$$

Note that Lagrangian of a relativistic particle in a weak gravitational field is:

$$L(t) = -mc^2\sqrt{1 - \frac{\mathbf{v}^2(t)}{c^2}} - m\varphi(x,y,z). \qquad (4.2.1.4)$$

In nonrelativistic limit $\mathbf{v}^2/c^2 \to 0$ from Eq.(4.2.1.4) one obtain

$$\mathbf{S} = \int L(t)dt = -mc\int\left(c - \frac{\mathbf{v}^2(t)}{2c} + \frac{\varphi(x,y,z)}{c}\right)dt. \qquad (4.2.1.5)$$

Take into account that $\mathbf{S} = -mc\int ds$ from Eq.(4.2.1.5) one obtain

$$ds = \left(c - \frac{\mathbf{v}^2(t)}{2c} + \frac{\varphi(x,y,z)}{c}\right)dt. \qquad (4.2.1.6)$$

Thus (take into account that $d\mathbf{r} = \mathbf{v}dt$) in nonrelativistic limit $\mathbf{v}^2/c^2 \to 0$ we obtain

$$\begin{aligned} ds^2 &= \left(c - \frac{\mathbf{v}^2(t)}{2c} + \frac{\varphi(x,y,z)}{c}\right)^2 dt^2 - dr^2 \approx \\ &\approx c^2\left(1 + \frac{2\varphi(x,y,z)}{c^2}\right)dt^2 - dr^2. \end{aligned} \qquad (4.2.1.7)$$

Finally Eq.(4.2.1.7) gives again the same *weak gravitational field limit:*

$$g_{00} \simeq -\left[1 + \frac{2\varphi(x,y,z)}{c^2}\right]. \qquad (4.2.1.8)$$

Einstein's field equation, written in terms of the Ricci tensor is

$$R^{ab} - \frac{1}{2}g^{ab}R = \kappa T^{ab}. \qquad (4.2.1.9)$$

In the case of a static body of uniform density, ρ, T^{ab} is

$$T^{ab} = \rho\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (4.2.1.10)$$

Contract the indices a and b, noting from the standard summation convention that $g_{ab}g^{ab} = \delta^a_a = 4$ one obtain

$$g_{ab}R^{ab} - \frac{1}{2}g_{ab}g^{ab}R = \kappa g_{ab}T^{ab},$$

$$R - 2R = \kappa\rho, \qquad (4.2.1.11)$$

$$R = -\kappa\rho.$$

Thus

$$R^{ab} = \kappa T^{ab} + \frac{1}{2}g^{ab}R = \kappa T^{ab} - \frac{1}{2}g^{ab}\kappa\rho,$$

$$R^{00} = \frac{1}{2}\kappa\rho. \tag{4.2.1.12}$$

The Ricci tensor is

$$R_{ab} = R^{c}_{acb} = \Gamma^{c}_{ab,c} - \Gamma^{c}_{ac,b} + \Gamma^{e}_{ab}\Gamma^{c}_{ec} - \Gamma^{e}_{ac}\Gamma^{c}_{eb}, \tag{4.2.1.13}$$

In the Newtonian approximation, the metric g, is slowly varying in space and constant in time. We may neglect terms of second order in derivatives of the metric, and set time derivatives to zero. Then from Eq.(4.2.1.12) one obtain

$$R_{00} \approx \Gamma^{c}_{00,c} \approx (g^{cd}\Gamma_{d00})_{c} = -\frac{1}{2}g^{cd}g_{00,dc} \tag{4.2.1.14}$$

In Cartesian, x,y,z -coordinates one obtain

$$R_{00} \approx \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)k = \Delta k,$$

$$\Delta k = \frac{1}{2}\kappa\rho(x,y,z). \tag{4.2.1.15}$$

Hence Poisson's equation for a Newtonian gravitational potential, $k,$ due to a mass distribution of density ρ is

$$\Delta k = \kappa\rho(x,y,z),$$
$$(4.2.1.16)$$
$$\kappa = 8\pi G.$$

IV.2.2.

Let us consider the motion of a charged particle with a charge $\pm e$ and masses m in any external electric field $\mathbf{E}^{Ext}(x,y,z,t)$. Note that Lagrangian of a relativistic particle in electric field is [65]:

$$L(t) = -mc^2\sqrt{1 - \frac{\mathbf{v}^2(t)}{c^2}} - e\varphi(x,y,z,t). \qquad (4.2.2.1)$$

In nonrelativistic approximation, i.e. $v/c \simeq 0$ from Eq.(4.2.2.1) one obtain

$$L(t) = -mc^2 + \frac{m\mathbf{v}^2(t)}{2} - e\varphi(x,y,z,t). \qquad (4.2.2.2)$$

The action for relativistic charged particle in electric field is:

$$\mathbf{S} = \int_{t_1}^{t_2} L(t)dt =$$

$$\int_{t_1}^{t_2}\left[-mc^2\sqrt{1-\frac{\mathbf{v}^2(t)}{c^2}} - e\bullet\varphi(x,y,z,t) \right]dt. \qquad (4.2.2.3)$$

In nonrelativistic approximation, i.e. $v/c \simeq 0$ from Eqs.(4.2.2.2)-(4.2.2.3) one obtain

$$\mathbf{S} = \int_{t_1}^{t_2} L(t)dt =$$

$$-mc\int_{t_1}^{t_2}\left(c - \frac{\mathbf{v}^2(t)}{2c} + \frac{e\bullet\varphi(x,y,z,t)}{m\bullet c} \right)dt. \qquad (4.2.2.4)$$

But from other side we have [65]:

$$\mathbf{S} = -mc\int ds. \qquad (4.2.2.5)$$

Thus from Eq.(4.2.2.4) and Eq.(4.2.2.4) one obtain

$$ds = \left[c - \frac{\mathbf{v}^2(t)}{2c} + \left(\tfrac{e}{mc}\right)\bullet\varphi(x,y,z,t) \right]dt. \qquad (4.2.2.6)$$

Thus (take into account that $d\mathbf{r} = \mathbf{v}dt$) in nonrelativistic limit $\mathbf{v}^2/c^2 \to 0$ we obtain

$$ds^2 = \left(c - \frac{\mathbf{v}^2(t)}{2c} + \left(\frac{e}{mc}\right) \cdot \varphi(x,y,z,t)\right)^2 dt^2 =$$

$$c^2dt^2 + \frac{\mathbf{v}^4(t)}{4c^2}dt^2 + \left(\frac{e}{mc}\right)^2 \cdot \varphi^2(x,y,z,t)dt^2 - \mathbf{v}^2(t)dt^2 +$$

$$\left(\frac{2e}{m}\right) \cdot \varphi(x,y,z,t)dt^2 - \frac{\mathbf{v}^2(t)}{c}\left(\frac{e}{mc}\right) \cdot \varphi(x,y,z,t)dt^2 = \qquad (4.2.2.7.a)$$

$$c^2\left(1 + \frac{2e \cdot \varphi(x,y,z,t)}{m \cdot c^2} + \left(\frac{e}{mc^2}\right)^2 \cdot \varphi^2(x,y,z,t)dt^2 + \frac{\mathbf{v}^4(t)}{4c^4}\right)dt^2 - d\mathbf{r}^2 \simeq$$

$$c^2\left(1 + \frac{2e \cdot \varphi(x,y,z,t)}{m \cdot c^2} + \left(\frac{e}{mc^2}\right)^2 \cdot \varphi^2(x,y,z,t)dt^2\right)dt^2 - d\mathbf{r}^2$$

and

$$ds^2 \simeq c^2\left(1 + \frac{2e \cdot \varphi(x,y,z,t)}{m \cdot c^2}\right)dt^2 - d\mathbf{r}^2. \qquad (4.2.2.7.b)$$

Let us consider now a system of charges located the electric field generated by a charge distribution is

$$\varphi = \sum \frac{e_a}{|\mathbf{R}_0 - \mathbf{r}_a|} \qquad (4.2.2.8)$$

$$\varphi \approx \frac{\sum e_a}{|\mathbf{R}_0|} - \left(\sum e_a \mathbf{r}_a\right) \cdot \mathbf{grad}\left(\frac{1}{|\mathbf{R}_0|}\right) = \qquad (4.2.2.9)$$
$$\frac{\sum e_a}{|\mathbf{R}_0|} - \mathbf{d} \cdot \mathbf{grad}\left(\frac{1}{|\mathbf{R}_0|}\right),$$

where sum $\sum e_a \mathbf{r}_a \triangleq \mathbf{d}$ is colled dipole moment [65]. In the complete expansion of the φ in powers $|\mathbf{R}_0|^{-1}$

$$\varphi = \varphi_0 + \varphi_1 + \ldots + \varphi_n + \ldots, \qquad (4.2.2.10)$$
$$\varphi_n \sim |\mathbf{R}_0|^{-(n+1)}.$$

We saw that:

$$\varphi_0 = |\mathbf{R}_0|^{-1} \sum e_a, \qquad (4.2.2.11)$$
$$\varphi_1 = -\mathbf{d} \cdot \mathbf{grad}\left(\frac{1}{|\mathbf{R}_0|}\right).$$

If the total charge is thero, term φ_0 is vanishes, and

$$\varphi = -\mathbf{d}\cdot\mathbf{grad}\left(\frac{1}{|\mathbf{R}_0|}\right) + O(|\mathbf{R}_0|^{-3}). \qquad (4.2.2.12)$$

The second term φ_1 is colled dipole potential of the system.

Let us consider now a system of charges located in an external electric field $\mathbf{E}^{Ext}(x,y,z,t)$. We denote the potential of this external electric field by $\varphi^{Ext}(\mathbf{r},t) = \varphi^{Ext}(x,y,z,t)$. Total potential energy of the system is [65]:

$$U = \int j_0\varphi^{Ext}(x,y,z,t)dV =$$

$$\sum_{i=1}^{n}\int e_i\cdot\delta(\mathbf{r}-\mathbf{r}_i)\varphi^{Ext}(\mathbf{r},t)dV = \qquad (4.2.2.13)$$

$$\sum_{i=1}^{n} e_i\cdot\varphi^{Ext}(\mathbf{r}_i,t).$$

We introduce another coordinate system with its origin anywhere within the system of charges: $\mathbf{r}_i$ is the radius vector of the charge e_i in these coordinate.We assume that the external field $\mathbf{E}^{Ext}(\mathbf{r}_i,t)$ changes slowly: (1) over region of the system of charges and (2) over region of the time $t\in[0,\infty]$, i.e $\mathbf{E}^{Ext}(\mathbf{r}_i,t)\simeq\widetilde{\mathbf{E}}^{Ext}(\mathbf{r}_i)$. Then one can expand the energy U is powers of $\mathbf{r}_i$

$$U = U_0 + U_1 + \ldots \qquad (4.2.2.14)$$

In this expansion the first term is

$$U_0 = \varphi_0^{Ext} \cdot \sum_{i=1}^{n} e_i, \qquad (4.2.2.15)$$

where φ_0^{Ext} is the walue of the potential at the origin. In this approximation, the energy of the system is the same as it would be if all the charges were located at one point: the origin. The second term in the expansion is

$$U_1 = (\mathbf{grad}\varphi^{Ext})_0 \cdot \sum_{i=1}^{n} e_i \cdot \mathbf{r}_i. \qquad (4.2.2.16)$$

Hence

$$\begin{aligned} U_1 &= -\mathbf{d} \cdot \mathbf{E}_0^{Ext}, \\ & \qquad\qquad\qquad (4.2.2.17) \\ \mathbf{E}_0 &= (\mathbf{grad}\varphi^{Ext})_0. \end{aligned}$$

The total force acting on the system in the external quasiuniform field is (to the order $O(|\mathbf{R}_0|^{-3})$ we are considering above)

$$\mathbf{F} = \mathbf{E}_0 \cdot \sum_{i=1}^{n} e_i + [\nabla(\mathbf{d} \cdot \mathbf{E})]_0. \qquad (4.2.2.18)$$

If the total charge is thero, the first term in Eq.(4.2.2.18) is vanishes, and we obtain

$$\mathbf{F} = (\nabla \cdot \mathbf{d})\mathbf{E}_0, \qquad (4.2.2.19)$$

where the derivatives of the field intensity taken at the origin $\mathbf{R}_0$.
Eq.(4.2.2.7) gives *weak inertional field limit:*

$$\breve{g}_{00} \simeq -1 - \frac{2e\varphi(x,y,z)}{mc^2}. \qquad (4.2.2.20)$$

Let us consider Bimetric theory of gravitational-inertial field in pure inertial field approximation (see section IV.5.2). Field equations we take in the form:

$$\breve{R}_i^k \simeq \breve{\kappa}\left(\breve{T}_i^k - \frac{1}{2}\delta_i^k \breve{T}\right). \qquad (4.2.2.21)$$

Now consider any discrete distribution of charged matter, with a four-current charge density

$$s_i = \left(\frac{\rho \mathbf{u}}{c}, i\rho\right),$$

$$\rho = \frac{\rho^0}{\sqrt{1 - \frac{u^2}{c^2}}}, \qquad (4.2.2.22)$$

$$\rho^0 = \sum_{j=1}^{n} \delta(\mathbf{r} - \mathbf{r}_j) \cdot e_j$$

in a given external electromagnetic field F_{ik}^{Ext}, where ρ^0 is the charge density in the rest system $\mathbf{S}$.Consider a definite point in space at a definite time; the charged matter at this point is moving with a certain velocity.Now, let $\mathbf{S}^0$ be the momentary rest system of the matter at this point. The components of s_i in this system are then

$$s_i^0 = (0,0,0,i\rho^0) \qquad (4.2.2.23)$$

The action of the electromagnetic field F_{ik} with charged particles is $S = \int_\Omega LdV$ with

$$L = -\frac{1}{4}F_{ik}^{Ext}F_{ik}^{Ext} + A_i^{Ext}s_i - \mu^0c^2 =$$

$$-\frac{1}{4}\left(\frac{\partial A_k^{Ext}}{\partial x_i} - \frac{\partial A_i^{Ext}}{\partial x_k}\right)\left(\frac{\partial A_k^{Ext}}{\partial x_i} - \frac{\partial A_i^{Ext}}{\partial x_k}\right) + A_i^{Ext}s_i - \mu^0c^2, \qquad (4.2.2.24)$$

$$\mu^0 = \sum\nolimits_{j=1}^{n} \delta(\mathbf{r} - \mathbf{r}_j) \cdot \mu_j.$$

Note that Lagrangian L does not contain any derivatives of the metric $\breve{g}_{ik}$. Hence the energy-momentum tensor of the electromagnetic field F_{ik} with charged particles is

$$\breve{T}_{ik} = -2\frac{\partial L}{\partial \breve{g}_{ik}} + \breve{g}_{ik}L. \qquad (4.2.2.25)$$

In the weak (pure inertional) field approximation, the metric $\breve{g}$, is slowly varying in space and constant in time. We may neglect terms of second order in derivatives of the metric, and set time derivatives to zero.

Substitution Eq.(4.2.2.20) into Eq.(4.2.2.) gives

$$\breve{R}_{00} = -\breve{R}_0^0 = \frac{\partial \breve{\Gamma}_{00}^{\alpha}}{\partial x^{\alpha}},$$
$$\breve{\Gamma}_{00}^{\alpha} \simeq -\frac{1}{2} \breve{g}^{\alpha\alpha} \frac{\partial \breve{g}_{00}}{\partial x^{\alpha}} = \frac{e}{mc^2} \frac{\partial \varphi}{\partial x^{\alpha}}. \qquad (4.2.2.26)$$

Hence

$$\breve{R}_0^0 \simeq -\frac{e}{mc^2} \frac{\partial^2 \varphi}{\partial x^{\alpha 2}} = -\frac{e}{mc^2} \Delta \varphi. \qquad (4.2.2.27)$$

Suppose that:

- (1) $\sum_{i=1}^{n} e_i \neq 0,$
- (2) $(\mathbf{grad}\varphi^{Ext})_0 \ll 1,$ i.e. $U_1 \simeq 0,$
- (3) $u^{\alpha} \simeq 0, \alpha = 1,2,3; u^0 = -u_0 = 1.$

$$\breve{T}_i^k \simeq 0, i \neq k,$$
$$\breve{T}_0^0 \simeq -\frac{\rho^0}{\sqrt{-\breve{g}}} \varphi_0^{Ext} = -\frac{1}{\sqrt{-\breve{g}}} \varphi_0^{Ext} \cdot \sum_{j=1}^{n} \delta(\mathbf{r} - \mathbf{r}_j) \cdot e_j. \qquad (4.2.2.28)$$

From the field equations (4.2.2.21) for the case $i = k = 0$ one obtain

$$\breve{R}_0^0 \simeq -\breve{\kappa}\rho^0\varphi_0^{Ext} = -\breve{\kappa}\varphi_0^{Ext}\sum_{j=1}^{n}\delta(\mathbf{r}-\mathbf{r}_j)\cdot e_j. \qquad (4.2.2.29)$$

Substitution Eq.(4.2.2.29) into Eq.(4.2.2.27) gives

$$\frac{e}{mc^2}\Delta\varphi = \breve{\kappa}\rho^0(x,y,z)\varphi_0^{Ext}. \qquad (4.2.2.30)$$

Poisson's equation for a Newtonian gravitational-inertional potential, φ, due to a distribution of charge density ρ^0 is

$$\Delta\varphi = \left(\frac{e}{mc^2}\right)^{-1}\breve{\kappa}\rho^0(x,y,z)\varphi_0^{Ext}. \qquad (4.2.2.31)$$

Thus,

$$\breve{\kappa} = \frac{e}{\varphi_0^{Ext}mc^2}, \qquad (4.2.2.32)$$

and field equation is

$$\breve{R}_i^k \simeq \breve{\kappa}\left(\breve{T}_i^k - \frac{1}{2}\delta_i^k \breve{T}\right). \qquad (4.2.2.21)$$

IV.2.3. Weak field limit of the geodesic equation for the motion of a free test particle.

In the linear approximation, pure inertial field tensor $\breve{g}$ can be written as

$$\breve{g}_{\mu\nu} = \eta_{\mu\nu} + \breve{h}_{\mu\nu}, \qquad (4.2.3.1)$$

where $\eta_{\mu\nu}$ is the Minkowski metric tensor with signature $+2$ and $\breve{h}_{\mu\nu}$ is a first-order perturbation.Under transformation of the background coordinates $x^\mu = (ct, \vec{\mathbf{x}})$, $x^\mu \to x^\mu - \varepsilon^\mu$, the pure inertial field potentials $\breve{h}_{\mu\nu}$ transform as

$$\breve{h}_{\mu\nu} \to \breve{h}_{\mu\nu} + \varepsilon_{\mu,\nu} + \varepsilon_{\nu,\mu}. \qquad (4.2.3.2)$$

Henceforth, the nertial field potentials are considered to be gauge dependent, while the background global inertial coordinate system is in effect fixed. The accelerated spacetime curvature $\breve{R}(\breve{g})$ is, however, gauge invariant. It is useful to introduce the trace-reversed pure inertial potentials

$$\widetilde{h}_{\mu\nu} \triangleq \breve{h}_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu},$$
$$h = tr(h_{\mu\nu}). \qquad (4.2.3.3)$$

By imposing the transverse gauge condition $\widetilde{h}^{\mu\nu}_{,\nu} = 0,$ the gravitational-inertial field equations take the form

$$\Box \widetilde{h}_{\mu\nu} = -\breve{k} T^{\mathbf{EM}}_{\mu\nu}. \qquad (4.2.3.4)$$

Where $T^{\mathbf{EM}}_{\mu\nu}$ is the corresponding electromagnetic stress-energy tensor
The general solution of (4.2.3.4) is given by the special retarded solution

$$\widetilde{h}_{\mu\nu} = \widetilde{h}^{\mathbf{a.c.}}_{\mu\nu},$$
$$\widetilde{h}^{\mathbf{a.c.}}_{\mu\nu} = \breve{k} \int \frac{T^{\mathbf{EM}}_{\mu\nu}(ct - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}', \qquad (4.2.3.5)$$

plus a general solution of the homogeneous wave equation that we comlete ignore in this consideration.In the linear approximation, all terms of $O(c^{-4})$ are neglected in the metric tensor. Thus from Eqs.(4.2.3.5) for the sources under consideration here one obtain

$$\widetilde{h}^{\mathbf{a.c.}}_{00} = \frac{4\Phi^{\mathbf{a.c.}}(t,\mathbf{x})}{c^2},$$

$$\widetilde{h}^{\mathbf{a.c.}}_{0i} = -\frac{2A^{\mathbf{a.c.}}_i(t,\mathbf{x})}{c^2}, \qquad (4.2.3.6)$$

$$\widetilde{h}_{ij} = O(c^{-4}); i,j \neq 0.$$

Then the spacetime metric in the linear approximation is

$$ds^2 = -c^2\left(1 - 2\frac{\Phi^{\mathbf{a.c.}}(t,\mathbf{x})}{c^2}\right) - \frac{4}{c}(\mathbf{A}^{\mathbf{a.c.}}(t,\mathbf{x}) \cdot d\mathbf{x})dt + \qquad (4.2.3.7)$$
$$+\left(1 + 2\frac{\Phi^{\mathbf{a.c.}}(t,\mathbf{x})}{c^2}\right)\delta_{ij}dx^i dx^j,$$

The geodesic equation for the motion of a free test particle is

$$\frac{dU^\mu}{d\tau} = \breve{\Gamma}^{\mu}_{\rho\sigma}U^\rho U^\sigma, \qquad (4.2.3.8)$$

where τ/c is the proper time and $U^\mu = dx^\mu/d\tau$ is the unit four-velocity vector of the test particle. The Christoffel symbols are given by

$$c^2\breve{\Gamma}^0_{0\mu} = \Phi^{\mathbf{a.c.}}_{,\mu}; c^2\breve{\Gamma}^0_{ij} = 2A^{\mathbf{a.c.}}_{(i,j)} + \delta_{ij}\Phi^{\mathbf{a.c.}}_{,0},$$

$$c^2\breve{\Gamma}^i_{00} = -\Phi^{\mathbf{a.c.}}_{,i} - 2A^{\mathbf{a.c}}_{i,0}; c^2\breve{\Gamma}^i_{0j} = \delta_{ij}\Phi^{\mathbf{a.c.}}_{,0} + \epsilon_{ijk}B^k \qquad (4.2.3.9)$$

$$c^2\breve{\Gamma}^i_{jk} = \delta_{ij}\Phi^{\mathbf{a.c.}}_{,k} + \delta_{ik}\Phi^{\mathbf{a.c.}}_{,j} - \delta_{jk}\Phi^{\mathbf{a.c.}}_{,i}$$

The geodesic equation can be reduced via $U^\mu = \gamma(1, \boldsymbol{\beta})$ with $\boldsymbol{\beta} = \mathbf{V}/c$ to

$$\frac{c}{\gamma}\frac{d\gamma}{d\tau} = (1-\beta^2)\Phi^{\mathbf{a.c.}}_{,0} + 2\beta^i\left[\Phi^{\mathbf{a.c.}}_{,i} - A^{\mathbf{a.c.}}_{(i,j)}\beta^j\right],$$

$$\frac{dV^i}{dt} = (1+\beta^2)\Phi^{\mathbf{a.c.}}_{,i} - 2(\boldsymbol{\beta}\times\mathbf{B})_i + A^{\mathbf{a.c}}_{i,0} - \beta^i(3-\beta^2)\Phi^{\mathbf{a.c.}}_{,0} + \qquad (4.2.3.10)$$

$$+2\beta^i\beta^j\left[A^{\mathbf{a.c.}}_{(j,k)}\beta^k - 2\Phi^{\mathbf{a.c.}}_{,j}\right].$$

Moreover, $U^\mu_\mu U = -1$ implies that

$$\frac{1}{\gamma^2} = 1 - \beta^2 - \frac{2}{c^2}(1+\beta^2)\Phi + \frac{4}{c^2}\boldsymbol{\beta}\cdot\mathbf{A}^{a.c.} \qquad (4.2.3.11)$$

For a stationary source ($\partial_t\Phi \simeq 0$ and $\partial_t\mathbf{A}^{a.c.} \simeq 0$), equations of motion (4.2.3.10) reduces to

$$m\frac{d\mathbf{V}}{dt} = m\breve{k}\mathbf{E} - 2m\breve{k}\frac{\mathbf{V}}{c}\times\mathbf{B},$$

$$(4.2.3.12)$$

$$\breve{k} = \frac{e}{m}.$$

where velocity-dependent terms of order higher that $\beta = V/c$ are neglected.

Thus

$$m\frac{d\mathbf{V}}{dt} = e\mathbf{E} - 2e\frac{\mathbf{V}}{c}\times\mathbf{B}. \qquad (4.2.3.13)$$

In the case of a general nonstationary source, however, the equations of motion (4.2.3.10) does not reduces to the Lorentz force law.

IV.3.General Variational Action Principle in Bimetric Theory of Gravitational-Inertial Field.

Let's consider bimetric space-time $\Re_2(g_{\mu\nu}, \breve{g}_{\mu\nu})$ with $R(g_{\mu\nu}) \neq 0,\ \breve{R}(\breve{g}_{\mu\nu}) \neq 0.$
Action integral takes the form

$$\mathbf{S} = \frac{1}{c}\int \mathcal{L}\sqrt{-\breve{g}}\, d^4x,$$

$$(4.3.1)$$

$$\mathcal{L}(g_{\mu\nu}, \breve{g}_{\mu\nu}) = \mathcal{L}_m(g_{\mu\nu}, \breve{g}_{\mu\nu}) + \mathcal{L}_{f_1}(g_{\mu\nu}, \breve{g}_{\mu\nu}) + \mathcal{L}_{f_2}(\breve{g}_{\mu\nu}),$$

where

$$\mathcal{L}_m = \left[(c^2 + \Xi)\rho - p\right]_g \bullet K, K = \sqrt{\frac{-g}{\breve{g}}},$$

$$\mathcal{L}_{f_1} = \frac{c^4}{16\pi G} f_1\left(\chi^{\alpha\beta}, g_{\alpha\beta\|\lambda}; \breve{h}^{\alpha\beta}\right),$$

$$\mathcal{L}_{f_2} = \frac{c^4}{16\pi G} f_2(\breve{g}_{\alpha\beta}, \breve{g}_{\alpha\beta;\lambda}),$$

$$g_{\mu\lambda}\chi^{\lambda\nu} = \delta^\nu_\mu;\ \breve{g}_{\mu\lambda}\breve{h}^{\lambda\nu} = \delta^\nu_\mu,$$

$$g^{\alpha\beta} = g_{\mu\lambda}\breve{h}^{\mu\alpha}\breve{h}^{\nu\beta} \neq \breve{\chi}^{\alpha\beta}, \tag{4.3.2}$$

$$\chi_{\alpha\beta} = \chi^{\mu\nu}\breve{g}_{\mu\alpha}\breve{g}_{\nu\beta} \neq g_{\alpha\beta}.$$

$$\Xi = \int_0^{p(\rho)} \frac{dp}{p(\rho)},$$

$$p = p(\rho), \rho = \rho(p),$$

$$g = \det\|g_{\mu\lambda}\| < 0, \breve{g} = \det\|\breve{g}_{\mu\lambda}\| < 0.$$

Here, a subscripts g and $\breve{g}$ stands for specifying that the labelled quantity is defined by curved space-time metric $ds_1^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ and $ds_2^2 = \breve{g}_{\alpha\beta}dx^\alpha dx^\beta$ accordingly.
The $\breve{g}$-covariant derivatives are denoted by a bislash ($\|$) followed by a certain (Greek) subscript, or (equivalently) by a derivative symbol ($\breve{D}_\alpha$) followed by the same subscript. For example

$$g_{\alpha\beta\|\lambda} = \breve{D}_{\lambda} g_{\alpha\beta} = g_{\alpha\beta,\lambda} - \breve{G}^{\sigma}_{\alpha\lambda} g_{\alpha\beta} - \breve{G}^{\sigma}_{\beta\lambda} g_{\sigma\alpha},$$

$$\breve{G}^{\lambda}_{\mu\nu} = \frac{1}{2} \breve{h}^{\lambda\sigma} (\breve{g}_{\mu\sigma,\eta} + \breve{g}_{\nu\sigma,\rho} - \breve{g}_{\mu\nu,\sigma}). \tag{4.3.3}$$

For obtaining the field equations and the energy canonical tensor one obtain to the following identity

$$\frac{1}{\sqrt{-g}} \delta\left(\sqrt{-g} L\right) = D_{\lambda} q^{\lambda} -$$

$$\frac{1}{2} K\left(T_{\alpha\beta} + \frac{c^4}{8\pi G} E_{\alpha\beta}\right) \delta\chi^{\alpha\beta} - \frac{1}{2} \Im_{\alpha\beta} \delta h^{\alpha\beta}, \tag{4.3.4}$$

where

$$q^{\lambda} = \mathbf{P}^{\alpha\beta\|\lambda}\delta g_{\alpha\beta} + 2g_{\sigma\nu}\mathbf{P}^{\mu\nu\|\eta}\Omega_{\nu\mu\|\alpha\beta}\delta\breve{h}^{\alpha\beta},$$

$$\mathbf{P}^{\alpha\beta\|\lambda} = \frac{\partial L_{f_1}}{\partial g_{\alpha\beta\|\lambda}},$$

$$\breve{\Omega}^{\lambda\sigma}_{\nu\mu\|\alpha\beta} = \frac{1}{4}(\delta^{\lambda}_{\beta}\delta^{\sigma}_{\nu}\breve{g}_{\mu\alpha} + \delta^{\alpha}_{\lambda}\delta^{\sigma}_{\nu}\breve{g}_{\mu\beta} + \delta^{\lambda}_{\beta}\delta^{\sigma}_{\mu}\breve{g}_{\nu\alpha} +$$

$$+\delta^{\lambda}_{\alpha}\delta^{\sigma}_{\mu}\breve{g}_{\nu\beta} - \breve{h}^{\lambda\sigma}\breve{g}_{\mu\alpha}\breve{g}_{\nu\beta} - \breve{h}^{\lambda\sigma}\breve{g}_{\mu\beta}\breve{g}_{\nu\alpha}),$$

$$[T_{\alpha\beta}]_g = \{(c^2 + \Xi)\rho U_{\alpha}U_{\beta} - p\gamma_{\alpha\beta}\}_g,$$

$$\widetilde{\breve{E}}_{\alpha\beta} = \widetilde{\breve{R}}_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\left(\chi^{\mu\nu}\widetilde{\breve{R}}_{\mu\nu}\right), \qquad (4.3.5)$$

$$\widetilde{\breve{R}}_{\alpha\beta} = -\frac{1}{K}\left\{\left(\frac{\partial f_1}{\partial\chi_{\alpha\beta}} - \frac{1}{2}g_{\alpha\beta}\chi^{\mu\nu}\frac{\partial f_1}{\partial\chi^{\mu\nu}}\right) +\right.$$

$$\left.\left(g_{\mu\alpha}g_{\nu\beta} - \frac{1}{2}g_{\alpha\beta}g_{\mu\nu}\right)\breve{D}_{\lambda}p^{\mu\nu\|\lambda}\right\},$$

$$p^{\alpha\beta\|\lambda} = \frac{16\pi G}{c^4}\mathbf{P}^{\alpha\beta\|\lambda},$$

$$\Im_{\alpha\beta} = D_{\sigma}Q^{\sigma}_{\alpha\beta} - 2\left(\frac{\partial L_{f_1}}{\partial\breve{h}^{\alpha\beta}} - \frac{1}{2}\breve{g}_{\alpha\beta}L_{f_1}\right),$$

$$\breve{Q}^{\sigma}_{\alpha\beta} = 4g_{\lambda\zeta}\mathbf{P}^{\mu\zeta\|\nu}\breve{\Omega}^{\lambda\sigma}_{\mu\nu\|\alpha\beta}.$$

V.4.Linear Post-Newtonian Approximation to Bimetric Theory of Gravitational-Inertial Field. Gravito-Inertial Ectromagnetism (GIEM).

V.4.1.Linear Post-Newtonian Approximation to Bimetric Theory of Gravitational-Inertial Field. Gravito-Inertial Ectromagnetism in a purely inertial approximation.

In Maxwell's electromagnetism, the combined dynamics of charged particles and electromagnetic field are consistently described by Maxwell's field equations and the Lorentz force law. Well-known that general relativity does indeed contain induction effects. These effects turn out to be, despite the differences, on the whole closely analogous to electromagnetic induction effects.
Let's consider the curved bimetric accelerated spacetime $\mathcal{L}(V_4, g, \breve{g})$ ($K_{ij}(g) \simeq 0$ $\|K_{ij}(g)\| \ll \|\breve{R}_{ij}(\breve{g})\|$) generated by an external pure nongravitational "nonrelativistic" Lorentz force [53] and sufficiently small gravitational force [54]. Suppose that the gravitational-Inertial field $g_{ij}(t,x)$ is governed by either:
(1) massive gravitational source with mass density $\rho(t,x)$,
(2) ectromagnetic field $\{\mathbf{A}(t,x), \Phi(t,x)\}$ and
(3) charged massive particles with mass density $\mu(t,x)$ and charge density $\rho_{ch}(t,x)$.
Here $\Phi(t,x)$ is the electric potential and $\mathbf{A}(t,x)$ is the magnetic vector potential. In the linear approximation, pure inertial field tensor $\breve{g}$ can be written as

$$\breve{g}_{\mu\nu} = \eta_{\mu\nu} + \breve{h}_{\mu\nu}, \qquad (4.4.1.1)$$

where $\eta_{\mu\nu}$ is the Minkowski metric tensor with signature $+2$ and $\breve{h}_{\mu\nu}$ is a first-order perturbation.Under transformation of the background coordinates $x^\mu = (ct, \vec{\mathbf{x}})$, $x^\mu \to x^\mu - \varepsilon^\mu$,the pure inertial field potentials $\breve{h}_{\mu\nu}$ transform as

$$\breve{h}_{\mu\nu} \to \breve{h}_{\mu\nu} + \varepsilon_{\mu,\nu} + \varepsilon_{\nu,\mu}. \qquad (4.4.1.2)$$

Henceforth, the potentials are considered to be gauge dependent, while the background global inertial coordinate system is in effect fixed. The accelerated spacetime curvature $\breve{R}(\breve{g})$ is, however, gauge invariant. It is useful to introduce the trace-reversed pure inertial potentials

$$\widetilde{h}_{\mu\nu} \triangleq \breve{h}_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu},$$
$$h = tr(h_{\mu\nu}). \qquad (4.4.1.3)$$

By imposing the transverse gauge condition $\widetilde{h}^{\mu\nu}_{,\nu} = 0,$ the gravitational-inertial field equations take the form

$$\Box\widetilde{h}_{\mu\nu} = -\breve{k}T^{\mathbf{a.c.}}_{\mu\nu} - \frac{4G}{c^4}T_{\mu\nu}. \qquad (4.4.1.4)$$

Where $T^{\mathbf{a.c.}}_{\mu\nu} \triangleq T^{\mathbf{EM}}_{\mu\nu}$ is the corresponding electromagnetic stress-energy tensor. The general solution of (4.4.1.4) is given by the special retarded solution

$$\widetilde{h}_{\mu\nu} = \widetilde{h}^{\mathbf{a.c.}}_{\mu\nu} + \widetilde{h}^{\mathbf{Gr.}}_{\mu\nu} \simeq \widetilde{h}^{\mathbf{a.c.}}_{\mu\nu},$$

$$\widetilde{h}^{\mathbf{a.c.}}_{\mu\nu} = \breve{k}\int \frac{T^{\mathbf{EM}}_{\mu\nu}(ct - |\mathbf{x}-\mathbf{x}'|,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}',$$

$$\widetilde{h}^{\mathbf{Gr.}}_{\mu\nu} = \frac{4G}{c^4}\int \frac{T^{\mathbf{Gr.}}_{\mu\nu}(ct - |\mathbf{x}-\mathbf{x}'|,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}', \qquad (4.4.1.5)$$

$$\frac{4G}{c^4} \ll \breve{k},$$

plus a general solution of the homogeneous wave equation that we ignore in this consideration.In the linear GIEM approximation, all terms of $O(c^{-4})$ are neglected in the metric tensor. Thus from Eqs.(4.4.1.4) for the sources under consideration here one obtain

$$\widetilde{h}_{00}^{\mathbf{a.c.}} = \frac{4\Phi^{\mathbf{a.c.}}(t,\mathbf{x})}{c^2}; \widetilde{h}_{00}^{\mathbf{Gr.}} = \frac{4\Phi^{\mathbf{Gr.}}(t,\mathbf{x})}{c^2},$$

$$\widetilde{h}_{0i}^{\mathbf{a.c.}} = -\frac{2\mathbf{A}^{\mathbf{a.c.}}(t,\mathbf{x})}{c^2}; \widetilde{h}_{0i}^{\mathbf{Gr.}} = -\frac{2\mathbf{A}^{\mathbf{Gr.}}(t,\mathbf{x})}{c^2}, \qquad (4.4.1.6)$$

$$\widetilde{h}_{ij} = O(c^{-4}); i,j \neq 0.$$

Where:

- $\Phi^{\mathbf{a.c.}}(t,\mathbf{x})$ is the inertiaelectric potential,
- $\mathbf{A}^{\mathbf{a.c.}}(t,\mathbf{x})$ is the inertiamagnetic vector potential,
- $\Phi^{\mathbf{Gr.}}(t,\mathbf{x})$ is the gravitoelectric potential,
- $\mathbf{A}^{\mathbf{Gr.}}(t,\mathbf{x})$ is the gravitomagnetic vector potential.

Where far from the gravitational source potentials $\Phi^{\mathbf{Gr.}}(t,\mathbf{x})$ and $\mathbf{A}^{\mathbf{Gr.}}(t,\mathbf{x})$ can be expressed as [54]:

$$\Phi^{\mathbf{Gr.}}(t,\mathbf{x}) = \frac{GM}{r},$$

$$(4.4.1.7)$$

$$\mathbf{A}^{\mathbf{Gr.}}(t,\mathbf{x}) = \frac{G}{c}\frac{\mathbf{J}\times\mathbf{x}}{r^3},$$

Here M and $\mathbf{J}$ are the inertial mass and angular momentum of the source, $r = |\mathbf{x}|$, $r \gg GM/c^2$ and $r \gg J/(Mc)$.

The spacetime metric in the linear GIEM approximation is

$$ds^2 = -c^2\left(1 - 2\frac{\Phi^{\mathbf{Gr.I}}(t,\mathbf{x})}{c^2}\right) - \frac{4}{c}(\mathbf{A}^{\mathbf{Gr.I}}(t,\mathbf{x}) \bullet d\mathbf{x})dt +$$

$$+\left(1 + 2\frac{\Phi^{\mathbf{Gr.I}}(t,\mathbf{x})}{c^2}\right)\delta_{ij}dx^i dx^j, \qquad (4.4.1.8)$$

$$\Phi^{\mathbf{Gr.I}}(t,\mathbf{x}) = \Phi^{\mathbf{a.c.}}(t,\mathbf{x}) + \Phi^{\mathbf{Gr.}}(t,\mathbf{x}),$$

$$\mathbf{A}^{\mathbf{Gr.I}}(t,\mathbf{x}) = \mathbf{A}^{\mathbf{a.c.}}(t,\mathbf{x}) + \mathbf{A}^{\mathbf{Gr.}}(t,\mathbf{x}).$$

Let us note that the gauge condition implies that

$$\frac{1}{c}\partial_t\Phi^{\mathbf{Gr.}} + \nabla \bullet \left(\frac{1}{2}\mathbf{A}^{\mathbf{Gr.}}\right) = 0,$$

$$\frac{1}{c}\partial_t\Phi^{\mathbf{a.c.}} + \nabla \bullet \left(\frac{1}{2}\mathbf{A}^{\mathbf{a.c.}}\right) = 0, \qquad (4.4.1.9)$$

$$\frac{1}{c}\partial_t\Phi^{\mathbf{Gr.I}} + \nabla \bullet \left(\frac{1}{2}\mathbf{A}^{\mathbf{Gr.I}}\right) = 0.$$

This is related to the conservation of mass-energy of the gravitational-inertional sources via Eq.(4.4.1.4).That is,let

$$T^{(\mathbf{Gr.})00} = (\rho + \mu)c^2$$

$$(4.4.1.10)$$

$$T^{(\mathbf{Gr.})0i} = cj^{(\mathbf{Gr.})i},$$

where $j^{(\mathbf{Gr.})\nu} = (c\rho, \mathbf{j})$ is the mass-energy current of the gravitational source.
Hence equations (4.4.1.9) is equivalent to

$$j^{(\mathbf{Gr.I})\mu}_{;\mu} = 0,$$

$$j^{(\mathbf{Gr.I})\mu} = j^{(\mathbf{Gr.})\mu} + j^{(\mathbf{EM})\mu}. \qquad (4.4.1.11)$$

Thus one can to define:

- the gravitoelectric field $\mathbf{E}^{\mathbf{Gr.}}$,
- the gravitomagnetic field $\mathbf{B}^{\mathbf{Gr.}}$
- the inertialelectric field $\mathbf{E}^{\mathbf{a.c.}}$,
- the inertialmagnetic field $\mathbf{B}^{\mathbf{a.c.}}$
- the gravitoinertialelectric field $\mathbf{E}^{\mathbf{Gr.I}}$
- the gravitoinertialmagnetic field $\mathbf{B}^{\mathbf{Gr.I}}$

in complete analogy with electrodynamics

$$\mathbf{E}^{\mathbf{Gr.}} = -\nabla\Phi^{\mathbf{Gr.}} + \frac{1}{c}\partial_r\left(\frac{1}{2}\mathbf{A}^{\mathbf{Gr.}}\right),$$

$$\mathbf{B}^{\mathbf{Gr.}} = \nabla \times \mathbf{A}^{\mathbf{Gr.}},$$

$$\mathbf{E}^{\mathbf{a.c.}} = -\nabla\Phi^{\mathbf{a.c.}} + \frac{1}{c}\partial_r\left(\frac{1}{2}\mathbf{A}^{\mathbf{a.c.}}\right), \qquad (4.4.1.11)$$

$$\mathbf{B}^{\mathbf{a.c.}} = \nabla \times \mathbf{A}^{\mathbf{a.c.}}.$$

$$\mathbf{E}^{\mathbf{Gr.I}} = \mathbf{E}^{\mathbf{Gr.}} + \mathbf{E}^{\mathbf{a.c.}}.$$

From Eq.(4.4.1.11) one obtain

$$\begin{aligned}\nabla \times \mathbf{E}^{\mathbf{GI}} &= -\frac{1}{c}\partial_r\left(\frac{1}{2}\mathbf{B}^{\mathbf{GI}}\right),\\ \nabla \bullet \left(\frac{1}{2}\mathbf{B}^{\mathbf{GI}}\right) &= 0.\end{aligned} \tag{4.4.12}$$

Eq.(4.4.12) and the gravitational-inertial field equations (4.4.1.4) imply

$$\begin{aligned}\nabla \bullet \mathbf{E}^{\mathbf{Gr.I}} &= 4\pi G(\rho + \mu) + 4\pi\breve{k}\rho_{ch.},\\ \nabla \times \left(\frac{1}{2}\mathbf{B}^{\mathbf{Gr.I}}\right) &= \frac{1}{c}\partial_t\mathbf{E}^{\mathbf{Gr.I}} + \frac{4\pi G}{c} \bullet \mathbf{j}^{\mathbf{Gr.}} + \frac{4\pi\breve{k}}{c} \bullet \mathbf{j}^{\mathbf{EM}}.\end{aligned} \tag{4.4.13}$$

IV.5. Bimetric theory of gravitational-inertial field in a purely inertial field approximation.

IV.5.1.Bimetric theory of gravitational-inertial field in a purely inertial field approximation. Rosen type approximation.

IV.5.2.Bimetric theory of gravitational-inertial field in a

purely inertial field approximation. Einstein type approximation.

The Gravitational-Inertial field equations in Einstein approximation is

$$\breve{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \breve{R} \simeq \breve{\kappa} \breve{T}_{\mu\nu}, \qquad (4.5.2.1)$$

where $\breve{R}_{\mu\nu}$ is the Ricci tensor and where $\breve{T}_{\mu\nu}$ is the energy-momentum tensor which in our farther consideration is the one for electromagnetism

$$\breve{T}_{\mu\nu} = F_{\mu\rho} F_{\nu}^{\rho} - \frac{1}{4} \breve{g}_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}, \qquad (4.5.2.2)$$

where $F_{\mu\nu}$ is the electromagnetic field strength tensor.Note that $\breve{T}_{\mu\nu}$ has zero trace, $\breve{T} = \breve{g}^{\mu\nu} F_{\mu\nu} = 0$. Eq. (4.5.2.2) allows us to rewrite the Eq. (4.5.2.1) in the following form

$$\breve{R}_{\mu\nu} \simeq \breve{\kappa} \breve{T}_{\mu\nu}. \qquad (4.5.2.3)$$

Finally, the Maxwell's equations are

$$\breve{g}^{\mu\nu}\nabla_\mu F_{\nu\sigma} = 0,$$

$$(4.5.2.4)$$

$$\nabla_{[\mu}F_{\nu\rho]} = 0.$$

IV.5.3. Gravitational-inertial black hole in a purely inertial field approximation. Einstein type approximation.

In General Relativity one of famous static solutions to the Einstein's field equations is the Reissner-Nordstrom metric describing the geometry of the spacetime surrounding a non-rotating charged spherical black hole. In this section we obtain completely *purely inertial analog* of the Reissner-Nordstrom black hole in Einstein approximation.

Canonical form for the metric in 4D spherical coordinates (t,r,θ,ϕ) is

$$d\breve{s}^2 = -e^{2\alpha(r,t)}dt^2 + e^{2\beta(r,t)}dr^2 + r^2 d\Omega^2,$$

$$(4.5.3.1)$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi.$$

Since there is spherical symmetry, the only non-zero components of the magnetic and electric fields are the radial components which should be independent of θ and ϕ. Therefore the radial component of the electric field has a form of

$$E_r = F_{tr} = -F_{rt} = f_1(r,t) \qquad (4.5.3.2)$$

The radial component of the magnetic field has a form of

$$B_r = \frac{2\breve{g}_{rr}}{\sqrt{|g|}} F_{\theta\phi}, \qquad (4.5.3.3)$$

$$F_{\theta\phi} = -F_{\phi\theta} = f_2(r,t) r^2 \sin^2\theta.$$

All the remaining components of the electromagnetic field strength tensor are either zero or related to these two through symmetries. Therefore for the electromagnetic field strength tensor one obtain

$$F_{\mu\nu} = \begin{bmatrix} 0 & f_1(r,t) & 0 & 0 \\ -f_1(r,t) & 0 & 0 & 0 \\ 0 & 0 & 0 & f_2(r,t) r^2 \sin^2\theta \\ 0 & -f_2(r,t) r^2 \sin^2\theta & 0 & 0 \end{bmatrix} \qquad (4.5.3.4)$$

For the $\theta\theta$-component of the the Riemann tensor $\breve{R}_{\mu\nu}$ and of the electromagnetic stress tensor $\breve{T}_{\mu\nu}$ one obtain

$$R_{\theta\theta} = e^{-2\beta(r,t)}[r(\partial_r\beta(r,t) - \partial_r\alpha(r,t)) - 1] + 1$$

$$\breve{T}_{\theta\theta} = \frac{1}{2}r^2 f_2(r,t) + f_1(r,t)e^{-2(\alpha(r,t)+\beta(r,t))}, \qquad (4.5.3.5)$$

$$\alpha(r,t) = \alpha(r) = -\beta(r).$$

Now lets solve the Maxwell equations for the form of the electromagnetic field strength tensor given in Eq.(4.5.3.4).Finally, for the electromagnetic field strength tensor one obtain

$$F_{\mu\nu} = \frac{1}{\sqrt{4\pi}}\begin{bmatrix} 0 & Qr^{-2} & 0 & 0 \\ -Qr^{-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & P\sin\theta \\ 0 & 0 & -P\sin\theta & 0 \end{bmatrix} \qquad (4.5.3.6)$$

Lets consider the $\theta\theta$ component of the Eq.(4.5.2.3).

$$\breve{R}_{\theta\theta} \simeq 8\pi\breve{\kappa}\breve{T}_{\theta\theta}. \qquad (4.5.3.7)$$

Substituting Eq.(4.5.3.5) into Eq.(4.5.3.7) we obtain

$$\partial_r(re^{2\alpha}) = 1 - \frac{\breve{\kappa}}{r^2}(Q^2 + P^2). \qquad (4.5.3.6)$$

By integration we obtain

$$e^{2\alpha(r)} = 1 + \frac{const}{r} + \frac{\breve{\kappa}}{r^2}(Q^2 + P^2). \qquad (4.5.3.7)$$

Take into account Eq.(4.2.2.7.a) we obtain

$$e^{2\alpha(r)} = 1 + \frac{2\sqrt{\breve{\kappa}}\,Q}{r} + \frac{\breve{\kappa}}{r^2}(Q^2 + P^2),$$

$$(4.5.3.7)$$

$$\breve{\kappa} = \left(\frac{e}{mc^2}\right)^2,$$

or

$$e^{2\alpha(r)} = 1 + \frac{2\mu Q}{r} + \frac{\mu^2}{r^2}(Q^2 + P^2),$$

$$(4.5.3.7)$$

$$\mu = \frac{e}{mc^2}.$$

Finally, upon substitution of Eq.(4.5.3.7) into Eq.(4.5.3.1) the metric of the purely inertial Reissner-Nordstrom black hole is readily found:

$$d\breve{s}^2 = -\Delta dt^2 + \Delta^{-1}dr^2 + r^2 d\Omega^2,$$

$$\Delta = 1 + \frac{2\mu Q}{r} + \frac{\mu^2}{r^2}(Q^2 + P^2), \qquad (4.5.3.8)$$

$$\mu = \frac{e}{mc^2}.$$

Note that in the absence of charges, this should reduce to the flat metric and hence purely inertial analog of the Schwarzschild black hole is absent.

V.Noninertial Pure Accelerated Curved Reference Frame in Bimetric theory of gravitational-inertial field.

Recall the basic concept and definitions of the accelerated reference frame in canonical GTR [43],[44],[45].Let us considered flat (curved) basic Lorentzian space-time $\mathcal{L}_4 = \mathcal{L}(V_4, g)$ and any timelike congruence C, in a certain domain $\mathrm{A}_4 \subseteqq V_4$, defined in a coordinate system $\{x^\alpha\}$ by

$$\mathrm{C}\ : x^\alpha = x^\alpha(t, \lambda^i), \qquad (5.1)$$
$$\alpha, \beta = 0,1,2,3; i = 1,2,3$$

where $\{\lambda^i\}$ are three parametr marking the specific curve in C an t is any parametr along these curves in C.
Some canonical intristict element defined by the congruence C are:
(**i**) The quotient space $V_3(\mathrm{C})$ associated to C or the internal space of C given by the equivalence relation: $V_3(\mathrm{C}) \triangleq V_3/\mathrm{C}$.
(**ii**) The natural projection $j : \mathrm{A}_4 \to V_3/\mathrm{C}$

$$x^\alpha \to \xi^i = \lambda^i(x^\alpha) \qquad (5.2)$$

where $\lambda^i(x^\alpha)$ are the inverted functions of (2.1.1) and $\{\xi^i\}$ are a coordinate sistem of $V_3(\mathrm{C})$. The pull-back and push-forward of j allow us to define certain objects on A_4 and $V_3(\mathrm{C})$ respectively. For instance, three one-form fields $\{d\xi^i\}$ of the naturalco- basis in $V_3(\mathrm{C})$ can be pulleed back to the tree one-form fields $\{\omega^i\}$ defined by

$$\omega^i = \frac{\partial\lambda^i(x^\alpha)}{\partial x^\alpha}dx^\alpha. \qquad (5.3)$$

Thus 3-dimensional subspase Δ_3 of $T^*_{V_4}$ spanned by $\{\omega^i\}$, i.e.$\Delta_3 \triangleq \mathbf{span}\{\omega^1,\omega^2,\omega^3\}$ is invariantly characterized by C. Concerning the push-forward projection, defined for example as follows

$$\begin{aligned} j' : T_x &\to T_{j(x)}, \\ & \qquad\qquad\qquad\qquad (5.4) \\ t^\alpha(x) \to t^i[\lambda(x)] &= t^\alpha(x)\omega^i_\alpha \end{aligned}$$

(**iii**) The proper time τ and 4-velocity vector $\vec{u}$ defined respectively by formulae

$$\tau = \int\sqrt{-g_{\alpha\beta}\frac{\partial x^\alpha}{\partial t}\frac{\partial x^\beta}{\partial t}} + C(\lambda^\alpha), \qquad (5.5)$$

and

$$\vec{\mathbf{u}} = \left(\frac{1}{\sqrt{-g_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial t}\frac{\partial x^{\beta}}{\partial t}}} \frac{\partial x^{\sigma}}{\partial t} \right) \frac{\partial}{\partial x^{\sigma}}. \qquad (5.6)$$

Where $C(\lambda^{\alpha})$ being an arbitrary function of arguments $\{\lambda^{\alpha}\}$ [43].Notice that $\vec{\mathbf{u}}$ is orthogonal to the three $\omega^{i}, i = 1,2,3$ which means that the functions $\lambda^{i}(x^{\alpha})$ are three independent first integrals of $\vec{\mathbf{u}}$.Of course, given any timelike unit vector $\vec{\mathbf{u}}$ one can build its associated congruences locally by constructing the curves tangent to $\vec{\mathbf{u}}$ by means of the canonical integration. Therefore one can identifies a timelike congruence and its tangent for-velocity vector field.The canonical kinematical quantities associated with $\vec{\mathbf{u}}$ are intistinc to congruence C. For example, the projection tensor orthogonal to $\vec{\mathbf{u}}$: $P_{\alpha\beta} = g_{\alpha\beta} + u_{\alpha}u_{\beta}$, the acceleration $\vec{\mathbf{a}}$, the deformation tensor $\Sigma_{\alpha\beta}$ and the rotation tensor $\omega_{\alpha\beta}$.

Let us considered flat basic Lorentzian space-time $\mathcal{L}_4 = \mathcal{L}(V_4, \eta_{\alpha\beta})$ where $\eta_{\alpha\beta} = diag(1,-1,-1,-1)$.Any timelike congruence C, in a certain domain $\mathrm{A}_4 \subseteqq V_4$, formed by Eq.(5.1) by using a regular parametrized timelike curves $\Gamma(s, \lambda^{i})$. It is also convenient to restrict ourselves to timelike curves $x^{\alpha} = x^{\alpha}(s)$ i.e. those for which $\eta_{\alpha\beta}\frac{dx^{\alpha}}{ds}\frac{dx^{\beta}}{ds} = 1$,where now s denotes the arc length parameter in the sense of Minkowski metric $\eta_{\alpha\beta}$.Accordingly,if we denote the tetrad vectors by $u^{\alpha}_{(A)}$ $(A = 0,1,2,3)$, then the orthonormality conditions read $u^{\alpha}_{(A)}u_{(A)\beta} = \eta_{\alpha\beta}u^{\alpha}_{(A)}u^{\beta}_{(B)} = \eta_{AB}$.

If we chose $u^{\alpha}_{(0)} = \frac{dx^{\alpha}}{ds}$ then we can easily construct an orthonormal basis of vectors $\left\{u^{\alpha}_{(A)}\right\}$,defined along the curve, which obey the following four-dimensional Serret-Frenet equations,given in matrix representation by

$$\begin{bmatrix} \dfrac{du_0(s)}{ds} \\ \dfrac{du_1(s)}{ds} \\ \dfrac{du_2(s)}{ds} \\ \dfrac{du_3(s)}{ds} \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 & 0 \\ \kappa(s) & 0 & \tau_1(s) & 0 \\ 0 & -\tau_1(s) & 0 & \tau_2(s) \\ 0 & 0 & -\tau_2(s) & 0 \end{bmatrix} \begin{bmatrix} u_0(s) \\ u_1(s) \\ u_2(s) \\ u_3(s) \end{bmatrix} \quad (5.7)$$

Theorem.5.1. [46] Given differentiable functions $\kappa(s) > 0, \tau_1(s)$ and $\tau_2(s)$, there exists a regular parametrized timelike curve Γ such that $\kappa(s)$ is the curvature, $\tau_1(s)$ and $\tau_2(s)$ are, respectively, the first and second torsion of Γ. Any other curve $\widetilde{\Gamma}$ satisfying the same conditions, differs from Γ by a Poincaré transformation, i.e. by a transformation of the type $x'_\mu = \Lambda^\nu_\mu x_\nu + a_\mu$, where μ represents a proper Lorentz matrix and a_μ is a constant four-vector.

As we have seen above, the canonical introduction of reference (comovin) frame formulation the crucial role in their mathematical discription has to by played in a study of congruences by the world lines of particles forming bodies of reference, i.e. the physical time lines for the chosen reference frame.The congruence concept is essential because for the sake of regularity of the mathematical description of the frame,these lines have not to mutually intersect, and they must cover completely the space-time region under consideration so that at every world point one has to find one and only one line passing throus it.Exactly the same approach is used for description of a regular continuous media,i.e. in relativistic gydrodynamics of a perfect (magnetic) fluid [44].

Remark *5.1. However it is important to note that in canonical GTR in contrast with GIFT, curvature of the basic Lorentzian space-time $\mathcal{L}(V_4, g)$ does not depend from accelerations of the particles forming fluid or bodies of reference.*

V.1. Non Inertial Anholonomic Accelerated Frame of Refferences in Finsler-Lagrange Approximation. Hollands type comovin frame.

Holland was studied a unified formalism which uses a anholonomic frame (nonintegrable 1-form) on space-time, *a sort of plastic deformation*, arising from consideration of a charged particle moving in an external electromagnetic field in the background space-time viewed as a strained medium [10]-[11]. In fact, Ingarden [12] was first to point out that the Lorentz force law, in this case, can be written as a geodesic equation on a Finsler space called Randers space [13] i.e., the physical space with a metric:

$$ds = \sqrt{g_{ij}dx^i dx^j} + a_k(x)dx^k,$$

$$\det\|g_{ij}\| \neq 0. \tag{5.1.1}$$

The metric given by Eq.(5.1.1) is defined by the pair (g_{ij}, a_k) of the tensor field g_{ij} and vector field a_k, where g_{ij} influences the local inhomogeneity of the space, a_k changes the local anisotropy.

Remark *5.1.1.Note that the additional term in the geodesic equation acts as repulsive force against the gravity [21].*

For complete references on these Finsler spaces see [14]-[17].This results in geometrical entities which depend on the electromagnetic field (vector potential), particle (velocity) and background space-time parameters. The Finsler structure implies the existence of a global anholonomic (Holland) frame which in turn yields a connection with torsion and vanishing Finsler curvatures.His differential geometric method is based on fundamental work of S. Amari on a Finsler approach to crystal dislocation theory [19]. Amari and Holland's idea conduct one to considered Holland

type frames as non inertial anholonomic accelerated frame of refferences in Finsler-Lagrange approximation.

V.2. Non Inertial Anholonomic Accelerated Frame of Refferences in Riemannian Approximation.Bravais Type Relativistic Comovin Frame with Curvature and Torsion.

Let M be a differentiable manifold of dimension n. At a point $p \in M$, let $\{e^i\}, i = 1,\ldots,n$ constitute the basis of the cotangent space $T_p^*(M)$ and let $\{e_i\}$ be the base vectors of the tangent space $T_p(M)$. The local coordinate form of the bases of $T_x^*(M)$ and $T_x(M)$ at $p = x$ are $\{dx^i\}$ and $\left\{\partial_i = \frac{\partial}{\partial x^i}\right\}$ respectively. Let ω_k^i be the connection 1-form of M. Then the description of M is given by Cartan's structure equations:

$$\begin{aligned} T^i &= De^i = de^i + \omega_k^i \wedge e^k = \tfrac{1}{2} T_{kl}^i e^k \wedge e^l, \\ R_k^i &= D\omega_k^i = d\omega_k^i + \omega_l^i \wedge \omega_k^l = \tfrac{1}{2} R_{klm}^i e^l \wedge e^m. \end{aligned} \qquad (5.2.1)$$

The integrability conditions of the above equations are given by

$$DT^i = R^i_k \wedge e^k,$$
$$DR^i_k = 0. \qquad (5.2.2)$$

These are known as Bianchi identities. The Cartan equations and Bianchi identities are present in both Yang-Mills and gravity-type gauge theories. However, the latter type has the following additional structural features. A symmetric metric tensor $g = g_{ik} e^i \otimes e^k,\ g_{ik} = g_{ki} = e_i \cdot e_k$ is introduced on M. In local coordinates, the metric is used to describe the distance element: $ds^2 = g_{ik} dx^i dx^k$. The inverse metric g^{kl} is such that $g^{kl} g_{li} = \delta^k_i$. The metric and the connection are so far two independent fields, defined at each point of M. A manifold in which the covariant derivative of the metric tensor vanishes is singled out by the property that the angle between two vectors and their lengths remain unchanged by the operation of parallel displacement of vectors on M. It is this property which guarantees locally Euclidean structure of the manifold. A connection is called metric compatible if

$$Dg_{ik} = dg_{ik} - g_{il}\omega^l_k - g_{kl}\omega^l_i = 0. \qquad (5.2.3)$$

In general, the connection ω^i_k can have a torsion-free part $\tilde{\omega}^i_k$ and an additional part τ^i_k which represents the non-Riemannian part, called the contorsion 1-form. The local coordinate representations of these geometrical objects are:

$$\omega^k_l = \Gamma^k_{ml} dx^k, \tilde{\omega}^k_l = \left\{ {k \atop ml} \right\} dx^m,$$
$$\tau^k_l = S^k_{ml} dx^m, T^k = \tfrac{1}{2} T^k_{ml} dx^m \wedge dx^l. \qquad (5.2.4)$$

Here $\left\{ {k \atop ml} \right\} = g^{ks}(\partial_m g_{sl} - \partial_s g_{lm} + \partial_l g_{ms})$, and $S^k_{ml} = g^{ks}(S_{msl} - S_{slm} + S_{lms})$ are respectively the Christoffel symbol of the second kind and the contorsion tensor. We next relate the above structure to that of a non inertial accelerated frame (comoving to accelerated body).

Definition *5.2.1.Non inertial accelerated frame (comoving to accelerated body) of refferences is identified with a four dimensional differentiable manifold M embedded in the real four-dimensional linear space* $\mathbb{R}^4$*.The current coordinates of the manifold of the accelerated frame (accelerated deformed body)* M' *are* x^i *(* $i,j,k,l,m,n,\ldots = 1,2,3,4$*) and the cartesian coordinates of the anholonomy-free configuration (reference manifold M) are* x^a $(a,b,c,d,\ldots = 1,2,3,4)$*.*

Definition *5.2.2.The current configuration of the accelerated frame (comoving to accelerated deformed body)* M' *is anholonomy-free iff functions* $x^i = x^i(x^a)$ *and* $x^a = x^a(x^i)$ *are well behaved, single-valued and differentiable functions of their respective arguments. The matrix* $\beta_a^i = \partial_a x^i = \dfrac{\partial x^i}{\partial x^a}$ *is the holonomy deformation (distortion) matrix.Its inverse matrix is* $\beta_i^a = \partial_i x^a = \dfrac{\partial x^a}{\partial x^i}$*.*

Anholonomy-free manifold M (defect-free body) is characterized by a global coordinate basis of the reference manifold (reference body). The metric $e_a \bullet e_b = \delta_{ab}$ is Euclidean and the connection $\omega_b^a = \Gamma_{cb}^a dx^c$ vanishes identically.The metric and the connection of the current configuration are

$$g_{ik} = \beta_i^a \beta_k^b \delta_{ab},$$
$$(5.2.5)$$
$$\omega_k^i = \beta_a^i d\beta_k^a.$$

Holonomic (defect-free body) manifold M is characterized by a global coordinate basis $e^i = dx^i$, and a (metric compatible) flat connection $\omega_k^i = \beta_a^i d\beta_k^a$.These equations may be regarded as a set of differential equations for e^i and ω_k^i. In this case the torsion and curvature tensors are zero and the integrability equations are (5.2.1) with the right sides set equal to zero. Torsion and curvature in general case represent anholonomic deformation of the accelerated frame (or defects). Anholonomic deformation (defects) are obstructions to diffeomorphisms from M to M'.

In order to relate the mathematical structure to the description of the accelerated frame (accelerated deformed body), consider the infinitesimal transformation

$$x^a \to x^m = (x^a + u^a(x^b))\delta_a^m \qquad (5.2.6)$$

where the total displacement u^a consists of an holonomy (elastic) part and an anholonomy (plastic) part. The elastic part is integrable and the plastic part is not. The total deformation (distortion) tensors are

$$\beta_a^i = \delta_a^i + \partial_a u_i, \qquad (5.2.7)$$
$$\beta_i^a = \delta_i^a - \partial_i u_a.$$

The metric gets related to the total "strain" tensor

$$g_{ik} = \beta_{ai}\beta_k^a =$$
$$\delta_{ik} - \partial_i u_k - \partial_k u_i = \qquad (5.2.8)$$
$$\delta_{ik} - 2e_{ik}.$$

1. The present internal geometry of the accelerated frame (comoving to accelerated deformed body) is Riemannian if :

$$Dg = 0,$$

$$R^i_k = D\omega^i_k \neq 0, \qquad (5.2.9)$$

$$DR^i_k = 0.$$

2. The present internal geometry of the accelerated frame (comoving to accelerated deformed body) is non Riemannian (tele-parallel) if :

$$Dg = 0,$$

$$R^i_k = 0,$$

$$DR^i_k = 0, \qquad (5.2.10)$$

$$DT^i = 0,$$

$$T^i = De^i \neq 0.$$

3. In general case the present internal geometry of the accelerated frame (comoving to accelerated deformed body) is characterized by Eq.(5.2.1), the Bianchi identities Eq.(5.2.3) is non Riemannian. It is also called Riemann-Cartan geometry.

Definition *5.2.3.Cartan's structure equations are nothing but the very definition of "dislocation" and "disclination" density tensors of manifold:*

$$\alpha^{ij} = \varepsilon^{ikm} T^{j}_{km},$$

$$(5.2.11)$$

$$\theta^{ij} = \frac{1}{2}\varepsilon^{imn}\varepsilon^{jkl} R_{klmn}.$$

In general case (in particular the presence of "defects"), the coordinate system of M' is nonholonomic. Denoting the anholonomic coordinates by e^{a} instead of e^{i}, we may write $e^{a} = \beta^{\alpha}_{a} dx^{a}$ but β^{α}_{a} is no longer a gradient field.

VI.Gauge theories of accelerated comovin frame.

VI.1.General mathematical structure of gauge theories.

Gauge theories are divided into two different classes:

- **1.**Yang-Mills type gauge theories and
- **2.**Gravity type gauge theories.
- **3.**Mixed type gauge theories.

Let us first consider their general mathematical structure. We shall refrain from a terse mathematical presentation of the principal fibre-bundle structure since this can be done away with.Let $u^{i}(x), i = 1,2,\ldots,n$ be a system of initial fields, called matter fields. Here x is a space-time point on a base Lorentzian manifold $M.$ To each point x of M is attached a fibre space V whose elements are values of u^{i}. This may be regarded as an *internal space.* The functions $u_{i}(x)$ are cross sections on the fibre-bundle $M \times V.$ Further we assume that a space-time group $\mathbf{P}_{0}$ and an internal group $\mathbf{G}_{0}$ act on M and V respectively.

- The group $\mathbf{P}_{0}$ could be, for example, the Poincaré group or one of its sub-groups (translation, rotation,etc.).
- The group $\mathbf{G}_{0}$ could be another Lie group such as a rotation or a unitary group.
- The group actions are $\mathbf{P}_{0} : M \to M$ and $\mathbf{G}_{0} : V \to V.$ Thus both groups are

continuous transformation groups. Both these groups are said to act globally, i.e., their actions do not depend on x.

Let a matter field model be given by a Lagrangian $\mathcal{L}_0 = \mathcal{L}_0(\partial u, u)$ which is invariant with respect to $\mathbf{P}_0$ and $\mathbf{G}_0$. This global symmetry is a necessary prerequisite of any gauge theory. The basic idea of gauging is to extend the global invariance group G_0 or P_0 to a local gauge group $\mathbf{G}$ (or $\mathbf{P}$) by allowing the transformations $\mathbf{G} \times V \to V$ and $\mathbf{P} \times M \to M$ to be x dependent.The gauge theory based on $\mathbf{G}_0 \to \mathbf{G}$ is of Yang-Mills type and that based on $\mathbf{P}_0 \to \mathbf{P}$ is of gravity type. A mixed type could be based on gauging of both $\mathbf{G}_0 \to \mathbf{G}$ and $\mathbf{P}_0 \to \mathbf{P}$. In order to ensure local invariance, the Lagrangian must contain, in addition to fields u^i, a set of connection fields or gauge potentials $A_\mu(x)$. these are a set of compensating fields coupled (minimally) to the matter fields u^i.The values of A_μ belong to the Lie algebra $\mathcal{L}(\mathbf{G}_0)$ of $\mathbf{G}_0$ (or $\mathbf{P}_0$).
These fields are called connections on the corresponding principal fibre bundles.
To obtain a closed system of equations for ui and A_μ, the gauge approach prescribes two recipes. Firstly, the derivatives ∂_μ are to be replaced by covariant derivatives $D_\mu = \partial_\mu + A_\mu(x)$.
Secondly, the new Lagrangian $\widetilde{\mathcal{L}}$ is supposed be given by $\widetilde{\mathcal{L}} = \mathcal{L}_0(Du, u) + \mathcal{L}_1(F)$ (minimal coupling) where $F_{\lambda\mu} = D_\lambda A_\mu$ is the Yang-Mills field (curvature field associated with the connection field). The piece $\mathcal{L}_1$ is usually chosen as $\mathbf{Tr}(FF^\dagger)$.

VI.2. Charged Particles as Defects In Bimetric Lorentzian Manifold

Kleinert [34],[48] demonstrates that a space with torsion and curvature can be generated from a Minkowski space via singular coordinate transformations $x^i = x^i(x^\mu)$ and is completely equivalent to a crystal which has undergone plastic deformation being filled with dislocations and disclinations. Typycally canonical transformation associated with dislocation can be described multivalued function [34]:

$$x^{\bar{1}} = x^1,$$

$$x^{\bar{2}} = x^2 - \frac{b}{2\pi}\tan^{-1}\left(\frac{x^2}{x^1}\right), \qquad (6.2.1)$$

where the function $\tan^{-1}\left(\frac{x^2}{x^1}\right)$ defined to be equal $\pm\pi$ for $x^1 < 0, x^2 = \pm\epsilon$.

$$dx^{\bar{1}} = dx^1,$$

$$\left(dx_\epsilon^{\bar{2}}\right)_\epsilon = dx^2 - \tag{6.2.2}$$

$$\frac{b}{2\pi}(x^2dx^1 - x^1dx^2)\left(\frac{1}{(x^1)^2 + (x^2)^2 + \epsilon}\right)_\epsilon$$

with the components of the Colombeau vielbein $(e^i_{\mu,\epsilon})_\epsilon = \left(\frac{\partial x^i_\epsilon}{\partial x_\mu}\right)_\epsilon$

$$(e^i_{\mu,\epsilon})_\epsilon =$$

$$\begin{bmatrix} 1 & 0 \\ \frac{b}{2\pi}\left(\frac{x^2}{(x^1)^2 + (x^2)^2 + \epsilon}\right)_\epsilon & 1 - \frac{b}{2\pi}\left(\frac{x^1}{(x^1)^2 + (x^2)^2 + \epsilon}\right) \end{bmatrix} \tag{6.2.3}$$

$$S_{12}{}^{\bar{2}} = \partial_1 e^{\bar{2}}{}_2 - \partial_2 e^{\bar{2}}{}_1 = \tag{6.2.4}$$

The associated Cartan curvature tensor $R_{\mu\nu\lambda}{}^{\kappa}$:

$$R_{\mu\nu\lambda}^{\ \ \ \ \kappa} = e_i^{\ \kappa}(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)e_\lambda^i \qquad (6.2.5)$$

vanishes, making the connection affine-flat. The antisymmetric part of the connection

$$S_{\alpha\beta}^{\ \ \ \gamma} = \frac{1}{2}\left(\Gamma_{\alpha\beta}^{\ \ \ \gamma} - \Gamma_{\beta\alpha}^{\ \ \ \gamma}\right), \qquad (6.2.6)$$

which is a tensor, is nonzero giving rise to a nonvanishing Riemann curvature tensor $\widetilde{R}_{\mu\nu\lambda}^{\ \ \ \ \kappa} \neq 0.$ The latter is formed by canonical manner from the Levi-Cevita connection $\widetilde{\Gamma}_{\mu\nu\lambda},$ also called Christoffel symbol:

$$\widetilde{R}_{\mu\nu\lambda}^{\ \ \ \ \kappa} = \partial_\mu\widetilde{\Gamma}_{\nu\lambda}^{\ \ \ \kappa} - \partial_\nu\widetilde{\Gamma}_{\mu\lambda}^{\ \ \ \kappa} - \left[\widetilde{\Gamma}_\mu, \widetilde{\Gamma}_\nu\right]_\lambda^{\ \kappa},$$

$$\widetilde{\Gamma}_{\mu\nu\lambda} = \frac{1}{2}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}), \qquad (6.2.7)$$

$$\left[\widetilde{\Gamma}_\mu, \widetilde{\Gamma}_\nu\right]_\lambda^{\ \kappa} = \widetilde{\Gamma}_{\mu\lambda}^{\ \ \ \sigma}\widetilde{\Gamma}_{\nu\sigma}^{\ \ \ \kappa} - \widetilde{\Gamma}_{\nu\lambda}^{\ \ \ \sigma}\widetilde{\Gamma}_{\mu\sigma}^{\ \ \ \kappa},$$

where

$$g_{\mu\nu} = e_\mu^i e_\nu^i \qquad (6.2.8)$$

is the Riemann metric in the space of anholonomic coordinates. The relation between the two curvature tensors is

$$R_{\mu\nu\lambda}^{\ \ \ \ \kappa} - \widetilde{R}_{\mu\nu\lambda}^{\ \ \ \ \kappa} = D_\mu K_{\nu\lambda}^{\kappa} - D_\nu K_{\mu\lambda}^{\kappa} - [K_\mu, K_\nu]_\lambda^{\ \kappa},$$

$$[K_\mu, K_\nu]_\lambda^{\ \kappa} = K_{\mu\lambda}^{\sigma} K_{\nu\sigma}^{\kappa} - K_{\nu\lambda}^{\sigma} K_{\mu\sigma}^{\kappa}, \qquad (6.2.9)$$

$$K_{\mu\nu}^{\lambda} = \Gamma_{\mu\nu}^{\lambda} - \widetilde{\Gamma}_{\mu\nu}^{\lambda}.$$

Where the symbols D_μ denote the covariant derivatives formed with the Christoffel symbol. From either of the two curvature tensors,i.e. Cartan curvature tensor $R_{\mu\nu\lambda}^{\ \ \ \ \kappa}$ and Riemann curvature tensor $\widetilde{R}_{\mu\nu\lambda}^{\ \ \ \ \kappa}$ one can form the oncecontracted tensors of rank 2, the Ricci tensors and the curvature scalars:

$$R_{\mu\nu} = R_{\mu\nu\lambda}^{\ \ \ \ \mu}; R = g^{\nu\lambda} R_{\nu\lambda},$$

$$(6.2.10)$$

$$\widetilde{R}_{\mu\nu} = \widetilde{R}_{\mu\nu\lambda}^{\ \ \ \ \mu}; \widetilde{R} = g^{\nu\lambda} \widetilde{R}_{\nu\lambda}.$$

It is possible to map a flat x-space locally into a curved y-space with $\widetilde{R}_{\mu\nu\lambda}^{\ \ \ \ \kappa} \neq 0$ via an infinitesimal anholonomic transformation

$$dx^i = e_\mu^i(y) dy^\mu \qquad (6.2.11)$$

with coefficient functions $e_\mu^i(y)$ which are not integrable,i.e.

$$\partial_\mu e_\nu^i(y) - \partial_\nu e_\mu^i(y) \neq 0. \qquad (6.2.12)$$

VI.3.Gravity type gauge theories of accelerated comovin frame formed by elastic media with defects.

Let's consider infinite three dimensional elastic media with defects.A theory of relativistic elastic media with defects based on gravity type gauge theories in three dimensional case (two space plus one time) is considered by Katanaev and Volovich [41].They introduce a metric affine space with a metric constructed from distortion e^i_μ and a $\mathbf{SO}(3)$ - connection ω^{ij}_μ. Here the index μ is a general curvilinear coordinate label of the material manifold and i labels the coordinate X^i of the current configuration manifold. Using simple and physically reasonable assumptions they define a two-parameter static Lagrangian which is the sum of the Hilbert-Einstein Lagrangian for the distorsion and the square of the antisymmetric part of the Ricci tensor [42]:

$$\frac{1}{e}\mathcal{L} = -\kappa\widetilde{R} + 2\gamma R^A_{ij}R^{Aij}, \qquad (6.3.1)$$

$$e = \det(e^i_\mu),$$

which is the sum of the Hilbert–Einstein Lagrangian for the vielbein and the square of the antisymmetric part of the Ricci tensor. The vielbein e^i_μ and $\mathbf{SO}(3)$ connection ω^{ij}_μ are basic and independent variables in the geometric approach.

Remark.6.3.1.Note that $\widetilde{R} = \widetilde{R}(e)$ and $R = R(e,\omega)$ are constructed from differentmetrical connections and the identity (6.3.2) is valid in the Riemann–Cartan

geometry in an arbitrary number of dimensions:

$$\widetilde{R}(e) = R(e,\omega) + \frac{1}{4} T_{ijk} T^{ijk} -$$

$$-\frac{1}{2} T_{ijk} T^{kij} - T_i T^i - \frac{2}{e} \partial_\mu (e T^\mu), \qquad (6.3.2)$$

$$e = \det(e^i_\mu).$$

One assume that equations of equilibrium must be covariant under general coordinate transformations and local rotations, be at most of the second order, and follow from a variational principle. The expression for the free energy leading to the equilibrium equations must then be equal to a volume integral of the scalar function (the Lagrangian) that is quadratic in torsion and curvature tensors. There are three independent invariants quadratic in the torsion tensor and three independent invariants quadratic in the curvature tensor in three dimensions [41]. It is possible to add the scalar curvature and a “cosmological” constant Λ. One thus obtain a general eight-parameter Lagrangian [41]:

$$\frac{1}{(e_\epsilon)_\epsilon} (\mathcal{L}_\epsilon)_\epsilon = -\kappa (R_\epsilon)_\epsilon +$$

$$\frac{1}{4} (T_{\epsilon,ijk})_\epsilon \cdot \left(\beta_1 \left(T_\epsilon^{ijk} \right)_\epsilon + \beta_2 \left(T_\epsilon^{kij} \right)_\epsilon + \beta_3 \left(T_\epsilon^{j} \right)_\epsilon \cdot \delta^{ik} \right) + \qquad (6.3.3)$$

$$+\frac{1}{4} (R_{\epsilon,ijkl})_\epsilon \left(\gamma_1 \left(R_\epsilon^{ijkl} \right)_\epsilon + \gamma_2 \left(R_\epsilon^{klij} \right)_\epsilon + \gamma_3 (R_\epsilon^{ik})_\epsilon \cdot \delta^{jl} \right) - \Lambda,$$

$$(e_\epsilon)_\epsilon = \det(e^i_{\epsilon,\mu})_\epsilon,$$

where $\kappa, \beta_{1,2,3}$ and $\gamma_{1,2,3}$ are some constants, and we have introduced the trace of the generalized torsion tensor $(T_{\epsilon,j})_\epsilon = (T^i_{\epsilon,ij})_\epsilon$ and the generalized Ricci tensor $(R_{\epsilon,ik})_\epsilon = \left(R_{\epsilon,ijk}^{\ \ j} \right)_\epsilon$.

The particular feature of three dimensions is that the full curvature tensor is in a one to

one correspondence with Ricci tensor $(R_{\epsilon,ijkl})_{\epsilon}$ and has three irreducible components. Therefore, the Lagrangian contains only three independent invariants quadratic in curvature tensor. We do not need to add the Hilbert–Einstein Lagrangian $\widetilde{R}$, also yielding second-order equations, to the free energy given by Eq.(6.3.3)
The Lagrangian (6.3.3) gives equations of the accelerated relativistic media [42]:

$$\frac{1}{(e_\epsilon)_\epsilon}\frac{\delta(\mathcal{L}_\epsilon)_\epsilon}{\delta(e^i_{\epsilon,\mu})_\epsilon} = -\kappa\Big[(R_\epsilon)_\epsilon \cdot (e^\mu_{\epsilon,i})_\epsilon - 2(R^\mu_{\epsilon,i})_\epsilon\Big] +$$

$$\beta_1\Big[\nabla_\nu (T^{\nu\mu}_{\epsilon,i})_\epsilon - \frac{1}{4}(T_{\epsilon,jkl})_\epsilon \cdot \Big(T^{jkl}_\epsilon\Big)_\epsilon \cdot (e^\mu_{\epsilon,i})_\epsilon + \Big(T^{\mu jk}_\epsilon\Big)_\epsilon \cdot (T_{\epsilon,ijk})_\epsilon\Big] +$$

$$\beta_2\Big[-\frac{1}{2}\nabla_\nu\Big((T^{\mu\nu}_{\epsilon,i})_\epsilon - (T^{\nu\mu}_{\epsilon,i})_\epsilon\Big) - \frac{1}{4}(T_{\epsilon,jkl})_\epsilon \cdot \Big(T^{ljk}_\epsilon\Big)_\epsilon \cdot (e^\mu_{\epsilon,i})_\epsilon$$

$$-\frac{1}{2}\Big(T^{j\mu k}_\epsilon\Big)_\epsilon \cdot (T_{\epsilon,kij})_\epsilon + \frac{1}{2}\Big(T^{jk\mu}_\epsilon\Big)_\epsilon \cdot (T_{\epsilon,kij})_\epsilon\Big] +$$

$$\beta_3\Big[-\frac{1}{2}\nabla_\nu\Big((T^\nu_\epsilon)_\epsilon \cdot (e^\mu_{\epsilon,i})_\epsilon - (T^\mu_\epsilon)_\epsilon \cdot (e^\nu_{\epsilon,i})_\epsilon\Big)$$

$$-\frac{1}{4}(T_{\epsilon,j})_\epsilon \cdot \Big(T^j_\epsilon\Big)_\epsilon \cdot (e^\mu_{\epsilon,i})_\epsilon + \frac{1}{2}(T^\mu_\epsilon)_\epsilon \cdot (T_{\epsilon,i})_\epsilon + \frac{1}{2}\Big(T^j_\epsilon\Big)_\epsilon \cdot (T^\mu_{\epsilon,ij})_\epsilon\Big] + \qquad (6.3.4)$$

$$\gamma_1\Big[-\frac{1}{4}(R_{\epsilon,jklm})_\epsilon \cdot \Big(R^{jklm}_\epsilon\Big)_\epsilon \cdot (e^\mu_{\epsilon,i})_\epsilon + \Big(R^{\mu jkl}_\epsilon\Big)_\epsilon \cdot (R_{\epsilon,ijkl})_\epsilon\Big] +$$

$$\gamma_2\Big[-\frac{1}{4}(R_{\epsilon,jklm})_\epsilon \cdot \Big(R^{lmjk}_\epsilon\Big)_\epsilon \cdot (e^\mu_{\epsilon,i})_\epsilon + \Big(R^{klmj}_\epsilon\Big)_\epsilon \cdot (R_{\epsilon,ijkl})_\epsilon\Big] +$$

$$\gamma_3\Big[-\frac{1}{4}(R_{\epsilon,jk})_\epsilon \cdot \Big(R^{jk}_\epsilon\Big)_\epsilon \cdot (e^\mu_{\epsilon,i})_\epsilon + \frac{1}{2}\Big(R^{\mu j}_\epsilon\Big)_\epsilon \cdot (R_{\epsilon,ij})_\epsilon +$$

$$+\frac{1}{2}\Big(R^{jk}_\epsilon\Big)_\epsilon \cdot (R^\mu_{\epsilon,jik})_\epsilon\Big] +$$

$$+\Lambda(e^\mu_{\epsilon,i})_\epsilon = (\mathbf{T}_{\epsilon,ij})_\epsilon,$$

where $\mathbf{T}_{ij}$

$$\frac{1}{(e_\epsilon)_\epsilon}\frac{\delta(\mathcal{L}_\epsilon)_\epsilon}{\delta\left(\omega^{ij}_{\epsilon,\mu}\right)_\epsilon} =$$

$$\kappa\left[\frac{1}{2}\left(T_{\epsilon,ij}^{\ \ \mu}\right)_\epsilon + (T_{\epsilon,i})_\epsilon \cdot \left(e^{\mu}_{\epsilon,\ \ j}\right)_\epsilon\right] + \beta_1\frac{1}{2}\left(T^{\mu}{}_{\epsilon,ji}\right)_\epsilon$$

$$+\beta_2\frac{1}{4}\left[\left(T^{\mu}_{\epsilon,i\ \ j}\right)_\epsilon - \left(T_{\epsilon,ij}^{\ \ \mu}\right)_\epsilon\right] + \tag{6.3.5}$$

$$\beta_3\frac{1}{4}(T_{\epsilon,j})_\epsilon \cdot \left(e^{\mu}_{\epsilon,\ \ i}\right)_\epsilon + \gamma_1\frac{1}{2}\nabla_\nu\left(R^{\nu\mu}_{\epsilon,\ \ ij}\right)_\epsilon + \gamma_2\frac{1}{2}\nabla_\nu\left(R_{\epsilon,ij}^{\ \ \nu\mu}\right)_\epsilon +$$

$$\gamma_3\frac{1}{4}\nabla_\nu\left[\left(R^{\nu}_{\epsilon,\ \ i}\right)_\epsilon \cdot \left(e^{\mu}_{\epsilon,\ \ j}\right)_\epsilon - \left(R^{\mu}_{\epsilon,\ \ i}\right)_\epsilon \cdot \left(e^{\nu}_{\epsilon,\ \ j}\right)_\epsilon\right] - (i \leftrightarrow j) = 0,$$

where the covariant derivative acts with the $\mathbf{SO}(3)$ connection on the Latin indices and with the Christoffel symbols on the Greek ones.

Remark.6.3.2.The most general Lagrangian () depends on eight constants: κ, β_1, $\beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3, \Lambda$ and leads to very complicated equations of accelerated relativistic media. Note that these constants is not apriori fixed and we do not know precisely

what values of the constants describe this or that accelerated relativistic media and corresponding comovin frame.
Therefore, we make physically reasonable assumptions to simplify matters at least for the case $R_{\mu\nu}{}^{ij} = 0$.
Remark.6.3.3.Note that curvature $R_{\mu\nu}{}^{ij}$ of the comovin frame of the any accelerated relativistic media in contrast with [42] satisfies the inequality $R_{\mu\nu}{}^{ij} \neq 0$.

Thus, one obtain that equations of the accelerated relativistic media must admit the following three types of solutions:

- **1**.There are solutions describing the relativistic accelerated media with only "dislocations":

$$\left(R_{\epsilon,\mu\nu}{}^{ij}\right)_{\epsilon} = 0,$$

$$\left(T_{\epsilon,\mu\nu}{}^{i}\right)_{\epsilon} \neq 0, \qquad (6.3.6)$$

$$\left(\widetilde{R}_{\epsilon,\mu\nu}\right)_{\epsilon} \neq 0.$$

- **2**.There are solutions describing the relativistic accelerated media with only "disclinations":

$$\left(R_{\mu\nu}{}^{ij}\right)_{\epsilon} \neq 0,$$

$$\left(T_{\mu\nu}{}^{i}\right)_{\epsilon} = 0, \qquad (6.3.7)$$

$$\left(\widetilde{R}_{\mu\nu}\right)_{\epsilon} \neq 0.$$

- 3.There are solutions describing the relativistic accelerated media almost without "dislocations" and "disclinations":

$$\left\| R_{\mu\nu}{}^{ij} \right\| \simeq 0,$$

$$\left\| T_{\mu\nu}{}^{i} \right\| \simeq 0, \qquad (6.3.8)$$

$$\widetilde{R}_{\mu\nu} \neq 0.$$

Substitution of the condition $R_{\mu\nu}{}^{ij} = 0$ into Eq.(6.3.4) for the $\mathbf{SO}(3)$ connection gives

$$(12\kappa + 2\beta_1 - \beta_2 - 2\beta_3)T_i = 0,$$

$$(\kappa - \beta_1 - \beta_2)T^* = 0, \qquad (6.3.9)$$

$$(4\kappa + 2\beta_1 - \beta_2)W_{ijk} = 0.$$

Here T_i, T^*, and W_{ijk} are the irreducible components of the torsion tensor.In a general case of dislocations all irreducible components of torsion tensor differ from zero ($T_i, T^*, W_{ijk} \neq 0$) and Eqs. (6.3.9) have a unique solution

$$\beta_1 = \kappa, \beta_2 = 2\kappa, \beta_3 = 4\kappa. \qquad (6.3.10)$$

For these coupling constants, the first four terms in Lagrangian (6.3.3) are equal to the Hilbert–Einstein Lagrangian $\kappa\widetilde{R}(e)$ up to a total divergence due to identity(6.3.2). Equation (6.3.2) then reduces to the Einstein equations with a cosmological constant

$$\widetilde{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\widetilde{R} - \frac{\Lambda}{2\kappa}g_{\mu\nu} = 0 \qquad (6.3.11)$$

According to the second condition,the equations of accelerated equilibrium must allow solutions with zero torsion $T_{\mu\nu}{}^i = 0$.In this case, the curvature tensor has additional symmetry $R_{ijkl} = R_{klij}$, and Eq.(6.3.5) becomes

$$\left(\gamma_1 + \gamma_2 + \frac{\gamma_3}{4}\right)\nabla_\nu\left(R^{\mathbf{S}\,\nu}{}_i e^\mu{}_j - R^{\mathbf{S}\,\mu}{}_i e^\nu{}_j + R^{\mathbf{S}\,\mu}{}_j e^\nu{}_i\right) +$$
$$+\frac{1}{6}(\gamma_1 + \gamma_2 + 4\gamma_3)\left(e^\nu{}_i e^\mu{}_j - e^\mu{}_i e^\nu{}_j\right)\nabla_\nu R \simeq 0, \tag{6.3.12}$$

where

$$R_{ij} = R^{\mathbf{S}}_{ij} + R^{\mathbf{A}}_{ij} + \frac{1}{3}R\delta_{ij},$$
$$R^{\mathbf{S}}_{ij} = R^{\mathbf{S}}_{ji}, R^{\mathbf{S}}{}_i = 0, R^{\mathbf{A}}_{ij} = -R^{\mathbf{A}}_{ji}. \tag{6.3.13}$$

Note that for almost zero torsion, the Ricci tensor is symmetrical,i.e.$R^{\mathbf{A}}_{ij} = 0$. Contraction of Eq.(6.3.12) with e^j_μ gives

$$\left(\gamma_1 + \gamma_2 + \frac{\gamma_3}{4}\right)\nabla_\nu R^{\mathbf{S}\,\nu}{}_\mu +$$
$$\frac{1}{3}(\gamma_1 + \gamma_2 + 4\gamma_3)\nabla_\mu R = 0. \tag{6.3.14}$$

Note that in the general case of nonvanishing curvature, the covariant derivatives $\nabla_\nu R^{\mathbf{S}\,\nu}{}_\mu$ and $\nabla_\mu R$ differ from zero and are independent. Therefore, one obtain two equations for the coupling constants,

$$\gamma_1 + \gamma_2 + \frac{\gamma_3}{4} = 0,$$

$$(6.3.15)$$

$$\gamma_1 + \gamma_2 + 4\gamma_3 = 0,$$

which have a unique solution

$$\gamma_1 = -\gamma_2 = \gamma, \gamma_3 = 0. \qquad (6.3.16)$$

The last requirement for the noexistence of solutions with zero curvature and torsion is satisfied only for the non zero cosmological constant

$$\frac{1}{e}\mathcal{L} = -\kappa\widetilde{R} + 2\gamma R^A_{ij}R^{Aij} + \Lambda, \qquad (6.3.17)$$

$$\Lambda \neq 0.$$

VII.Bimetrical interpretation some exact solutions of the Einstein field equations.

VII.1. General consideration.

Let us consider that in the Minkovsky space with the signature the continuum medium moves in some force field, the motion law of this continuum in the Lagrange variables has the form [70]:

$$x^{\mu} = x^{\mu}(y^{k}, \xi^{0}), \qquad (7.1.1)$$

where x^{μ} are the Euler coordinates, y^{k} are the Lagrange coordinates constant along each fixed world line of the medium particle, is the some time parameter. The greek indexes are changed from zero to three, the latin indexes are changed from unit to three. We consider that the medium particles do not interact with each other and they interact only with the external field.

Similarly to electrodynamics the actions for the probe particle in the force field we specify in the form

$$S = -mc\int_{a}^{b}(ds + \alpha A_{\mu}dx^{\mu}),$$
$$\alpha = \frac{e}{mc^{2}}, \qquad (7.1.2)$$

where for each medium particle the ds interval along the world lines is $ds = V_{\mu}dx^{\mu},\ V^{\mu}$ is the four dimensional velocity.From the action variation the motion equation follows

$$\frac{DV_\mu}{ds} = \alpha F_{\mu\nu} V^\nu, \qquad (7.1.3)$$

where the field tensor $F_{\mu\nu}$ is determined as

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \qquad (7.1.4)$$

On the other hand one can to introduce the effective interval $d\widetilde{s} = ds + \alpha A_\mu dx^\mu$
so (7.1.2) is represented in the form

$$S = -mc \int d\,\widetilde{s}, \qquad (7.1.5)$$

variation of this action results in the motion of the probe particle on the geodesic line in some pseudo Riemannian space $\Re$

$$\frac{dU_\mu}{d\widetilde{s}} + \tilde{\Gamma}_{\mu,\nu\epsilon} U^\nu U^\epsilon = 0. \qquad (7.1.6)$$

Claim.**7**.**1**.**1**. Suppose that the dynamic equations (7.1.3) and (7.1.6) is equivalent,i.e. $(\mathbf{7.1.3}) \Leftrightarrow (\mathbf{7.1.6})$.
It follows from the expression for the effective interval $d\widetilde{s}$ along the geodesic line that

$$d\tilde{s} = (V_\mu + \alpha A_\mu)dx^\mu \equiv U_\mu dx^\mu, \qquad U_\mu \equiv V_\mu + \alpha A_\mu,$$
$$U^\mu = \frac{dx^\mu}{d\tilde{s}} = \frac{dx^\mu}{ds}\frac{ds}{d\tilde{s}} = PV^\mu, \qquad P \equiv (1 + \alpha A_\epsilon V^\epsilon)^{-1}. \tag{7.1.7}$$

Besides the connection between covariant U_ν and the contravariant U^μ vectors of the 4-velocity in the pseudo Riemannian space has the form

$$U_\nu = g_{\nu\mu}U^\mu = V_\nu + \alpha A_\nu. \qquad (7.1.8)$$

Conditions (7.1.3), (7.1.4), (7.1.6), (7.1.7) and (7.1.8) will be is self consistent if and only if the metric tensor $g_{\nu\mu}$ of the pseudo Riemannian space $\Re$ will have the form:

$$g_{\nu\mu} = \gamma_{\mu\nu} + \alpha^2 A_\mu A_\nu + \alpha A_\mu V_\nu + \alpha A_\nu V_\mu, \qquad (7.1.9)$$

Thus, one can consider the motion of the probe particle as motion in Rosen bimetric space $\Re_2 = \Re(g_{\nu\mu}, \gamma_{\mu\nu})$ i.e. from two points of view:

- The motion on the world line in the Minkovsky space in the force field (7.1.3) with the metrics $\gamma_{\mu\nu}$.
- The motion in the Riemannian space on the geodesic line with the metrics $g_{\nu\mu}$ determined in accordance with (7.1.9).

The correlations between the 4-velocities in the different spaces are determined with the formulas (7.1.7) and (7.1.8). Herewith in the two spaces the general coordination has been selected. Unlike electrodynamics the tensor field $F_{\mu\nu}$structure in (7.1.4) has not been concreted, that is for the tensor field $F_{\mu\nu}$ equations are not specified.

Let the probe particles move in the Einstein gravitational field. Then the “charge”

$e = m$, and the metrics (7.1.9) has to satisfy to the Einstein equations with the dusty stress energy tensor.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi\kappa}{c^4} \varepsilon U_\mu U_\nu. \qquad (7.1.10)$$

If as a result of the solution of the equations (7.1.10) obtained $g_{\mu\nu}$ and $U_\nu$will provide the equalities (7.1.8) and (7.1.9), then we can to find the field of 4-velocity V_μ, the potentials A_μ and the field tensor $F_{\mu\nu}$in the Minkovsky space, that is the mapping of the curvature field of the Riemannian space on the force field of the plane space- time will be constructed.
Let us ascertain the connection between the congruencies of the world lines in the Minkovsky space and the congruencies of the geodesic lines in the Riemannian space which in the general coordination are determined with the Eq.(7.1.1). Because of the Eq.(7.1.9) in the space-time two metric tensors $g_{\mu\nu}$ and $\gamma_{\mu\nu}$ have been introduced, and, consequently, two connections $\tilde{\Gamma}^{\epsilon}{}_{\mu\nu}$ and $\Gamma^{\epsilon}{}_{\mu\nu}$ exist, the first connection relates to the pseudo Riemannian space $\Re$, and the second one relates to the Minkovsky space M_4. In the Minkovsky space the curvature coordinates can be introduced. Thus, in the general coordination two different covariant derivatives $\tilde{\nabla}_\nu$ and ∇_ν arise.From the

Eq.(7.1.8) we have

$$\begin{aligned} \tilde{\nabla}_\nu U_\mu &= -S^{\epsilon}{}_{\nu\mu} U_\epsilon + \nabla_\nu V_\mu + \alpha \nabla_\nu A_\mu, \\ S^{\epsilon}{}_{\nu\mu} &= \tilde{\Gamma}^{\epsilon}{}_{\nu\mu} - \Gamma^{\epsilon}{}_{\nu\mu}, \end{aligned} \qquad (7.1.11)$$

where $S^{\epsilon}{}_{\nu\mu}$ is the tensor of the affine connectivity deformation.From (7.1.11) we find

$$2\tilde{\nabla}_{[\nu} U_{\mu]} = 2\nabla_{[\nu} V_{\mu]} - \alpha F_{\mu\nu}, \qquad (7.1.12)$$

For geodesic congruensies without rotations the equalities take place

$$\tilde{\nabla}_{[\nu}U_{\mu]} = 0; \qquad 2\nabla_{[\nu}V_{\mu]} = \alpha F_{\mu\nu}. \qquad (7.1.13)$$

Convoluting (7.1.13) with V^{ν} we once again obtain the equation (7.1.13). From the equalities (7.1.13) and (7.1.7) we have

$$U_{\mu} = \frac{\partial\Phi}{\partial x^{\mu}} = V_{\mu} + \alpha A_{\mu} \qquad (7.1.14)$$

that permits the representation of the (7.1.9) metrics in the form

$$g_{\mu\nu} = \gamma_{\mu\nu} + \frac{\partial\Phi}{\partial x^{\mu}}\frac{\partial\Phi}{\partial x^{\nu}} - V_{\mu}V_{\nu} \qquad (7.1.15)$$

For the contravariant components we have

$$g^{\mu\nu} = \gamma^{\mu\nu} + P^{2}V^{\mu}V^{\nu}\left(1 + \gamma^{\alpha\beta}\frac{\partial\Phi}{\partial x^{\alpha}}\frac{\partial\Phi}{\partial x^{\beta}}\right) - \qquad (7.1.16)$$

where in accordance with (7.1.7)

$$P = (1 + \alpha\gamma_{\epsilon\sigma}A^{\epsilon}V^{\sigma})^{-1} = \qquad (7.1.17)$$

It follows from the equalities (7.1.9), (7.1.14) and (7.1.15)

$$g_{\mu\nu} - U_{\mu}U_{\nu} = \gamma_{\mu\nu} - V_{\mu}V_{\nu}, \qquad (7.1.18)$$

i.e. is the projection operators determining the space geometry of the hypersurfaces orthogonal to the world lines in the Minkovsky space and the hupersurfaces orthogonal to the geodesic lines in the Riemannian space are the invariants of the correspondence.

VII.2. Bimetrical interpretation of the Shvartzshild solution.

Let us consider some particular cases of the general mapping considered above.Let in the Minkovsky space the dust continuum moves on the radius to the centre. We consider the case of the stationary motion that means time independence of the velocity field in the Euler variables and the potentials A_{μ}. In the GIFT language this corresponds to the constant gravitational-inertional field.
In order to the metric tensor (7.1.15) does not obviously depend from the time and pass at the infinity to the Galilean form it is necessary that the velocity at the infinity becomes zero. Thus the next equalities have to be satisfied:

$$\Phi = x^0 + \Psi(x^k),$$
$$V_a = -V(r)n_a = -V(r)\frac{x_a}{r} \qquad (7.1.19)$$

Using formulas (7.1.15) and (7.1.19) we find the expressions for three-dimensional metric tensor: $\tilde{\gamma}_{kl} = -g_{kl} + g_{0k}g_{0l}/g_{00}$; three-dimensional vector: $g_l = g_{0l}/g_{00} = g_{0l}/h$; three-dimensional antisymmetric tensor: $f_{kl} = \partial g_l/\partial x^k - \partial g^k/\partial x_l$. As a result we have

$$g_{00} = 1 - V^2,\ g_l = n_l\frac{\frac{\partial\Phi}{\partial r} + V_0 V}{h},\ f_{kl} = 0,\ V_0^2 - V^2 = 1,$$

$$\tilde{\gamma}_{kl} = \delta_{kl} + D(r)n_k n_l,$$

$$D = \frac{2V^2 + 2V_0 V\frac{\partial\Phi}{\partial r} + V^2\left(\frac{\partial\Phi}{\partial r}\right)^2}{1 - V^2} \qquad (7.1.20)$$

$$\tilde{\gamma}^{kl} = -g^{kl} = \delta^{kl} + Tn^k n^l,\ n^k = n_k,\ V^0 = V_0,\ \tilde{\gamma}_{kl}\tilde{\gamma}^{ln} = \delta^k_n,$$

$$T = \frac{2V^2 + 2V_0 V\frac{\partial\Phi}{\partial r} + V^2\left(\frac{\partial\Phi}{\partial r}\right)^2}{\left(V^0 + V\frac{\partial\Phi}{\partial r}\right)^2}.$$

Einstein equations for the case of the constant gravitational field in the vacuum (we consider that the dusty medium is strongly discharged and itself does not create the field) will result to two independent expressions:

$$\frac{\partial}{\partial r}\left(\frac{r^2 \frac{\partial F}{\partial r}}{\sqrt{1+D}}\right) = 0,\ F = \sqrt{h} = \sqrt{1-V^2},$$

$$D + \frac{r}{2}\frac{\partial D}{\partial r}\frac{1}{1+D} = \frac{r}{F}\frac{\partial F}{\partial r} \tag{7.1.21}$$

the solution of which has the form

$$D = \frac{r_g/r}{1-r_g/r}, F = \sqrt{g_{00}} = \sqrt{1-\frac{r_g}{r}}, r_g \equiv \frac{2kM}{c^2}. \tag{7.1.22}$$

From correlations (7.1.20) and (7.1.22) we find zero and radial field components of the 4-velocity in the Minkovsky space in the Euler variables and also function Φ :

$$V_0 = V^0 = \left(1+\frac{r_g}{r}\right)^{1/2}, V^1 = V = -\sqrt{\frac{r_g}{r}},$$

$$V_0 + V\frac{\partial \Phi}{\partial r} = V^\epsilon \frac{\partial \Phi}{\partial x^\epsilon} = \frac{\partial \Phi}{\partial s} = 1. \tag{7.1.23}$$

Thus,$\Phi/c = \tau = s/c$ coincides with the own time of the basis particles in the Minkovsky and Riemann space.It follows from (7.1.23), (7.1.7) and (7.1.14) that $(1+\alpha A_\mu V^\mu) = P^{-1} = 1$,this results in the equality of the contravariant components of 4-velocities $U^\mu = V^\mu$ of the basis particles in the plane and curved space-time. Covariant components U_μ and V_μ are connected with the equation (7.1.14). Integrating equation (7.1.23) for Φ taking into account (7.1.19) we find

$$\Phi = c\tau = s = x^0 + \frac{2}{3} r_g \left(\frac{r}{r_g} + 1 \right)^{3/2} - \frac{2}{3} \frac{r^{3/2}}{r_g{}^{1/2}}. \qquad (7.1.24)$$

Using (7.1.20), (7.1.23) and (7.1.24) we obtain the expression for the interval element of the "original" in the spherical Euler coordinates and time T of the Minkovsky space

$$d\tilde{s}^2 = c^2 dT^2 \left(1 - \frac{r_g}{r} \right) -$$

$$dr^2 \left\{ 2 \left[\frac{r}{r_g} \left(\frac{r}{r_g} + 1 \right)^{1/2} - 2 \frac{r}{r_g} + \frac{r_g}{r} \right] \right\} +$$

$$2cdTdr \left[\left(1 + \frac{r}{r_g} \right)^{1/2} - \left(\frac{r}{r_g} \right)^{1/2} - \frac{r_g}{r} \left(1 + \frac{r}{r_g} \right)^{1/2} \right] - \qquad (7.1.25)$$

$$r^2 (\sin^2 \Theta d\varphi^2 + d\Theta^2).$$

Known the field of the 4-velocity in the Euler variables we find the motion law of the continuum in the Lagrange variables (7.1.1) selecting as a time parameter ξ^0 the own time $\tau = \Phi/c = s/c$. From (7.1.23) we have $dr/ds = V = -(r_g/r)^{1/2}$. Integrating we obtain $R - s = 2/3(r^{3/2}/r_g{}^{1/2})$,where R is the constant of integration. Taking into account (7.1.24) as a result we find

$$r = \left[\frac{3}{2}(R - c\tau)\right]^{2/3} r_g{}^{1/3},$$

$$x^0 = cT = R - \frac{2}{3} r_g \left\{ \left[\frac{3}{2r_g}(R - c\tau)\right]^{2/3} + 1 \right\}^{3/2}, \tag{7.1.26}$$

that determines the sought motion law in the Lagrange variables, substitution of this law to the expression (7.1.25) results in the Lemetr interval element.Formulas (7.1.23), (7.1.26) determine the kinematics of the dust medium moving with the acceleration on the radius to the centre in the Minkovsky space in the gravitational field of the central body. For the field of the three-dimensional velocity v, 4-acceleration g, three-dimensional acceleration a and the three-dimensional force N we have:

$$\frac{dr}{dT} = v = -c\left(1 + \frac{r}{r_g}\right)^{-1/2}, \frac{1}{c^2}\frac{d^2r}{d\tau^2} = g = -\frac{r_g}{2r^2},$$

$$a = \frac{d^2r}{dT^2} = -\frac{c^2 r_g}{2r^2}\left(1 + \frac{r_g}{r}\right)^{-2}, \tag{7.1.27}$$

$$N = \frac{d}{dT}\left[\frac{mv}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}}\right] = -\frac{m r_g c^2}{2r^2(1 + r_g/r)^{1/2}}.$$

Movement of the Lemetr basis in the Minkovsky space is described with the functions continuous in the region $0 < r < \infty$ not having the particularities at the gravitational radius. Three-dimensional velocity v and three-dimensional acceleration a are restricted at the origin of the coordinates,$v(0) = -c, a(0) = -c^2/(2r_g)$. . The value of the three-dimensional force N (7.1.27) influencing on the probe mass from the side of the central body is smaller then in the Newton gravitation theory

$$N = -\frac{kmM}{r^2\left(1 + \frac{2kM}{c^2 r}\right)^{1/2}}. \qquad (7.1.28)$$

It is evident that the space components of the 4-velocity cV^1 (7.1.23) and 4-acceleration gc^2 (7.1.27) exactly coincide with the usual velocity and acceleration in the non-relativistic Newton mechanics, when the radial fall of the dust having zero velocity at the infinity on the force centre is considered. From the formulas (7.1.23) and (7.1.27) we find the time of the basis particles fall from the distance $r_1 > r$ up to $r \geq 0$ in accordance with the clock of the falling particle and in accordance with the Minkovsky space clock T

$$\delta\tau = \frac{2}{3}\left[\frac{r_1}{c}\left(\frac{r_1}{r_g}\right)^{1/2} - \frac{r}{c}\left(\frac{r}{r_g}\right)^{1/2}\right], \qquad (7.1.29)$$

$$\delta T = \frac{2}{3}\left[\left(1 + \frac{r_1}{r_g}\right)^{3/2} - \left(1 + \frac{r}{r_g}\right)^{3/2}\right]\frac{r_g}{c}. \qquad (7.1.30)$$

Eqn. (7.1.29) coincides with the result of the Newton theory.It follows from the formulas (7.1.29), (7.1.30) that the time of the particle fall is finite for any r from the range $0 \leq r \leq r_1$,both in accordance with the clock of the fallen particle and in accordance with the clock of the Minkovsky space.

Usually in GR the time coordinate t including in the Shvartzshild solution is introduced as a time of the external observer. The connection between the t coordinate and T time of the Minkovsky space is determined with the formula

$$T = t - \frac{1}{c}\int\left[\left(1+\frac{r}{r_g}\right)^{1/2}\left(1-\frac{r_g}{r}\right)-\left(\frac{r}{r_g}\right)^{1/2}\right]\left(1-\frac{r_g}{r}\right)^{-1}dr =$$

$$t - \frac{r_g}{c}\left[\frac{2}{3}\left(1+\frac{r}{r_g}\right)^{3/2} - 2\left(\frac{r}{r_g}\right)^{1/2}\left(1+\frac{1}{3}\frac{r}{r_g}\right) - \ln\left|\frac{1-\left(\frac{r}{r_g}\right)^{1/2}}{1+\left(\frac{r}{r_g}\right)^{1/2}}\right|\right] \qquad (7.1.31)$$

Substitution of the formula to the interval (7.1.25) forms the Schwarzschild interval. The velocity field dr/dt of the Lemetr basis in the Schwarzschild metrics is connected with the velocity field $dr/dT = v$ (7.1.7) in the Minkovsky space with the equation

$$\frac{dr}{dt} = \frac{dr}{dT}\frac{dT}{dt} = \frac{dr}{dT}\left(\frac{dT}{dt} + \frac{dT}{dr}\frac{dr}{dt}\right) \qquad (7.1.32)$$

Whence using (4.21.31) we find

$$\frac{dr}{dt} = \frac{\frac{dr}{dT}\frac{dT}{dt}}{1 - \frac{dr}{dT}\frac{dT}{dr}} \qquad (7.1.33)$$

that coincides with the “coordinate” parabolic velocity of the free fall in the Shvartzshild field obtained from the equations for the geodesic. If the “coordinate” velocity in the Shvartsshild field goes to zero when approximation to the gravitational radius then the velocities of the particles in the Minkovsky space in the force field (7.1.28) are always smaller than the light velocity in the vacuum and their tend to the light velocity when $r \to 0$, and at the gravitational radius $|v| = c/\sqrt{2}$ .It follows from (4.21.33) that if the external observer uses the time Shvartsshild coordinate as a time of the removed observer then the approximation to the gravitational radius demands the infinite value t [7, 60]. The later becomes clear from the form of the formula (4.21.31) when at

$r \to r_g, t \to \infty$ at any finite $T.$ From our view point T should be taken as the time of the removed observer, T in accordance with the image construction is the time in the Minkovsky space and interval (4.21.25) is written in the "primary" coordinate system where the radial $r,$ angle Θ, φ and time T coordinates have evident metric sense and they determine the interval in the Minkovsky space in the form

$$ds^2 = c^2 dT^2 - dr^2 - r^2(\sin^2\Theta d\varphi^2 + d\Theta^2). \qquad (7.1.34)$$

At $r_g/r \ll 1$ the interval element (7.1.25) passes to the interval of the plane space-time (7.1.34). Naturally besides interval (7.1.25) one can to consider any other coordinate systems but from our view point the coordinates entering to (7.1.25) coincide with the STR Galilean coordinates and so they are stood out with their clarity from all other coordinate systems.

As is well known when moving the particle in the constant field its energy is kept W_0, is the time component of the covariant 4-vector of the pulse. From (7.1.14), (7.1.24) we have for the basis particles

$$W_0 = m_0 c^2 U_0 = m_0 c^2 = m_0 c^2 (V_0 + \alpha A_0). \qquad (7.1.35)$$

whence using (7.1.23), (7.1.24), (7.1.35), (7.1.19) we find

$$\alpha A_0 = 1 - \left(1 + \frac{r_g}{r}\right)^{1/2},$$
$$\alpha A_k = \frac{\partial \Phi}{\partial x^k} - V_k = \left[\left(1 + \frac{r_g}{r}\right)^{1/2} - \left(\frac{r}{r_g}\right)^{1/2} - \left(\frac{r_g}{r}\right)^{1/2}\right] n_k \qquad (7.1.36)$$

It follows from (7.1.36) that $\alpha A_\mu V^\mu = 0,$ that is in agreement with (7.1.23).

Thus, the solution of the Einstein equations determined the metric $g_{\mu\nu}$ (7.1.25) in the coordinates of the Minkovsky space, field velocity V_μ and vector-potential A_μ.
From (7.1.36) we find the tensor of the constant gravitational field $F_{\mu\nu}$ in the Minkovsky space

$$F_{\mu\nu} = \left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}\right), \qquad F_{kl} = 0,$$

$$F_{0k} = -\frac{\partial A_0}{\partial x^k} = -\frac{r_g n_k}{2\alpha r^2 \sqrt{1 + (r_g/r)}}. \tag{7.1.37}$$

Similarly to the electrodynamics one can see that the tensor $F_{\mu\nu}$ for the case of the spherical symmetry does not contain the analogue of the "magnetic" field $\vec{H}$. The intensity of the gravitational field E_k taking into account (7.1.28) has the form

$$E_k = F_{0k} = \frac{N}{m_0} n_k = -\frac{kMn_k}{r^2\left(1 + \frac{2kM}{c^2 r}\right)^{1/2}}. \tag{7.1.38}$$

Let us introduce the "induction" vector $D_k = \varepsilon E_k$, where

$$\varepsilon \equiv -\left(1 + \frac{2kM}{c^2 r}\right)^{1/2} \frac{1}{k}, \qquad D_k = \frac{M}{r^2} n_k. \tag{7.1.39}$$

Thus, for the case of the spherically-symmetrical gravistatic field outside of the creating mass the expressions are valid

$$\vec{\nabla} \times \vec{E} = 0, \qquad \vec{\nabla} \bullet \vec{D} = 0, \qquad \vec{H} = 0. \qquad (7.1.40)$$

Whence the energy density of the gravistatic field ρ in analogy with the electrostatics is calculated in accordance with the formula

$$\rho = \frac{ED}{8\pi} = -\frac{kM^2}{8\pi r^4\left(1 + \frac{2kM}{c^2 r}\right)^{1/2}}. \qquad (7.1.41)$$

Note that energy density has no a particularity at the gravitational radius.Field energy W outside of the sphere with the radius r_0 is determined with the equation

$$W = \int_{r_0}^{\infty} \rho 4\pi r^2 dr = -\frac{Mc^2}{2}\left[\left(1 + \frac{r_g}{r_0}\right)^{1/2} - 1\right], \qquad (7.1.42)$$

which passes to the Newton expression $W = -(kM^2)/(2r_0)$ at $r_g/r \ll 1$.
Calculation of the known GR effects in accordance with the metrics (7.1.25) connected with the path form results in the same result as in the Shvartsshild field. The difference reveals in the expressions depending on the time and on the time derivatives. For the light beams from (7.1.25) spreading on the radius at $d\tilde{s}^2 = 0$ we have:

$$\left(\frac{dr}{dT}\right)_1 = c_1(r) = c\left[1 - \left(\frac{r}{r_g}\right)^{1/2}\right]$$

$$(7.1.43)$$

$$\left[\left(1 + \frac{r}{r_g}\right)^{1/2}\left(\left(\frac{r}{r_g}\right)^{1/2} - 1\right) - \frac{r}{r_g}\right]^{-1},$$

$$\left(\frac{dr}{dT}\right)_2 = c_2(r) = c\left[1 + \left(\frac{r}{r_g}\right)^{1/2}\right]$$

$$\left[\frac{r}{r_g} - \left(1 + \frac{r}{r_g}\right)^{1/2}\left(\left(\frac{r}{r_g}\right)^{1/2} + 1\right)\right]^{-1}, \qquad (7.1.44)$$

where (7.1.43) corresponds to the velocity of the spreading beams, and (7.1.44) corresponds to the velocity of the converging ones. At $r < r_g$ the expressions (7.1.43), (7.1.44) are negative that is the beams spread only in one direction inside.
Note that $c_1(r_g) = 0.$ So the time of the light signal spreading from $r = r_g$ to $r_0 > r_g$ tends to infinity.

$c_1|_{r>r_g} > 0; \quad |c_1| \leq c$ equal sign takes place at $r \to 0; \quad r \to \infty.$
$c_1|_{r>r_g} < 0; \quad |c_1| \geq c$ equal sign takes place at $r \to 0; \quad r \to \infty.$
$|c_2|$ has a maximum at the $r = 3r_g, c_2(3r_g) = c\left(7 + \sqrt{3}\right)/11.$
For converging beams the time of the signal spreading between any $0 \leq r_1 < \infty$ and $0 \leq r_2 < \infty$ is finite. If $r_g/r \ll 1,$ then

$$c_1 \simeq \left(1 - 0.5(r_g/r)^{1/2} - r_g/r\right)c,$$

$$c_2 \simeq -\left(1 - 0.5(r_g/r)^{1/2} - r_g/r\right)c.$$

VIII.The Mössbauer experiment in a rotating system and the extraenergy shift between emission and absorption lines using Bimetric theory of gravitational-inertial field in Riemannian approximation explained

VIII.1.The Mössbauer experiment in a rotating system and the extraenergy shift between emission and absorption lines.

- In a series of papers published during the past decade with respect to Mössbauer experiments in a rotating system [71]-[75],it has been experimentally shown that the relative energy shift $\Delta E/E$ between the source of resonant radiation (situated at the center of the rotating system) and the resonant absorber (located on the rotor rim) is described by the relationship

$$\Delta E/E = -ku^2/c^2, \tag{8.1.1}$$

where u is the tangential velocity of the absorber, c the velocity of light in vacuum, and k some coefficient, which – contrary to what had been classically predicted equal $1/2$ (see for example [35]) – turns out to be substantially larger than $1/2$.

It cannot be stressed enough that the equality $k = 1/2$ had been predicted by general theory of relativity (GTR) on account of the special relativistic time dilation effect delineated by the tangential displacement of the rotating absorber, where the "clock hypothesis" by Einstein (i.e., the non-reliance of the time rate of any clock on its acceleration [35]) was straightly adopted. Hence, the revealed inequality $k > 1/2$ indicates the presence of some additional energy shift (next to the usual time dilation effect arising from tangential displacement alone) between the emitted and absorbed resonant radiation.

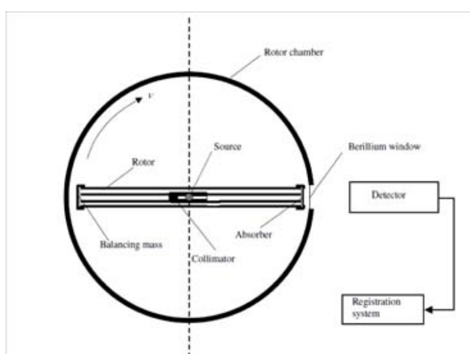

Fig.1.General scheme of Mössbauer experiment sinrotating systems.A source of resonant radiation is located on the rotational axis;an absorber is located on the rotor rim, while a detector of gamma-quanta is placed outside the rotor system,and it counts gamma-quanta at the time moment, when source,absorber and detector are aligned in a straight line. Adapted from [75].

VIII.2.The inertional field equation in Riemannian approximation

We write the inertional field equations in Riemannian approximation in the form

$$R_i^{\mathbf{ac}\,k} = \frac{\eta^{\mathbf{ac}}}{c^2}\left(T_i^{\mathbf{ac}\,k} - \frac{1}{2}T^{\mathbf{ac}}\right), \tag{8.2.1}$$

where $\eta^{\mathbf{ac}}$ is dimensional constant with absolute value equal to 1.
We intoduce now 4-potential $U_\mu, \mu = 0,1,2,3$ in 4-D Minkovski space-time

$$U_\mu = (U_0, U_1, U_2, U_3) \tag{8.2.2}$$

We define a tensor of the accelerations by $a_{\nu\mu} = \partial_\nu U_\mu, \mu, \nu = 0,1,2,3$

$$a_{\nu\mu} = \begin{pmatrix} \frac{\partial U_0}{\partial x_0} & \frac{\partial U_0}{\partial x_1} & \frac{\partial U_0}{\partial x_2} & \frac{\partial U_0}{\partial x_3} \\ \frac{\partial U_1}{\partial x_0} & \frac{\partial U_1}{\partial x_1} & \frac{\partial U_1}{\partial x_2} & \frac{\partial U_1}{\partial x_3} \\ \frac{\partial U_2}{\partial x_0} & \frac{\partial U_2}{\partial x_1} & \frac{\partial U_2}{\partial x_2} & \frac{\partial U_2}{\partial x_3} \\ \frac{\partial U_3}{\partial x_0} & \frac{\partial U_3}{\partial x_1} & \frac{\partial U_3}{\partial x_2} & \frac{\partial U_3}{\partial x_3} \end{pmatrix} \tag{8.2.3}$$

In 3-D we obtain $a_{\nu\mu} = \partial_\nu U_\mu, \mu, \nu = 0,1,2$

$$a_{\nu\mu} = \begin{bmatrix} \frac{\partial U_0}{\partial x_0} & \frac{\partial U_0}{\partial x_1} & \frac{\partial U_0}{\partial x_2} \\ \frac{\partial U_1}{\partial x_0} & \frac{\partial U_1}{\partial x_1} & \frac{\partial U_1}{\partial x_2} \\ \frac{\partial U_2}{\partial x_0} & \frac{\partial U_2}{\partial x_1} & \frac{\partial U_2}{\partial x_2} \end{bmatrix} \tag{8.2.4}$$

In polar coordinates we obtain

$$a_{\nu\mu} = \begin{bmatrix} \frac{\partial U_0}{\partial x_0} & \frac{\partial U_0}{\partial r} & \frac{\partial U_0}{\partial \theta} \\ \frac{\partial U_1}{\partial x_0} & \frac{\partial U_1}{\partial r} & \frac{\partial U_1}{\partial \theta} \\ \frac{\partial U_2}{\partial x_0} & \frac{\partial U_2}{\partial r} & \frac{\partial U_2}{\partial \theta} \end{bmatrix} \tag{8.2.5}$$

We assume now that $U_0 = 0, U_1 = U_1(r), U_2 = U_2(\theta)$. From Eq.(8.2.5) we obtain

$$a_{\nu\mu} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\partial U_1}{\partial r} & 0 \\ 0 & 0 & \frac{\partial U_2}{\partial \theta} \end{bmatrix} \tag{8.2.6}$$

We assume now that $U_2(\theta) = const$

$$a_{\nu\mu} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\partial U_1}{\partial r} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8.2.7}$$

VIII.3.The Mössbauer experiment in a rotating system explained

In the inertional field equations (8.2.1) we now carry out the transition to the limit of nonrelativistic mechanics.This is, for instance, the case in the nonrelativistic rotating system considered above in subsection VIII.1.Thus the acceleration of a particle of zero velocity lies in the direction of increasing r and is equal to

$$a = \omega^2 r. \tag{8.3.1}$$

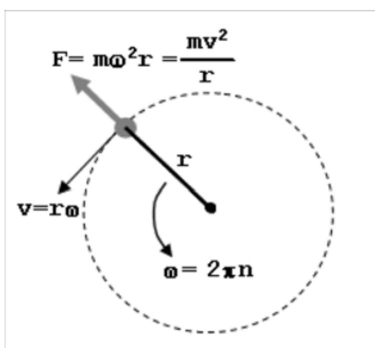

Fig.2.

$$U_1 = \omega^2 r^2/2. \tag{8.3.2}$$

This formula (8.3.2) is in accordance with the usual expression for the centrifugal force. We remind that the expression for the component $g_{00}^{\mathbf{ac}}$ of the metric tensor (the only one which we need) was found, for the limiting case which we are considering, in subsection II

$$g_{00}^{\mathbf{ac}}(r) = 1 + \frac{2\Phi^{\mathbf{ac}}(r)}{c^2}. \tag{8.3.3}$$

Further, we can use for the components of the inertion tensor the expression (8.2.7), where $U_1 = \omega^2 r^2/2$. Of all the components $T_i^{\mathbf{ac}\,k}$, there thus remains only

$$T_1^{\mathbf{ac}\,1} = \frac{1}{2}\omega^2 r. \tag{8.3.4}$$

The scalar $T^{\mathbf{ac}} = T_i^{\mathbf{ac}\,i}$ will be equal to the value $T_1^{\mathbf{ac}\,1} = \partial U_1/\partial r = \omega^2 r$.
We write the field equations in the form (8.2.1). For $i = k = 0$ we get

$$R_0^{\mathbf{ac}\,0} = -\frac{1}{2c^2}\omega^2 r \tag{8.3.5}$$

and $i = k = 1$ we get

$$R_1^{\mathbf{ac}\,1} = \frac{1}{2c^2}\omega^2 r. \tag{8.3.6}$$

Remark 8.3.1. Note that in the approximation we are considering all the other equations vanish identically.

Remark 8.3.2. For the calculation of $R_0^{\mathbf{ac}\,0}$ from the canonical general formula, we note that terms containing derivatives of the quantities Γ_{ki}^i are in every case quantities of the second order. Terms containing derivatives with respect to $x^0 = ct$ are small (compared with terms with derivatives with respect to the coordinates $x^\alpha, \alpha = 1,2,3$) since they contain extra powers of $1/c$. As a result, there remains

$$R_{00}^{\mathbf{ac}} = R_0^{\mathbf{ac}\,0} = \partial\Gamma_{00}^\alpha/\partial x^\alpha, \tag{8.3.7}$$

where

$$\Gamma_{00}^\alpha \simeq -\frac{1}{2}g^{\mathbf{ac}\,\alpha\beta}\frac{\partial g_{00}^{\mathbf{ac}}}{\partial x^\beta}. \tag{8.3.8}$$

Substituting (8.3.8) into (8.3.7) we get

$$R_0^{\mathbf{ac}\,0} \simeq \frac{1}{c^2}\Delta\Phi^{\mathbf{ac}}(r) = -\frac{1}{2c^2}\omega^2 r. \tag{8.3.9}$$

Finally we obtain radial Poisson equation

$$\frac{d(r\Phi^{\mathbf{ac}}(r))}{dr} = -\frac{1}{2c^2}\omega^2 r. \tag{8.3.10}$$

By integration one obtains

$$\Phi^{\mathbf{ac}}(r) = -\frac{1}{8c^2}\omega^2 r^2. \tag{8.3.11}$$

Substituting (8.3.11) into (8.3.3) we get

$$g_{00}^{\mathbf{ac}}(r) = 1 + \frac{2\Phi^{\mathbf{ac}}(r)}{c^2} = 1 - \frac{1}{4c^2}\omega^2 r^2. \tag{8.3.12}$$

Suppose that light flashes are emitted from a point $r = r_1$ at an interval Δt. The field being static, the flashes will reach the observer at $r = r_2$ after the same interval Δt. The ratio of the proper time intervals at these two points is

$$\frac{\Delta\tau_1}{\Delta\tau_2} = \sqrt{\frac{g_{00}^{\mathbf{ac}}(r_1)}{g_{00}^{\mathbf{ac}}(r_2)}}\,. \tag{8.3.13}$$

Hence, the ratio of frequences is

$$\frac{\omega_1}{\omega_2} = \frac{\Delta\tau_2}{\Delta\tau_1} = \sqrt{\frac{g^{\mathbf{ac}}_{00}(r_2)}{g^{\mathbf{ac}}_{00}(r_1)}} = \sqrt{\frac{1 - \dfrac{1}{4c^2}\omega^2 r_2^2}{1 - \dfrac{1}{4c^2}\omega^2 r_1^2}}\,. \tag{8.3.14}$$

Substituting $r_1 = 0$ into (8.3.14) we get

$$\frac{\omega_1}{\omega_2} = \frac{\Delta\tau_2}{\Delta\tau_1} = \sqrt{\frac{g^{\mathbf{ac}}_{00}(r_2)}{g^{\mathbf{ac}}_{00}(0)}} = \sqrt{1 - \frac{1}{4c^2}\omega^2 r_2^2} = 1 - \frac{1}{8c^2}\omega^2 r_2^2 = 1 - \frac{u^2}{8c^2}. \tag{8.3.15}$$

Therefore

$$\Delta E/E = -\frac{1}{2}\frac{u^2}{c^2} - \frac{u^2}{8c^2} = -\left(\frac{1}{2} + \frac{1}{8}\right)\frac{u^2}{c^2} = -0.625\frac{u^2}{c^2}\,. \tag{8.3.16}$$

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