On the general solution to the Bratu and generalized Bratu equations

L. H. Koudahoun [a], J. Akande [a], D.K.K. Adjaï [a],
Y.J.F. Kpomahou [b], M. Nonti [a] and M. D. Monsia [a 1]

[a] *Department of Physics, University of Abomey- Calavi,*
Abomey-Calavi, 01.B.P.526, Cotonou, BENIN
E-mail: monsiadelphin@yahoo.fr

[b] *Department of Industrial and Technical Sciences, ENSET-Lokossa,*
University of Abomey, Abomey, BENIN
E-mail : kpomaf@yahoo.fr

This work shows that the Bratu equation belongs to a general class of Liénard-type equations for which the general solution may be exactly and explicitly computed within the framework of the generalized Sundman transformation. In this perspective the exact solution of the Bratu nonlinear two-point boundary value problem as well as of some well-known Bratu-type problems have been determined.

keywords: Bratu equation, boundary value problem, initial value problem, general solution, generalized Sundman transformation.

Introduction

The mathematical modeling of many processes in physics as in other applied disciplines of science is often achieved in terms of differential equations supplemented by initial or boundary conditions. An initial problem requires the specification of the solution at a point, while several points of specification are needed for the solution to a boundary value problem. In this regard, solving explicitly and exactly a boundary value problem becomes more mathematically complicated than a problem with initial conditions. This complication is accentuated when there is a nonlinear boundary value problem since, up to now there is no explicit and exact general method that can account for the individual behavior of each nonlinear process. The Bratu nonlinear two-point boundary value problem is one of those nonlinear problems whose explicit and exact solution for a wide variety of initial and boundary conditions remains very difficult to formulate. The Bratu problem is also one of the most investigated boundary value problems also in mathematics [1, 7]. This problem derives its importance first from the combustion theory where it has been used for several applications [2, 7] and secondly, from the fact that its exact solution is well known [1, 7] so that it has been widely applied to test the accuracy and efficiency of many approximate methods of different complexity like the Adomian decomposition approach [7], the Legendre wavelet method [5], the perturbation technique [6] and the virial theorem [4]. This solution exhibits also a bifurcation pattern, which only characterizes nonlinear differential equations . The one-dimensional Bratu boundary value problem may be written [3, 7]

$$u''(x) + \lambda e^{u(x)} = 0 \tag{0.1}$$

where

$$u(0) = u(1) = 0 \tag{0.2}$$

and λ is a constant. The Bratu type initial value problems have also been examined by a number of authors [5, 7]. Such an importance motivates the reason to investigate the explicit and exact general solution to

[1]corresponding author. E.mail: monsiadelphin@yahoo.fr

the Bratu equation (0.1). The usual way to solve a boundary or initial value problem consists of computing first the general solution to the differential equation and secondly of finding the arbitrary parameters by applying the boundary or initial conditions [8]. So, several methods are developed in mathematics for finding explicit and exact solutions to nonlinear differential equations. In this way the variables change like the point transformation, the contact transformation and the generalized Sundman transformation may be mentioned. The generalized Sundman linearization theory has been recently the object of many applications so that new first integrals [9] and general periodic solutions [10, 11] for well-known nonlinear differential equations have been computed explicitly. As such the generalized Sundman transformation recently developed by some authors of this work [10, 11] has successfully been applied to determine the explicit and exact general periodic solutions to various types of Liénard nonlinear differential equations. In this regard a general class of quadratic Liénard type equations whose exact general solutions are trigonometric functions has been for the first time highlighted by the application of the generalized Sundman transformation under consideration [10,11]. It has been, particularly, possible to show in this context that the well-known Painlevé-Gambier XVIII equation and its inverted version admit, for the first time, a trigonometric function as explicit and exact general periodic solution but with amplitude dependent frequency [10]. This is also shown for a reduced Painlevé-Gambier XII equation under an appropriate parametric choice but with a shift factor [11]. The same generalized Sundman transformation has been used to compute successfully the explicit and exact general periodic solutions of the famous cubic Duffing equation in terms of Jacobian elliptic functions, as expected. In spite of this progress in explicit and exact methods for solving nonlinear differential equations, it seems that a century after, the general solution to the Bratu differential equation (0.1) from which the exact solution to the Bratu initial and boundary value problems may be computed by determination of integration constants, is not, unfortunately, computed in the literature explicitly and exactly in a straightforward manner [12]. In this perspective, since it is almost impossible to find in the literature an explicit and exact general solution to the Bratu differential equation (0.1) in a straightforward fashion, there appears convenient to examine, regarding the above, such a problem of finding the general solution by means of the generalized Sundman transformation previously mentioned [10, 11]. In other words, within the framework of the generalized Sundman transformation under consideration, the problem of interest in this paper is to ask whether such a nonlocal transformation may be successfully applied to compute the explicit and exact general solution to the Bratu differential equation, from which the exact solution to the one-dimensional Bratu boundary value problem may be deduced. More precisely, in this context, the following question may be posed: Can we compute the general solution to the Bratu equation (0.1) from which the exact solution to the Bratu boundary value problem may be determined? Such a general solution is of high interest from theoretical point of view since it may allow one not only, to better understand the analytical properties of the Bratu equation under various types of initial and boundary conditions but also to detect the connection between the Bratu equation and other differential equations. From a practical point of view, it may allow the use of the Bratu equation adequately and satisfactorily as a simulation model for a large variety of engineering applications under various types of initial and boundary conditions, and may serve to better test the accuracy and effectiveness of various approximation theories. In this work, it is assumed that such a general solution may be computed explicitly and exactly by the application of the generalized Sundman transformation. To demonstrate, the generalized Sundman transformation theory needed is first reviewed (section 2) and secondly the generalized Bratu equation of interest (section 3) as well as its explicit and exact general solution are established (section 4) such that the well-known exact solution to the one-dimensional Bratu boundary value problem may be deduced (section 5). Finally the explicit and exact solutions to some Bratu type initial and boundary value problems examined by Wazwaz [7] using Adomian decomposition method and also by Boyd [13] are easily computed (section 6) so that a discussion of results (section 7) and a conclusion may be addressed.

1 Review of the generalized Sundman linearization theory

In this section the generalized Sundman linearization theory recently introduced by Akande et al. [10] is considered. The application of this linearizing transformation requires to consider the general class of quadratic Liénard type nonlinear differential equations

$$u''(x) + \left(l\frac{g'(u)}{g(u)} - \gamma\varphi'(u)\right)u'^2(x) + a^2\frac{\exp(2\gamma\varphi(u))\int g(u)^l du}{g(u)^l} = 0, \tag{1.1}$$

which may be reduced under the conditions

$$y(\tau) = F(x,u), \quad d\tau = G(x,u)dx, \quad G(x,u)\frac{\partial F(x,u)}{\partial u} \neq 0 \tag{1.2}$$

with

$$F(x,u) = \int g(u)^l du, \qquad G(x,u) = \exp(\gamma\varphi(u))$$

to

$$y''(\tau) + a^2 y(\tau) = 0 \tag{1.3}$$

The equation (1.3) admits the solution

$$y(\tau) = A_0 \sin(a\tau + \alpha) \tag{1.4}$$

where prime denotes ordinary differentiation of the dependent variable with respect to the argument, A_0, α, a, l and γ are arbitrary parameters. The functions $\varphi(u)$ and $g(u) \neq 0$, are arbitrary functions of u. So with that the generalized Bratu equation of interest may be established.

2 Generalized Bratu equation

This section is devoted to carry out the generalized Bratu equation under question. To that end it is required that

$$l\frac{g'(u)}{g(u)} - \gamma\varphi'(u) = 0, \qquad l \neq 0 \tag{2.1}$$

that is

$$g(u) = e^{\frac{\gamma}{l}\varphi(u)} \tag{2.2}$$

such that (1.1) becomes

$$u''(x) + a^2 e^{\gamma\varphi(u)} \int e^{\gamma\varphi(u)} du = 0 \tag{2.3}$$

The application of $\varphi(u) = u$, to (2.3) yields as equation

$$u''(x) + \frac{a^2}{\gamma} e^{2\gamma u(x)} = 0 \tag{2.4}$$

The equation (2.4) is the desired generalized Bratu equation. To observe this, it suffices to choose $2\gamma = 1$, and $2a^2 = \lambda$, to obtain [3, 7] the celebrated Bratu equation (0.1). In this regard the explicit and exact general solution in question may, as one can see, be easily computed for various boundary and initial conditions, in other words for all $x \in \mathbb{R}$.

3 General solutions

In this section the general solutions to the Bratu equation and generalized Bratu equation (2.4) are explicitly and exactly computed under the conditions (1.2), such that one may deduce the well-known exact solution to the one-dimensional Bratu boundary value problem. So substituting the equation (2.2) into (1.2) leads to

$$y(\tau) = \frac{1}{\gamma} e^{\gamma u} \tag{3.1}$$

so that the general solution becomes

$$u(x) = \frac{1}{\gamma} \ln\left(\gamma y(\tau)\right) \tag{3.2}$$

Knowing (1.4), the equation (3.2) may be written as

$$u(x) = \frac{1}{\gamma} \ln\left(\gamma A_0 \sin(a\tau + \alpha)\right) \tag{3.3}$$

The problem is now to express $\sin(a\tau + \alpha)$ in terms of x. In this way the preceding relation $d\tau = \exp(\gamma\varphi(u))dx$ reduces to

$$\frac{d\tau}{A_0 \sin(a\tau + \alpha)} = \gamma dx$$

that is

$$\frac{d\tau}{\sin(a\tau + \alpha)} = A_0 \gamma dx \tag{3.4}$$

such that

$$a\tau + \alpha = 2tg^{-1}(K \exp(\gamma a A_0 x)) \tag{3.5}$$

where K is an arbitrary constant. In this context the general solution (3.3) may take the form

$$u(x) = \frac{1}{\gamma} \ln\left\{\gamma A_0 \sin[2tg^{-1}(K \exp(\gamma a A_0 x))]\right\} \tag{3.6}$$

where $\gamma > 0$. The equation (3.6) is the desired explicit and exact general solution to the generalized Bratu equation (2.4) for all $x \in \mathbb{R}$. The parametric choice $\gamma = \frac{1}{2}$, yields the general solution under question to the Bratu equation (0.1), that is

$$u(x) = 2\ln\left\{\frac{A_0}{2}\sin[2tg^{-1}(K\exp(\frac{aA_0}{2}x))]\right\} \tag{3.7}$$

for all $x \in \mathbb{R}$. In this context the integration constants K and A_0 may be determined for various initial and boundary conditions. In other words, the behavior of $u(x)$ depends on these conditions. The objective is now to show that the general solution (3.7) may yield the well-known exact solution to the one-dimensional Bratu boundary value problem under the conditions that $u(0) = u(1) = 0$.

4 Exact solution to the Bratu boundary value problem

This section is devoted to determine the exact solution of the one-dimensional Bratu boundary value problem, in other words to compute the two constants of integration A_0 and K under the conditions that $u(0) = u(1) = 0$. So the application of $u(0) = 0$ leads to

$$\ln\left\{\gamma A_0 \sin[2tg^{-1}(K)]\right\} = 0$$

that is

$$\sin(2tg^{-1}(K)) = \frac{1}{\gamma A_0} \tag{4.1}$$

Knowing that

$$\sin(2tg^{-1}(K)) = \frac{2K}{1+K^2} \tag{4.2}$$

the equation (4.1) becomes

$$\frac{2K}{1+K^2} = \frac{1}{\gamma A_0} \tag{4.3}$$

On the other hand the application of the condition $u(1) = 0$, gives

$$\ln\left\{\gamma A_0 \sin[2tg^{-1}(K\exp(\gamma a A_0))]\right\} = 0$$

that is

$$\sin[2tg^{-1}(K\exp(\gamma a A_0))] = \frac{1}{\gamma A_0} \tag{4.4}$$

which may be written in the form

$$\frac{2Ke^{\gamma a A_0}}{1+K^2e^{2\gamma a A_0}} = \frac{1}{\gamma A_0} \tag{4.5}$$

Equating (4.3) and (4.5) yields

$$\frac{2Ke^{\gamma a A_0}}{1+K^2e^{2\gamma a A_0}} = \frac{2K}{1+K^2} \tag{4.6}$$

which leads after a little mathematical treatment to

$$K = e^{-\frac{aA_0\gamma}{2}} \tag{4.7}$$

In this context the parameter A_0 may be computed, using (4.3) as

$$A_0 = \frac{1}{\gamma}\cosh(\frac{a\gamma A_0}{2}) \tag{4.8}$$

such that the general solution (3.6), that is, the exact solution to the generalized Bratu equation (2.4) under the boundary conditions $u(0) = u(1) = 0$, may take the expression

$$u(x) = \frac{1}{\gamma}\ln\left\{\frac{2\gamma A_0 e^{a\gamma A_0(x-\frac{1}{2})}}{1+e^{2a\gamma A_0(x-\frac{1}{2})}}\right\} \tag{4.9}$$

or

$$u(x) = -\frac{1}{\gamma}\ln\left\{\frac{1+e^{2a\gamma A_0(x-\frac{1}{2})}}{2\gamma A_0 e^{a\gamma A_0(x-\frac{1}{2})}}\right\} \tag{4.10}$$

Using the identity

$$\cosh q = \frac{e^{-q}}{2}(1+e^{2q}) \tag{4.11}$$

the equation (4.10) may be expressed as

$$u(x) = -\frac{1}{\gamma}\ln\left\{\frac{\cosh[a\gamma A_0(x-\frac{1}{2})]}{\cosh(\frac{a\gamma A_0}{2})}\right\} \tag{4.12}$$

The expression (4.12) is the desired explicit and exact solution to the generalized Bratu equation (2.4) under the boundary conditions in consideration. So the exact solution of the one-dimensional Bratu boundary value problem may, for the value $\gamma = \frac{1}{2}$, take the expression

$$u(x) = -2\ln\left\{\frac{\cosh[\frac{aA_0}{2}(x-\frac{1}{2})]}{\cosh(\frac{aA_0}{4})}\right\} \tag{4.13}$$

so that for the parametric choice

$$\theta = aA_0$$

that is for the transcendental equation

$$\theta = 2a\cosh(\frac{\theta}{4}) \tag{4.14}$$

the exact solution to the Bratu boundary value problem may be definitively written as

$$u(x) = -2\ln\left\{\frac{\cosh[\frac{\theta}{2}(x-\frac{1}{2})]}{\cosh(\frac{\theta}{4})}\right\} \tag{4.15}$$

After showing that the exact solution of the one-dimensional Bratu boundary value problem may be calculated from the general solution to the Bratu differential equation, the purpose, now, is to show that the current general theory may also be used to compute the explicit and exact solutions to the initial and boundary value problems investigated by Wazwaz [7] on the basis of Adomian decomposition method [7] and Boyd [13].

5 Bratu type initial and boundary value problems

In the investigation of the Bratu boundary value problem by Adomian decomposition method, Wazwaz [7] considered a number of Bratu type initial and boundary value problems. The results obtained by Wazwaz [7] are later used by several authors [5, 6] to test the accuracy and efficiency of some approximate methods for solving differential equations. In this section, the explicit and exact solutions of the Bratu type boundary value problem investigated by Boyd [13] and the Bratu type initial value problem considered by Wazwaz [7] are determined using the general solution established in this work.

5.1 Bratu-type problem 1

The Bratu-type problem considered by Boyd [13] may be written in the form

$$\begin{gathered} u''(x) + \lambda e^{u(x)} = 0, \qquad -1 < x < 1 \\ u(-1) = u(1) = 0 \end{gathered} \tag{5.1}$$

Although, here, the differential equation is that of Bratu, the boundary conditions are different from those usually used for the Bratu nonlinear two-point boundary value problem. In this regard under the condition $u(-1) = 0$, the general solution (3.6) yields

$$\ln\left\{\gamma A_0 \sin[2tg^{-1}(Ke^{-a\gamma A_0})\right]\} = 0$$

that is

$$\frac{1}{\gamma A_0} = \sin\left[2tg^{-1}(Ke^{-a\gamma A_0})\right] \tag{5.2}$$

which may be reduced to

$$\frac{1}{\gamma A_0} = \frac{2Ke^{-a\gamma A_0}}{1+K^2e^{-2a\gamma A_0}} \tag{5.3}$$

On the other hand, the application of $u(1) = 0$, turns (3.6) into

$$\ln\left\{\gamma A_0 \sin[2tg^{-1}(Ke^{a\gamma A_0})\right]\} = 0$$

which leads to

$$\frac{1}{\gamma A_0} = \sin\left[2tg^{-1}(Ke^{a\gamma A_0})\right] \tag{5.4}$$

such that

$$\frac{1}{\gamma A_0} = \frac{2Ke^{a\gamma A_0}}{1+K^2e^{2a\gamma A_0}} \tag{5.5}$$

The comparison of (5.3) with (5.5) allows, after a few mathematical treatments, to obtain

$$K = 1 \tag{5.6}$$

so that the second integration constant A_0 may take the form

$$A_0 = \frac{1}{2\gamma}\left(e^{-a\gamma A_0} + e^{a\gamma A_0}\right)$$

which may also be written

$$A_0 = \frac{1}{\gamma}\cosh(a\gamma A_0) \tag{5.7}$$

In this context the general solution (3.6) becomes

$$u(x) = \frac{1}{\gamma}\ln\left\{\gamma A_0 \sin[2tg^{-1}(\exp(\gamma a A_0 x))]\right\} \tag{5.8}$$

where A_0 is given by (5.7). Knowing that (0.1) is obtained for $\gamma = \frac{1}{2}$, the exact solution to the boundary value problem (5.1) becomes

$$u(x) = 2\ln\left\{\frac{A_0}{2}\sin[2tg^{-1}(\exp(\frac{aA_0}{2}x))]\right\} \tag{5.9}$$

where

$$A_0 = 2\cosh(\frac{aA_0}{2}) \tag{5.10}$$

Using the identity

$$\sin[2tg^{-1}(\exp(\frac{aA_0}{2}x))] = \frac{1}{\cosh(\frac{aA_0}{2}x)} \tag{5.11}$$

the solution (5.9) takes the form

$$u(x) = 2\ln\left[\frac{2}{A_0}\cosh(\frac{aA_0}{2}x)\right]$$

that is

$$u(x) = -2\ln\left[\frac{\cosh(\frac{aA_0}{2}x)}{\cosh(\frac{aA_0}{2})}\right] \tag{5.12}$$

where A_0 is given by (5.10). Now, a few algebraic manipulations is needed to compare the exact solution (5.12) to the problem (5.1) with the solution given by Boyd [13]. As $2a^2 = \lambda$, that is $a = \pm\sqrt{\frac{\lambda}{2}}$, it suffices to set $\frac{A_0}{2} = z$, which is equivalent to $\cosh(\frac{A_0}{2}\sqrt{\frac{\lambda}{2}}) = z$, that is to say $\cosh(z\sqrt{\frac{\lambda}{2}}) = z$, to write (5.12) in the form

$$u(x) = -2\ln\left[\frac{\cosh(z\sqrt{\frac{\lambda}{2}}x)}{\cosh(z\sqrt{\frac{\lambda}{2}})}\right] \tag{5.13}$$

or definitively under the expression

$$u(x) = \ln\left[z^2 sech^2(z\sqrt{\frac{\lambda}{2}}x)\right] \tag{5.14}$$

which is nothing but the form used by Boyd [13] to express the solution to the boundary value problem (5.1)

5.2 Bratu-type problem 2

The third Bratu-type problem solved by Wazwaz [7] is an initial value problem formulated as

$$\begin{gathered} u''(x) - 2e^{u(x)} = 0, \qquad 0 < x < 1 \\ u(0) = u'(0) = 0 \end{gathered} \tag{5.15}$$

It is convenient before solving (5.15) to consider the general solution (3.6) under the general initial conditions $u(0) = u_0$, and $u'(0) = v_0$. In this perspective, the application of $u(0) = u_0$, gives

$$e^{\gamma u_0} = \gamma A_0 \sin(2tg^{-1}K) \tag{5.16}$$

and the application of $u'(0) = v_0$, leads to

$$v_0 = \frac{2aA_0K\cos(2tg^{-1}K))}{(1+K^2)\sin(2tg^{-1}K)} \tag{5.17}$$

so that (5.17) may be rewritten as

$$v_0 = \frac{2\gamma aA_0^2K\cos(2tg^{-1}K))}{(1+K^2)e^{\gamma u_0}} \tag{5.18}$$

Therefore, for the initial conditions $u_0 = 0$, and $v_0 = 0$, one may find $K = 1$, and $A_0 = 2$, such that the general solution (3.6) under the above conditions reduces to

$$u(x) = \frac{1}{\gamma}\ln\left\{2\gamma\sin[2tg^{-1}(e^{2a\gamma x})]\right\} \tag{5.19}$$

As the Bratu type initial value problem (5.15) is obtained from (2.4) for $\gamma = \frac{1}{2}$ and $a^2 = -1$, the exact solution to (5.15) may take the expression

$$u(x) = 2\ln\left\{\sin[2tg^{-1}(e^{ix})]\right\} \tag{5.20}$$

where i is the purely imaginary number.

Using the identity

$$\sin[2tg^{-1}(e^{ix})] = \frac{1}{\cosh(ix)} \tag{5.21}$$

the solution (5.20) may be written as

$$u(x) = -2\ln(\cosh(ix)) \tag{5.22}$$

which takes the definitive expression

$$u(x) = -2\ln(\cos x) \tag{5.23}$$

This result (5.23) is identical to that obtained by Wazwaz using the Adomian decomposition method [7].

6 Discussion

Although the Bratu boundary value problem has been intensively investigated in the literature, one may unfortunately note that there doesn't exist an explicit and exact general solution to the differential equation which describes this problem. In this work the Bratu equation is investigated with the aim of finding general solution by using an explicit and exact method for solving nonlinear differential equations such that various initial and boundary conditions may be applied. A straightforward way to do so was to transform the Bratu nonlinear differential equation into a linear second order differential equation with well known properties. It is found in this regard that the generalized Sundman linearization theory recently developed by Akande et al. [10] is well convenient to solve this problem. In so doing it has been highlighted that the Bratu equation consists of a special case of a generalized nonlinear differential equation admitting an explicit and exact general solution for all $x \in \mathbb{R}$ so that various initial and boundary conditions may be applied. In such a situation, it is, for the first time, shown that the well-known exact solution to the one-dimensional Bratu boundary value problem may be, by applying the required boundary conditions, deduced from the explicit and exact general solution to the differential equation. In this perspective, it has been possible to compute the exact solutions to some Bratu type initial and boundary value problems examined by Boyd [13] and Wazwaz [7] using Adomian decomposition method [7], in the context of the current general theory. The method applied in this work shows as an advantage, the fact that the Bratu nonlinear differential equation is intimately and directly related, for the first time, to the linear harmonic oscillator differential equation with well known exact analytical solution. So from the above a conclusion may be formulated for the work.

Conclusion

While the exact solution of the one-dimensional Bratu boundary value problem is well known in the literature, the explicit and exact general solution of the Bratu nonlinear differential equation is unfortunately, after a century, an unresolved question. This constitutes a fundamental drawback in the understanding of analytical properties of Bratu equation. Fortunately this shortcoming has been, for the first time, overcome in this work, using the generalized Sundman transformation, which closely relates the Bratu differential equation to the linear harmonic oscillator equation. In so doing, it was possible to deduce from the computed general solution the well-known exact solution of the Bratu boundary value problem and to show that the Bratu equation is a special case of a more general equation. Therefore, one could compute the explicit and exact solutions of a large variety of Bratu type problems with a relative simplicity.

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