# Multiplicative Versions of Infinitesimal Calculus

What happens when you replace the **summation** of standard integral calculus with **multiplication**?

Compare the ***abbreviated*** definition of a standard integral

$$\int f(x)dx = \lim_{\Delta x \to 0} \sum f(x_i) * \Delta x$$

With

$$\prod f(x) \uparrow dx = \lim_{\Delta x \to 0} \prod f(x_i) \uparrow \Delta x$$

and

$$\prod (1 + f(x)dx) = \lim_{\Delta x \to 0} \prod (1 + f(x_i) * \Delta x)$$

Call these later two "integrals" **multigrals of Type I and II.**
**(***Note: unlike "normal" products, these products are not discrete but continuous over an interval)*.

Consider each in turn.

**<u>Multigrals (Type I)</u>**

By standard operations $\prod f(x) \uparrow dx = e \uparrow (\int \ln(f(x))dx)$

By not taking limits, a finite product approximation can be obtained.

For example, let f(x)= x from 0 to 1. Then the Type I multigral of x from 0 to 1 is:

$$\prod_0^1 x \uparrow dx = e \uparrow (\int_0^1 \ln(x)dx) = 1/e$$

This can be approximated by the sequence

$$[(\frac{1}{3})(\frac{2}{3})] \uparrow (\frac{1}{2}) = 0.4714...$$

$$[(\frac{1}{4})(\frac{2}{4})(\frac{3}{4})] \uparrow (\frac{1}{3}) = 0.4543...$$

$$[(\frac{1}{5})(\frac{2}{5})(\frac{3}{5})(\frac{4}{5})] \uparrow (\frac{1}{4}) = 0.4427...$$

....

$$[(\frac{1}{1000})(\frac{2}{1000})....(\frac{999}{1000})] \uparrow (\frac{1}{999}) = 0.369123...etc$$

Which tends to 1/e=0.36788… (use Stirling's Formula to support this).

For f(x)=tan(x) in radians from 0 to pi/2, $\prod_{0}^{pi/2} \tan(x)\uparrow dx = e^0 = 1$

And

| |
|---|
| $\left[\tan(\pi/6)*\tan(2\pi/6)\right]\uparrow(\frac{1}{2}) = 1$ <br> $\left[\tan(\pi/8)*\tan(2\pi/8)*\tan(3\pi/8)\right]\uparrow(\frac{1}{3}) = 1...etc.$ |

The above approximations of the multigral can be likened to the **mid-point-rule** when approximating standard integrals. Like standard integrals, multiplicative **analogs** of the **Trapezoidal Rule** and **Simpson's Rule** can be found, like:

**"Simpson's" Product:**

$$\prod_{a}^{b} f(x)\uparrow dx \approx$$

$$\left\{\begin{matrix}[f(a)*f(b)]* \\ \{[f(a+\Delta x)*f(a+3\Delta x)*...]\uparrow 4\}* \\ \{[f(a+2\Delta x)*...]\uparrow 2\}\end{matrix}\right\}\uparrow(\frac{\Delta x}{3})$$

Consider the following approximations:

| $Y = \prod_{1}^{2} x\uparrow dx = e\uparrow(\int_{1}^{2}\ln(x)dx = e\uparrow(2\ln(2)-2+1) = 4/e = 1.471517765...$ | | | |
|---|---|---|---|
| | Multiplicative Analog of …. | | |
| | Mid-point Rule | Trapezoidal Rule | Simpson's Rule |
| Δx=1 | 1.5 | [(1)(2)]^(1/2)<br>=1.4142…. | n.a. |
| Δx=1/2 | [(1.25)(1.75)]^(1/2)<br>=1.4790199… | [(1)(2)]^(1/4)<br>*[1.5]^(1/2)<br>=1.4564753… | [(1)(2)]^((1/2)(1/3))<br>*[1.5]^((4/3)(1/2))<br>=1.47084… |
| Δx=1/3 | [(7/6)(9/6)(11/6)]^(1/3)<br>=1.474890668… | [(1)(2)]^(1/6)<br>*[(4/3)(5/3]^(1/3)<br>=1.46476345…. | n.a. |
| Δx=1/4 | [(9/8)(11/8)(13/8)<br>(15/8)]^(1/4)<br>=1.473423… | [(1)(2)]^(1/8)<br>*[(5/4)(6/4)(7/4)]^(1/4)<br>=1.4677043…. | [(1)(2)]^((1/4)(1/3))<br>*[(1.25)(1.75)]^((4/3)(1/4))<br>*[1.5]^((2/3)(1/4))<br>=1.471466559…. |

Like standard calculus you can define a multiplicative analog of the derivative ( the m-derivative), construct a multiplicative version of the Fundamental Theorem of Calculus, construct a multiplicative analog of Maclaurin's Series, etc.

The m-derivative for Type I multigrals is:

$$f^*_I(x) = e \uparrow (\frac{f'(x)}{f(x)})$$

The Fundamental Theorem is:

$$\prod_a^b f^*_I(x) \uparrow dx = \prod_a^b e \uparrow ((\frac{f'(x)}{f(x)})dx) = \frac{f(b)}{f(a)}$$

Compare with the Fundamental Theorem of Standard Calculus:

$$\int_a^b f'(x)dx = f(b) - f(a)$$

Small programs can be written to approximate the above results by finite products for those who doubt.

Type I multigrals find application in the area of population dynamics. With stochastic birth- and death- rates, the conventional approach is to use means (ie: expectations). Without migration, mean populations E(P) remain constant ***iff*** mean birth-rates E(b)= mean death-rates E(d) under the stochastic recursive equation $P_{n+1} = (1+b-d)*P_n$ .

But, while mathematically correct, this result is ***misleading***.

In certain circumstances, simulations show that mean birth-rates can significantly exceed mean death-rates yet **MOST** population trials decline, even though the mean population of **many** trials stays constant. True.

Let G(x)= $\prod x \uparrow (p(x)dx)$ where X= the random variable of (1+b-d) and p(x) is its probability density function. It can be shown that the **MODE** of populations ($P_n$) tends to $\{G(x)\uparrow n\}*P_0$ as $n \to \infty$. In general G(x) is < E(x)=E(1+b-d). Thus when E(b)=E(d), the mode of $P_n \to 0$ as $n \to \infty$ even though $E(P_n) = P_0$.

Thus the stochastic recursive equations (where ran# is a random number between 0 and 1)

$$P_{n+1} = (2.718281828....*ran\#)*P_n$$
$$P_{n+1} = (2*ran\#+0.17696)*P_n$$
$$P_{n+1} = (ran\#+0.54421)*P_n$$

are all **constant** (in the long-term **mode**) unlike

$$P_{n+1} = (2*ran\#)*P_n$$
$$P_{n+1} = (ran\#+0.5)*P_n$$

which are constant in the long-term **mean** but tend to zero in the mode. (Try simulating using Excel if you don't believe).

Now consider…

**Multigrals (TypeII)**

Consider $\prod_0^1 (1+x*dx)$ which is the limit of the sequence:

$$\left(1+\frac{1}{2}*1\right)=1.5$$
$$\left(1+\frac{1}{4}*\frac{1}{2}\right)\left(1+\frac{3}{4}*\frac{1}{2}\right)=1.546875$$
$$\left(1+\frac{1}{6}*\frac{1}{3}\right)\left(1+\frac{3}{6}*\frac{1}{3}\right)\left(1+\frac{5}{6}*\frac{1}{3}\right)=1.573559671$$
$$\left(1+\frac{1}{8}*\frac{1}{4}\right)\left(1+\frac{3}{8}*\frac{1}{4}\right)\left(1+\frac{5}{8}*\frac{1}{4}\right)\left(1+\frac{7}{8}*\frac{1}{4}\right)=1.589455605...etc.$$

which tends to sqr(e)=1.648721271…

This is due to the non-standard integral $\int_0^1 \ln(1+x*dx)=0.5$

(which is ***not*** of the form $\int f(x)dx$ )

and thus

$$\prod_0^1 (1+x*dx)=e\uparrow(\int_0^1 \ln(1+x*dx))=e\uparrow(\int_0^1 x*dx)=\sqrt{e}$$

In general,

$$\prod(1+f(x)dx)=e\uparrow(\int f(x)dx) \text{ provided } \int(f(x)dx)\uparrow n=0 \text{ for } n\in \mathrm{N}\geq 2.$$

Functions f(x) which fail the later condition appear to be few.

For instance, f(x)=1/x fails this test from 0 to 1 as $\int_0^1 (\frac{dx}{x})\uparrow 2=\pi^2/6$ (look at the limit definition of the integral under equal Δx subintervals to see this).

But for most other functions $\int(f(x)dx)\uparrow n=0$ for $n\in \mathrm{N}\geq 2$

For Type II multigrals, the m-derivative is: $f_{II}^*(x)=\frac{f'(x)}{f(x)}$

And the Fundamental Theorem is: $\prod_a^b (1+f_{II}^{*}(x)dx) = \prod_a^b (1+\frac{f'(x)}{f(x)}dx) = \frac{f(b)}{f(a)}$

Higher order m-derivatives can be also used, like:

$$\prod_a^b (1+f_{II}^{**}(x)dx) = \prod_a^b (1+(\frac{f''(x)}{f'(x)}-\frac{f'(x)}{f(x)})dx) = \frac{f_{II}^{*}(b)}{f_{II}^{*}(a)} = \frac{\left\{\frac{f'(b)}{f(b)}\right\}}{\left\{\frac{f'(a)}{f(a)}\right\}} = \frac{f(a)*f'(b)}{f'(a)*f(b)}$$

And so on. In general, the Fundamental Theorem becomes more complicated for higher order m-derivatives, unlike (say) polynomials with standard calculus. For instance,

$$f_{II}^{***}(x) = \frac{\left[\frac{f'''(x)}{f'(x)}-\frac{f''(x)}{f(x)}-\left(\frac{f''(x)}{f'(x)}\right)^2+\left(\frac{f'(x)}{f(x)}\right)^2\right]}{\left[\frac{f''(x)}{f'(x)}-\frac{f'(x)}{f(x)}\right]}$$

And thus

$$\prod_a^b (1+f_{II}^{***}(x)dx) = \prod_a^b \left\{1+\frac{\left[\frac{f'''(x)}{f'(x)}-\frac{f''(x)}{f(x)}-\left(\frac{f''(x)}{f'(x)}\right)^2+\left(\frac{f'(x)}{f(x)}\right)^2\right]}{\left[\frac{f''(x)}{f'(x)}-\frac{f'(x)}{f(x)}\right]}dx\right\} = \frac{\left[\frac{f''(b)}{f'(b)}-\frac{f'(b)}{f(b)}\right]}{\left[\frac{f''(a)}{f'(a)}-\frac{f'(a)}{f(a)}\right]} = \frac{f_{II}^{**}(b)}{f_{II}^{**}(a)}$$

For example, let f(x)=ln(x+2) then f'(x)=1/(x+2), f''(x)=-1/(x+2)^2, f'''(x)=2/(x+2)^3 and f(0)=ln(2), f(1)=ln(3), etc. Then

$$\prod_0^1 (1+f^{***}(x)dx)$$

$$=\prod_0^1\left\{1+\frac{\left[(\frac{2/(x+2)\uparrow 3}{1/(x+2)})+(\frac{1/(x+2)\uparrow 2}{\ln(x+2)})-\left((\frac{-1/(x+2)\uparrow 2}{1/(x+2)})\uparrow 2\right)+\left(\left(\frac{1/(x+2)}{\ln(x+2)}\right)\uparrow 2\right)\right]}{\left[(\frac{-1/(x+2)\uparrow 2}{1/(x+2)})-\left(\frac{1/(x+2)}{\ln(x+2)}\right)\right]}*dx\right\}$$

$$=\prod_0^1\{1-\frac{1}{(x+2)}\left[\frac{\ln(x+2)+\frac{1}{\ln(x+2)}}{\ln(x+2)+1}\right]*dx\}$$

$$=f^{**}(1)\Big/f^{**}(0)$$

$$==\left[\frac{\frac{f''(1)}{f'(1)}-\frac{f'(1)}{f(1)}}{\frac{f''(0)}{f'(0)}-\frac{f'(0)}{f(0)}}\right]$$

$$=(2/3)((1+1/\ln(3))/(1+1/\ln(2)))=0.5213474447...$$

Approximating using N Δx subintervals gives:

| N | 10 | 100 | 1000 |
|---|---|---|---|
| approximation | 0.5096103 | 0.520198 | 0.5212327 |

Whacko!

Like standard calculus you can change variables in the standard way:

$$\begin{cases}\prod_a^b (1+f(x)dx)\rightarrow\prod_{f(a)}^{f(b)}(1+u\dfrac{du}{f'(f^{-1}(u))})\\ \text{from}\\ \text{let u=f(x) then du=f}'\text{(x)dx}\\ \text{x=a}\Rightarrow\text{u=f(a)}\\ \text{x=b}\Rightarrow\text{u=f(b)}\\ \text{dx=du/f}'\text{(x)=du/(f}'\text{(f}^{-1}(u)))\end{cases}$$

And thus, for example:

$$\prod_a^b(1+xdx)=\prod_0^1(1+((b-a)x+a)*(b-a)dx)$$

Product and Quotient Rules for Type I and II multigrals are:

| | Type I | Type II |
|---|---|---|
| Derivative | $f^* = e \uparrow (\frac{f'(x)}{f(x)})$ | $f^* = \frac{f'(x)}{f(x)}$ |
| Product Rule | $(fg)^* = f^* g^*$ | $(fg)^* = f^* + g^*$ |
| Quotient Rule | $(\frac{f}{g})^* = \frac{f^*}{g^*}$ | $(\frac{f}{g})^* = f^* - g^*$ |

Surprisingly Type II multigrals have the same sort of "Maclaurin's" Product as Type I. It is

$$f(x) = f(0) * e \uparrow (\frac{f'(0)}{f(0)} * x + \frac{1}{2!}\left(\frac{f'(0)}{f(0)}\right)' * x^2 + \frac{1}{3!}\left(\frac{f'(0)}{f(0)}\right)'' * x^3 + ...)$$

And the two types of multigral can be related by

$\prod f(x) \uparrow dx = \prod (1 + \ln(f(x))dx)$ for acceptable f(x).

**<u>Other Types of Multigral</u>**

With type II multigrals, problems arise for functions like f(x)=1/x due to the fact that $\int (f(x)dx) \uparrow n \neq 0$ for $n \in \mathbb{N} \geq 2$. But sometimes related multigrals can be evaluated using certain theta functions. For instance,

$$\prod_0^1 (1 + (\frac{dx}{x}) \uparrow 2) = 5(1 + (\frac{2}{3})^2)(1 + (\frac{2}{5})^2).... = \cosh(\pi) \approx 11.591...$$ and

$$\prod_0^1 (1 - (\frac{dx}{x}) \uparrow 4) = -\cosh(\pi) \approx -11.591....$$ and

$$\prod_0^1 (1 + (\frac{2dx}{x}) \uparrow 3) = (4 \uparrow 3 + 1)\left[\frac{\left\{\Gamma(\frac{3}{2})\right\} \uparrow 3}{\Gamma(\frac{7}{2})}\right]^{\frac{\cosh(\pi\sqrt{3})}{\pi}}$$ and the like.

However, these type III multigrals have certain unusual properties like

$$\prod_0^b (1+(\frac{dx}{x})\uparrow 2)) = \prod_0^1 (1+(\frac{dx}{x})\uparrow 2))$$

$$\prod_{ka}^{kb} (1+(\frac{dx}{x})\uparrow 2)) = \prod_a^b (1+(\frac{dx}{x})\uparrow 2))...etc.$$

So take care when playing around with.

**<u>Type IV Multigrals</u>**

Surprisingly the multigral

$$\prod_0^1 (1+x\uparrow(\frac{1}{dx})) = e\uparrow(\frac{\sqrt{e}}{(e-1)} - \frac{e}{(e\uparrow 2-1)} + \frac{e\uparrow 1.5}{(e\uparrow 3-1)} - ...) \approx 2.22...\text{ exists!}$$

This is thanks to the non-standard "standard" integrals of

$$\int_0^1 x\uparrow(\frac{1}{dx}) = \frac{\sqrt{e}}{(e-1)} = 0.959517...$$

$$\int_0^1 x\uparrow(\frac{k}{dx}) = \frac{e\uparrow(k/2)}{(e\uparrow k-1)}$$

These type of multigrals are more restricted (in range) than type I and II, but can still be used to derive certain stochastic limits such as

$$\operatorname*{mod}_{n\to\infty}\left\{\sum_{i=1}^{n}(ran\#_i)\uparrow n\right\} = \frac{\sqrt{e}}{(e-1)} \approx 0.959517...$$

$$\operatorname*{mod}_{n\to\infty}\left\{\prod_{i=1}^{n}(1+(ran\#_i)\uparrow n)\right\} = e\uparrow\left(\frac{\sqrt{e}}{(e-1)} - \frac{e}{(e\uparrow 2-1)} + \frac{e\uparrow 1.5}{(e\uparrow 3-1)} - ...\right) \approx 2.22...$$

Where mod is "the mode" and ran# is a random number between 0 and 1.

**<u>Unanswered Questions</u>**

1. How many types of multigrals are there? Do they all have m-derivatives, Fundamental Theorems, analogs of Simpson's Rule, Maclaurin Series, etc?
2. What do multigrals do in the complex plane?

Answers please. Happy multigrating!

All comments welcome. Please send to: everythingflows@hotmail.com

$$\prod_0^e x = \sqrt{2}$$