The Lorentz Transformation from Light-Speed Invariance Alone

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Abstract

The derivation of the Lorentz transformation normally rests on two a priori demands, namely that reversing the direction of the transformation's constant-velocity boost inverts the transformation, and that the transformation leaves light-speed invariant. It is notable, however, that the simple light-clock concept, which is rooted entirely in light-speed invariance, immediately implies reciprocal time dilation, and it has been demonstrated that reciprocal time dilation implies length contraction. Reciprocal time dilation and length contraction are the pivotal consequences of the Lorentz transformation, so demanding only inertial transformation-invariance of light-speed apparently already uniquely produces the Lorentz transformation. We show that the demand of x-direction inertial transformation-invariance of the light-cone space-time locus uniquely produces the x-direction Lorentz transformation, and also that the demand of x-direction inertial transformation-invariance of light speed itself has the same consequence; in the latter case the velocity, instead of the space-time, version of the general x-direction inertial transformation must be used. The x-direction Lorentz transformation is also uniquely produced by the reciprocal time dilation and length contraction implications of light clocks.

Introduction

The Lorentz transformation is normally derived on the basis of two a priori demands, namely that reversal of the direction of the transformation's constant-velocity boost inverts the transformation, and that the transformation leaves light-speed invariant. The Galilean transformation *shares* the "inversion by boost reversal" property of the Lorentz transformation, but leaves *time*, rather than light-speed, invariant. The version which transforms the inertial frame of reference by the addition of the x-direction constant-velocity boost $(v, 0, 0)$ is,

$$(t',\, x',\, y',\, z') = (t,\, (x - vt),\, y,\, z). \tag{1a}$$

Since $t' = t$, it clearly leaves *time* invariant. To grasp the "inversion by boost reversal" property *incorporated* into the Eq. (1a) Galilean transformation, we must *invert* it, which yields,

$$(t,\, x,\, y,\, z) = (t',\, (x' + vt')\,,\, y',\, z')\,. \tag{1b}$$

We see that this *inverse* of the Eq. (1a) Galilean transformation is *also* a Galilean transformation, one that transforms the inertial frame of reference *by the addition of the x-direction constant-velocity boost* $(-v, 0, 0)$, which is equal in magnitude and opposite in direction to the original boost velocity $(v, 0, 0)$. Thus *if the "inversion by boost reversal" property holds*, observer B, *who is at rest in a frame of reference that is moving at velocity* $(v, 0, 0)$ *relative to observer* A, makes *the transformation of relative velocity* $(-v, 0, 0)$ from his own (primed) space-time coordinates to understand what observer A perceives in *his* (unprimed) coordinates; of course observer A makes *the transformation of relative velocity* $(v, 0, 0)$ from his own (unprimed) coordinates to understand what observer B perceives in *his* (primed) coordinates.

Special relativity and Galilean relativity have the "inversion by boost reversal" property *in common*, but the Galilean transformations *leave time invariant* (i.e., $t' = t$), whereas Lorentz transformations *leave the speed of light invariant*. A straightforward consequence of the inertial transformation-invariance of light speed is the inertial transformation-invariance of the "light cone" space-time locus $x^2 + y^2 + z^2 = (ct)^2$, which is the locus of a spherical shell of light whose radius grows at the rate c from a light pulse of arbitrarily small duration and extent at the space-time point $(t = 0,\ x = 0,\ y = 0,\ z = 0)$; the *statement* of the inertial transformation-invariance of the "light cone" space-time locus is,

$$\text{if } x^2 + y^2 + z^2 = (ct)^2, \text{ then } (x')^2 + (y')^2 + (z')^2 = (ct')^2. \tag{1c}$$

We will see that the Eq. (1c) demand of inertial transformation-invariance of the light-cone space-time locus *by itself* uniquely yields the Lorentz transformation; *no separate demand for the "inversion by boost reversal" property of that transformation is needed*.

But *before* we formally derive the Lorentz transformation solely from the Eq. (1c) demand of inertial transformation-invariance of the light-cone locus, we peruse a less technical, more readily visualized consequence of light-speed invariance by itself, namely the increase in the"tick" time interval of the conceptual

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light clock [1, 2] when it is in motion at constant velocity $\mathbf{v}$. The effect is due to invariance of light-speed in conjunction with the patently longer path which the clock's light traverses between "ticks" when the clock is in motion; the light path is lengthened by the celebrated time dilation factor $(1-(|\mathbf{v}|/c)^2)^{-\frac{1}{2}}$. Furthermore, if two observers A and B who travel at relative constant velocity $\mathbf{v}$ *each* have a light clock, it is *obvious* that *each one* will perceive the *other's* light clock "tick" more slowly *by precisely this factor* than his *own* light clock which is *stationary* with respect to him. Thus for light clocks *it is obvious that time dilation is reciprocal between inertial frames*; this *particular* aspect of "inversion by boost reversal", namely that one merely takes $\mathbf{v} \to -\mathbf{v}$ in order to switch a pair of inertial frames, *is simply part and parcel of the impact of light-speed invariance on the transformation of time between inertial frames*. Incidentally, the reciprocity of time dilation between inertial frames is counterintuitive *not because of its nature* but because terrestrial creatures *have had no long-term ordinary experience of it*. Distance/size reciprocity, namely that *each* of two observers perceives *the other one* to decrease in size with distance, *isn't* counterintuitive because it has *for ages* been part of the ordinary experience of terrestrial creatures. However, at 11.2 km/s (about 40,000 km/hour) an object *could escape altogether from the earth's gravitational influence*, yet *at that extra-terrestrial speed* the time dilation of a light clock *is still only one part in a billion*!

It has been demonstrated that *reciprocal* time dilation implies *length contraction* (where the length contraction factor $(1-(|\mathbf{v}|/c)^2)^{\frac{1}{2}}$ is the *inverse* of the time dilation factor $(1-(|\mathbf{v}|/c)^2)^{-\frac{1}{2}}$) [3]. Thus informal perusal of the implications of the light clock strongly suggests that light-speed invariance *by itself* produces both reciprocal time dilation and length contraction, *which are the pivotal consequences of the Lorentz transformation*. Indeed we shall now show that the Eq. (1c) transformation-invariance of the light-cone space-time locus, when imposed *by itself* on the completely general x-direction space-time inertial transformation, *compels it to uniquely be the x-direction Lorentz transformation*.

The general x-direction inertial transformation

The *most general* homogeneously linear space-time transformation that is nontrivial *only* for the (t, x) pair has the four-parameter form,

$$(t',\, x',\, y',\, z') = \left(\gamma_0 \left(t - \left(v_0/c^2\right) x\right),\, \gamma(x - vt),\, y,\, z\right), \tag{2a}$$

where γ_0 and γ are dimensionless parameters which are independent of the value of (t, x, y, z), while v_0 and v are parameters that have the dimension of velocity and are likewise independent of the value of (t, x, y, z). The homogeneous linearity of Eq. (2a) ensures coincidence of the space-time coordinate origins, namely,

$$(t = 0,\, x = 0,\, y = 0,\, z = 0) \text{ transforms to } (t' = 0,\, x' = 0,\, y' = 0,\, z' = 0). \tag{2b}$$

The transformation of *velocity* which *corresponds* to the four-parameter general x-direction homogeneously linear transformation of *space-time* given by Eq. (2a) is,

$$(dx'/dt',\, dy'/dt',\, dz'/dt') = \frac{(dx'/dt,\, dy'/dt,\, dz'/dt)}{dt'/dt} = \frac{(\gamma((dx/dt) - v),\, dy/dt,\, dz/dt)}{\gamma_0 \left(1 - (v_0/c^2)\,(dx/dt)\right)}. \tag{3a}$$

Eq. (3a) shows that the transformation of *velocity* is in general a *rational* transformation rather than a *linear* one. In order to *ensure* that the rational *velocity* transformation given by Eq. (3a) *is well-defined*, we impose the following *two restrictions*,

$$\gamma_0 \neq 0, \tag{3b}$$

and,

$$|dx/dt| < \left(c^2/|v_0|\right). \tag{3c}$$

We now take note of *a key property* of the Eq. (3a) *transformation of velocity* (which of course *corresponds* to the Eq. (2a) *transformation of space-time*), namely,

$$(dx/dt,\, dy/dt,\, dz/dt) = (v, 0, 0) \text{ implies that } (dx'/dt',\, dy'/dt',\, dz'/dt') = (0, 0, 0). \tag{3d}$$

The result given by Eq. (3d) shows that the four-parameter general x-direction transformation described by Eq. (2a) or (3a) *expressly compensates for the x-direction constant velocity* $(v, 0, 0)$. Therefore, as long as,

$$|v| < \left(c^2/|v_0|\right), \tag{3e}$$

in accord with the Eq. (3c) restriction, the x-direction constant velocity $(v, 0, 0)$ ought to be identifiable as the *intrinsic* Eq. (2a) or (3a) transformation "boost" to the inertial frame of reference. There is one additional caveat, however: a *zero-velocity "boost"* to the inertial frame of reference *ought not to transform the space-time coordinates nor the velocities at all*. Therefore the general x-direction transformation must be *the identity transformation* when its intrinsic x-direction velocity parameter v equals zero. That reduction to *the identity transformation* when $v = 0$ clearly will be the case for *both* the Eq. (2a) general x-direction homogeneously linear space-time transformation *and* for its Eq. (3a) velocity counterpart if and only if,

$$\gamma_0(v=0) = 1,\ \ v_0(v=0) = 0 \ \text{ and } \ \gamma(v=0) = 1. \tag{4}$$

The Eq. (1a) Galilean transformation *manifestly obeys the rule set out in* Eq. (4). In fact, the Eq. (4) rule compels any physically legitimate x-direction homogeneously linear space-time transformation whose γ_0, v_0 and γ parameters *have completely fixed numerical values* to be *precisely the Galilean transformation*.

The inertial transformation that light-cone invariance by itself imposes

We *can't* extract *consequences* of the Eq. (1c) statement of light-cone locus transformation-invariance *directly* from Eq. (2a); we need to pare Eq. (2a) down to a form which involves *both* the transformed light-cone locus algebraic entity $(x')^2 + (y')^2 + (z')^2 - (ct')^2$ *and* its untransformed counterpart $x^2 + y^2 + z^2 - (ct)^2$. That is readily done, with the result,

$$\begin{gathered}(x')^2 + (y')^2 + (z')^2 - (ct')^2 = \gamma^2(x - vt)^2 + y^2 + z^2 - \gamma_0^2\left(ct - (v_0/c)\,x\right)^2 = \\ x^2 + y^2 + z^2 - (ct)^2 + \gamma^2(x - vt)^2 - \gamma_0^2\left(ct - (v_0/c)\,x\right)^2 - x^2 + (ct)^2.\end{gathered} \tag{5a}$$

The *imposition* of the Eq. (1c) requirement on Eq. (5a) *implies the particular consequence of the vanishing of both of the light-cone locus algebraic entities* $x^2 + y^2 + z^2 - (ct)^2$ *and* $(x')^2 + (y')^2 + (z')^2 - (ct')^2$ *in* Eq. (5a), the result of which is,

$$0 = \gamma^2(x - vt)^2 - \gamma_0^2\left(ct - (v_0/c)\,x\right)^2 - x^2 + (ct)^2. \tag{5b}$$

What we want to obtain from Eq. (5b) are *the constraints which it imposes on the three parameters* γ_0, v_0 *and* γ *that determine the* Eq. (2a) *general* x*-direction inertial transformation*. It is clearly feasible *to present* Eq. (5b) *as the vanishing of a linear combination of the three variable-value entities* x^2, xt *and* t^2, *which are clearly linearly independent*. Therefore *the three coefficients of the three linearly independent entities* x^2, xt *and* t^2 *must vanish*, so Eq. (5b) *produces three equalities which involve* c^2, γ_0^2, v_0, γ^2 *and* v. The *presentation of* Eq. (5b) *as a vanishing linear combination of the three linearly independent entities* x^2, xt *and* t^2 *and their unique coefficients is*,

$$\left(-\gamma_0^2(v_0/c)^2 + \gamma^2 - 1\right)x^2 + 2\left(\gamma_0^2 v_0 - \gamma^2 v\right)xt + \left(-\gamma_0^2 + \gamma^2(v/c)^2 + 1\right)c^2t^2 = 0. \tag{5c}$$

Eq. (5c) *clearly implies that* γ_0^2 *satisfies the following three equalities*,

$$\gamma_0^2 = (\gamma^2 - 1)/\left(v_0/c\right)^2 = \gamma^2\left(v/v_0\right) = \gamma^2(v/c)^2 + 1, \tag{5d}$$

which, in turn, imply that γ^2 satisfies *the following two equalities*,

$$\gamma^2 = \left(1 - \left(v_0 v/c^2\right)\right)^{-1} = \left((v/v_0) - (v/c)^2\right)^{-1}, \tag{5e}$$

which yield the following quadratic equation for the transformation parameter v_0 in terms of v and c,

$$\left(v_0\right)^2 - v_0\left(v + \left(c^2/v\right)\right) + c^2 = 0, \tag{5f}$$

whose left-hand side is readily *factored* as follows,

$$\left(v_0 - v\right)\left(v_0 - \left(c^2/v\right)\right) = 0, \tag{5g}$$

revealing the equation's two roots,

$$v_0 = v \text{ and } v_0 = \left(c^2/v\right). \tag{5h}$$

Inserting the root $v_0 = \left(c^2/v\right)$ into Eq. (5e) makes γ^2 *equal to the undefined inverse of zero*. Therefore *the only applicable root of* Eq. (5h) *is*,

$$v_0 = v, \tag{5i}$$

which obeys the Eq. (4) rule that $v_0(v = 0) = 0$. Inserted into Eq. (3e), $v_0 = v$ yields *the restriction*,

$$|v| < c, \tag{5j}$$

and inserted into Eq. (3c) it yields $|dx/dt| < (c^2/|v|)$, which together with Eq. (5j) implies *the restriction*,

$$|dx/dt| \leq c. \tag{5k}$$

Inserted into Eq. (5e), $v_0 = v$ yields,

$$\gamma^2 = \left(1 - (v/c)^2\right)^{-1}, \tag{5l}$$

for which the Eq. (5j) *restriction is needed to ensure that the parameter* γ *is a real-valued finite number*. Inserting Eq. (5l) and $v_0 = v$ into Eq. (5d) yields,

$$\gamma_0^2 = \left(1 - (v/c)^2\right)^{-1}. \tag{5m}$$

Although Eq. (5m) is compatible with,

$$\gamma_0 = \pm\left(1 - (v/c)^2\right)^{-\frac{1}{2}}, \tag{5n}$$

the Eq. (4) rule that $\gamma_0(v = 0) = 1$, *selects* $\pm = +$ in Eq. (5n). Likewise, although Eq. (5l) is compatible with,

$$\gamma = \pm\left(1 - (v/c)^2\right)^{-\frac{1}{2}}, \tag{5o}$$

Eq. (4), which requires that $\gamma(v = 0) = 1$, *selects* $\pm = +$ in Eq. (5o). We thus see that the Eq. (1c) inertial transformation-invariance of the light-cone locus, together with the Eq. (4) rule that an inertial-frame constant-velocity "boost" transformation must be the identity transformation when that "boost" velocity vanishes altogether, yields the three *completely-determined* x*-direction inertial transformation parameter values*,

$$\gamma_0 = \left(1 - (v/c)^2\right)^{-\frac{1}{2}},\ \ v_0 = v\ \ \text{and}\ \ \gamma = \left(1 - (v/c)^2\right)^{-\frac{1}{2}}, \tag{6a}$$

which, on insertion into the Eq. (2a) general x-direction homogeneously linear inertial transformation yield,

$$(t',\, x',\, y',\, z') = \left(\gamma\left(t - \left(v/c^2\right)x\right),\, \gamma(x - vt),\, y,\, z\right)\ \ \text{where}\ \ \gamma = \left(1 - (v/c)^2\right)^{-\frac{1}{2}}. \tag{6b}$$

Eq. (6b) *is precisely the* x*-direction space-time Lorentz transformation* [4]. Its *inverse* transformation is readily verified to be,

$$(t,\, x,\, y,\, z) = \left(\gamma\left(t' + \left(v/c^2\right)x'\right),\, \gamma(x' + vt'),\, y',\, z'\right), \tag{6c}$$

which *differs* from the *direct* Lorentz transformation of Eq. (6b) *only* in that $v \to -v$. Thus the Lorentz transformation *indeed conforms with the "inversion by boost reversal" property that was pointed out below* Eq. (1b) *in connection with the Galilean transformation*.

The above derivation of the Lorentz transformation, however, *manifestly doesn't assume that "inversion by boost reversal" holds*; it assumes *only* the Eq. (1c) inertial transformation-invariance of the light-cone space-time locus, which it *imposes* on the Eq. (5a) specialized consequence of the Eq. (2a) *general* x*-direction inertial transformation*.

The inertial transformation that light-speed invariance by itself imposes

Instead of demanding the Eq. (1c) inertial transformation-invariance of the light-cone space-time locus in conjunction with an appropriately specialized consequence of the Eq. (2a) general x-direction inertial space-time transformation (i.e., Eq. (5a)), *we can equally well simply demand inertial transformation-invariance of light-speed itself*, namely,

$$\text{if } (dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2 = c^2, \text{ then } (dx'/dt')^2 + (dy'/dt')^2 + (dz'/dt')^2 = c^2, \tag{7a}$$

in conjunction with an appropriately specialized consequence of the Eq. (3a) general x-direction inertial *velocity* transformation. Clearly, in order to be *germane* to the demand made by Eq. (7a), that specialized consequence of Eq. (3a) must *explicitly* include *both* the transformed speed squared $(dx'/dt')^2 + (dy'/dt')^2 + (dz'/dt')^2$ *and* the untransformed speed squared $(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2$. Such a specialized consequence is readily obtained from Eq. (3a), namely,

$$\begin{gathered}(dx'/dt')^2 + (dy'/dt')^2 + (dz'/dt')^2 = \frac{\gamma^2((dx/dt) - v)^2 + (dy/dt)^2 + (dz/dt)^2}{\gamma_0^2\left(1 - (v_0/c^2)(dx/dt)\right)^2} = \\ \frac{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2 + \gamma^2((dx/dt) - v)^2 - (dx/dt)^2}{\gamma_0^2\left(1 - (v_0/c^2)(dx/dt)\right)^2}.\end{gathered} \tag{7b}$$

Application of the the Eq. (7a) demand of inertial transformation-invariance *of light-speed itself* to Eq. (7b) clearly *has the particular consequence of replacing both* $(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2$ *and* $(dx'/dt')^2 + (dy'/dt')^2 + (dz'/dt')^2$ *in* Eq. (7b) *by* c^2, the result of which is,

$$-c^2\gamma_0^2\left(1 - \left(v_0/c^2\right)(dx/dt)\right)^2 + c^2 + \gamma^2((dx/dt) - v)^2 - (dx/dt)^2 = 0, \tag{7c}$$

What we want to obtain from Eq. (7c) are *the constraints which it imposes on the three parameters* γ_0*,* v_0 *and* γ *that determine the general* x*-direction inertial transformation which is set out in* Eq. (2a)/(3a). Eq. (7c) *can obviously be presented as the vanishing of a second-order polynomial in the variable-value entity* (dx/dt)*, and of course different powers of* (dx/dt) *are linearly independent*. Therefore *the three coefficients of that second-order polynomial in* (dx/dt) *must vanish*, so Eq. (7c) *produces three equalities which involve* c^2*,* γ_0^2*,* v_0*,* γ^2 *and* v. The presentation of Eq. (7c) as a vanishing second-order polynomial in (dx/dt), *organized into uniquely-presented linearly independent powers of* (dx/dt) *and their coefficients is*,

$$\left(-\gamma_0^2(v_0/c)^2 + \gamma^2 - 1\right)(dx/dt)^2 + 2\left(\gamma_0^2 v_0 - \gamma^2 v\right)(dx/dt) + \left(-\gamma_0^2 + \gamma^2(v/c)^2 + 1\right)c^2 = 0. \tag{7d}$$

Eq. (7d) *clearly implies that* γ_0^2 *satisfies the following three equalities*,

$$\gamma_0^2 = (\gamma^2 - 1)/\left(v_0/c\right)^2 = \gamma^2\left(v/v_0\right) = \gamma^2(v/c)^2 + 1, \tag{7e}$$

The three equalities of Eq. (7e) *are identical to those of* Eq. (5d), which implies that the Eq. (7a) direct demand of the inertial transformation-invariance of light-speed itself *uniquely implies the* Eq. (6b) x*-direction Lorentz transformation in precisely the same manner as the* Eq. (1c) *demand of the inertial transformation-invariance of the light-cone space-time locus uniquely implies the* Eq. (6b) x*-direction Lorentz transformation*.

The Eq. (7a) demand of the inertial transformation-invariance of light-speed itself *also manifestly doesn't assume that "inversion of the transformation by boost reversal" holds*; that property of the Lorentz transformation *simply doesn't need to be separately demanded if inertial transformation-invariance of light-speed itself or of the light-cone space-time locus is demanded*. This state of affairs dovetails with our informal discussion in the Introduction of how all the salient implications of Lorentz transformation can be induced from reflecting on the implications of the light clock. We now proceed to fill in some of the particulars which weren't gone into in detail in that informal discussion of the light clock.

The inertial transformation that light clocks by themselves impose

When a light clock of natural at-rest length l_0 has velocity $(v, 0, 0)$ in the x-direction, *and is oriented perpendicular to that velocity*, it is of course *well-known* that its natural at-rest "tick" time interval $\Delta t_0 = (2l_0/c)$ is *dilated* by the factor $(1 - (v/c)^2)^{-\frac{1}{2}}$ [1, 2]. We denote the result of that Δt_0 dilation as $(\Delta t_0)'$,

$$(\Delta t_0)' = \left(1 - (v/c)^2\right)^{-\frac{1}{2}}\Delta t_0 = \left(1 - (v/c)^2\right)^{-\frac{1}{2}}(2l_0/c). \tag{8a}$$

Furthermore, as was emphasized in the Introduction, the time dilation factor $(1 - (v/c)^2)^{-\frac{1}{2}}$ *is patently reciprocal* between two inertial observers, *given that each one has a light clock*. A *crucial consequence of reciprocal time dilation* is the phenomenon *of length contraction* [3]. We now paraphrase the simple "time-interval for rod passage" calculation *based on reciprocal time dilation* which is given in Ref. [3] to obtain the contraction of a rod of natural at-rest length l_0 which is oriented lengthwise along the x-axis and is moving along that axis with constant velocity $(v, 0, 0)$. Given a mark on the x-axis of the unprimed stationary inertial reference frame, the rod, moving at constant speed $|v|$, will pass over that mark in a certain time interval Δt, as is recorded by the light clock which is stationary in the unprimed inertial reference frame. From that Δt "time-interval for passage over the mark at speed $|v|$", the rod's length $l(l_0)$ in the unprimed stationary inertial reference frame will of course be inferred to be,

$$l(l_0) = |v|\Delta t, \tag{8b}$$

In the primed moving inertial reference frame *in which the rod is at rest*, the *mark* is seen to pass over *that stationary rod of length* l_0 *with exactly the same speed* $|v|$ because of the *reciprocity* of the two inertial reference frames *that is enforced by dual light-clock time-keeping*. The mark's speed $|v|$ implies that the mark's $(\Delta t)'$ "time-interval for passage over the rod of length l_0 at speed $|v|$" is given by,

$$(\Delta t)' = l_0/|v|, \tag{8c}$$

as is recorded by the light clock which is stationary in the primed moving inertial reference frame.

However, time intervals recorded by the light clock which is stationary in the primed moving inertial reference frame *are dilated by the factor* $(1-(v/c)^2)^{-\frac{1}{2}}$ *from the standpoint of the light clock which is stationary in the unprimed stationary inertial reference frame* (e.g., see Eq. (8a) for the Δt_0 case), namely,

$$(\Delta t)' = \left(1-(v/c)^2\right)^{-\frac{1}{2}} \Delta t. \tag{8d}$$

We now eliminate $(\Delta t)'$ between Eqs. (8d) and (8c), and thereby obtain,

$$\Delta t = \left(1-(v/c)^2\right)^{\frac{1}{2}} l_0/|v|. \tag{8e}$$

Insertion of Eq. (8e) into Eq. (8b) yields the celebrated *length contraction factor*,

$$(l(l_0)/l_0) = \left(1-(v/c)^2\right)^{\frac{1}{2}}. \tag{8f}$$

This length contraction result *plus reciprocal time dilation* in fact *yields the Lorentz transformation*.

But *before* we show that light clocks *thus entail the Lorentz transformation*, it is worth showing that *due to the contraction of its natural at-rest length* l_0, a light clock *oriented at any angle* θ *to its x-direction velocity* $(v,0,0)$ has a dilated "tick" time interval $(\Delta t_0)'_\theta$ *which doesn't vary at all with* θ, but is *fixed* to the Eq. (8a) value $(\Delta t_0)' = (1-(v/c)^2)^{-\frac{1}{2}}(2l_0/c)$. The *contracted length* $l_\theta(l_0)$ *of the moving* θ*-oriented light clock* arises *only* from the contracted length $(l_\theta(l_0)\cos\theta)$ *of its x-component*, so its at-rest natural length l_0 is obtained by the following *inverse application to* $l_\theta(l_0)$ of the Eq. (8f) contraction factor $(1-(v/c)^2)^{\frac{1}{2}}$,

$$l_0 = l_\theta(l_0)\left[\left(\cos\theta/\left(1-(v/c)^2\right)^{\frac{1}{2}}\right)^2 + (\sin\theta)^2\right]^{\frac{1}{2}} = l_\theta(l_0)\left[\left(1-(v\sin\theta/c)^2\right)^{\frac{1}{2}}/\left(1-(v/c)^2\right)^{\frac{1}{2}}\right], \tag{9a}$$

which in turn yields the *moving* θ*-oriented* light clock's *contracted length* $l_\theta(l_0)$ in terms of l_0, v and θ,

$$l_\theta(l_0) = \left[\left(1-(v/c)^2\right)^{\frac{1}{2}}/\left(1-(v\sin\theta/c)^2\right)^{\frac{1}{2}}\right](l_0). \tag{9b}$$

The pair of light-pulse traversal times $(\Delta t_0)'^{(\pm)}_\theta$ of the *moving* θ*-oriented length-*$l_\theta(l_0)$ light clock adhere to,

$$\left(c(\Delta t_0)'^{(\pm)}_\theta\right)^2 = \left(l_\theta(l_0)\cos\theta \pm v(\Delta t_0)'^{(\pm)}_\theta\right)^2 + \left(l_\theta(l_0)\sin\theta\right)^2, \tag{9c}$$

an equation pair that yields the following *unique positive pair* of light-pulse traversal times of the light clock,

$$(\Delta t_0)'^{(\pm)}_\theta = \left(\left(1-(v\sin\theta/c)^2\right)^{\frac{1}{2}} \pm (v\cos\theta/c)\right)\left(1-(v/c)^2\right)^{-1}(l_\theta(l_0)/c). \tag{9d}$$

The *traversal-time pair sum* $[(\Delta t_0)'^{(+)}_\theta + (\Delta t_0)'^{(-)}_\theta]$ is the light clock's dilated *"tick" time interval* $(\Delta t_0)'_\theta$,

$$(\Delta t_0)'_\theta = \left(1-(v\sin\theta/c)^2\right)^{\frac{1}{2}}\left(1-(v/c)^2\right)^{-1}(2l_\theta(l_0)/c) = \left(1-(v/c)^2\right)^{-\frac{1}{2}}(2l_0/c) = (\Delta t_0)', \tag{9e}$$

where the last two equalities reflect the insertion of the Eq. (9b) *value of* $l_\theta(l_0)$ and *fusion with* Eq. (8a).

Listing now the facts we *know* about any light clock which has x-direction velocity $(v,0,0)$: (1) its *time dilation factor* $(1-(v/c)^2)^{-\frac{1}{2}}$, (2) the *reciprocity* between inertial frames of that factor and (3) the light-clock imposed *length contraction factor* $(1-(v/c)^2)^{\frac{1}{2}}$. We proceed to *use* these three facts *to evaluate the three* Eq. (2a) *general x-direction inertial-transformation parameters* γ_0*,* v_0 *and* γ *in the case of light clocks*; the specific values of γ_0, v_0 and γ that result in the light-clock case are those of the Lorentz transformation.

We now extract the *general* x-direction *time-interval inertial-transformation factor at fixed untransformed location* from the Eq. (2a) transformation *in terms of its* v*,* γ_0*,* v_0 *and* γ *parameters*, and then *in the case of light clocks we set that general result equal to the light clock time dilation factor* $(1-(v/c)^2)^{-\frac{1}{2}}$. We extract that *general time-interval factor* from Eq. (2a) by noting that *at fixed location in the unprimed space-time coordinates*, namely,

$$\text{when } x_2 = x_1,\ (t'_2 - t'_1) = \gamma_0\,(t_2 - t_1). \tag{10a}$$

Eq. (10a) implies that the *general* x-direction *time-interval inertial-transformation factor at fixed untransformed location* is γ_0, *so for light clocks we set* γ_0 *to the light clock time dilation factor* $(1-(v/c)^2)^{-\frac{1}{2}}$,

$$\gamma_0 = \left(1-(v/c)^2\right)^{-\frac{1}{2}} \text{ in the case of light clocks.} \tag{10b}$$

To obtain the implications of *the reciprocity of* γ_0 we first carry out the *analog* of Eq. (10a) with the *inverse* of the Eq. (2a) transformation, which is,

$$(t,\, x,\, y,\, z) = \left(\frac{t' + (\gamma_0/\gamma)\,(v_0/c^2)\, x'}{\gamma_0\,(1 - (v_0 v/c^2))},\ \frac{x' + (\gamma/\gamma_0)\, v t'}{\gamma\,(1 - (v_0 v/c^2))},\ y',\ z' \right). \tag{10c}$$

The *analog* of Eq. (10a) obtained from Eq. (10c) is that when $x'_2 = x'_1$, $(t_2 - t_1) = [\gamma_0(1-(v_0 v/c^2))]^{-1}(t'_2 - t'_1)$. Thus *the reciprocity of* γ_0 requires that,

$$\gamma_0(-v) = \left[\gamma_0(v)\left(1 - \left(v_0(v)\, v/c^2\right)\right)\right]^{-1}, \tag{10d}$$

which, *together with the* Eq. (10b) *result that* $\gamma_0(-v) = \gamma_0(v) = \left(1 - (v/c)^2\right)^{-\frac{1}{2}}$ *for light clocks yields that,*

$$v_0 = v \text{ in the case of light clocks.} \tag{10e}$$

Having evaluated γ_0 and v_0 for light clocks, we now tackle γ for light clocks by working out the *general* x-direction *length-interval inertial-transformation factor at fixed transformed time* in terms of v, γ_0, v_0 and γ, *which in the case of light clocks we set equal to the light clock length contraction factor* $\left(1 - (v/c)^2\right)^{\frac{1}{2}}$. We *begin* by noting from Eq. (2a) that,

$$(x'_2 - x'_1) = \gamma\left((x_2 - x_1) - v\,(t_2 - t_1)\right). \tag{10f}$$

We *as well* obtain from Eq. (2a) *that at fixed time in the primed space-time coordinates*, namely,

$$\text{when } t'_2 = t'_1,\ \ (t_2 - t_1) = \left(v_0/c^2\right)(x_2 - x_1), \tag{10g}$$

which together with Eq. (10f) yields that,

$$\text{when } t'_2 = t'_1,\ \ (x'_2 - x'_1) = \gamma\left(1 - \left(v_0 v/c^2\right)\right)(x_2 - x_1). \tag{10h}$$

Eq. (10h) implies that the *general* x-direction *length-interval inertial-transformation factor at fixed transformed time* is $\gamma(1 - (v_0 v/c^2))$. Mindful of the Eq. (10e) result that $v_0 = v$ in the case of light clocks, *for the light-clock case we now set* $\gamma(1 - (v_0 v/c^2))$*, with* $v_0 = v$*, equal to the light-clock length contraction factor* $(1 - (v/c)^2)^{\frac{1}{2}}$*, and thereby obtain,*

$$\gamma = \left(1 - (v/c)^2\right)^{-\frac{1}{2}} \text{ in the case of light clocks.} \tag{10i}$$

From Eqs. (10b), (10e) and (10i), we have that in the case of light clocks,

$$\gamma_0 = \left(1 - (v/c)^2\right)^{-\frac{1}{2}},\ \ v_0 = v \ \ \text{and}\ \ \gamma = \left(1 - (v/c)^2\right)^{-\frac{1}{2}}, \tag{10j}$$

which is exactly the same as Eq. (6a), and therefore of course produces the Lorentz transformation of Eq. (6b). Thus *light clocks uniquely yield the Lorentz transformation*.

We have now seen thrice that *whatever encompasses the inertial transformation-invariance of the speed of light*, such as the inertial transformation-invariance of the light-cone space-time locus, *enough accumulated consequences* of inertial transformation-invariant light-speed in the case of light clocks, or outright inertial transformation-invariance of the speed of light itself *automatically uniquely yields the Lorentz transformation without need for any additional assumption*.

The Lorentz invariance of the Minkowski quadratic form

We note that a very important characteristic of the Eq. (6b) special-relativistic Lorentz space-time transformation is that *it leaves the Minkowski quadratic form* $(ct)^2 - x^2 - y^2 - z^2$ *outright invariant regardless of what value that quadratic form happens to have*, namely,

$$\begin{gathered}(ct')^2 - (x')^2 - (y')^2 - (z')^2 = \gamma^2\left[(ct - (v/c)x)^2 - (x - vt)^2\right] - y^2 - z^2 = \\ \left[1/\left(1 - (v/c)^2\right)\right]\left[(ct)^2\left(1 - (v/c)^2\right) - x^2\left(1 - (v/c)^2\right)\right] - y^2 - z^2 = (ct)^2 - x^2 - y^2 - z^2.\end{gathered} \tag{11}$$

In other words, the Lorentz transformation leaves the Minkowski quadratic form $(ct)^2 - x^2 - y^2 - z^2$ invariant *whether it has the light-cone value of zero or any other value—and that is the case notwithstanding that derivation of the Lorentz transformation needs only to demand transformation-invariance of the light-cone value of zero of* $(ct)^2 - x^2 - y^2 - z^2$.

References

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