Moduli spaces of Special Lagrangian submanifolds with singularities

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Abstract

We try to give a formulation of Strominger-Yau-Zaslow conjecture on mirror symmetry by studying the singularities of special Lagrangian submanifolds of 3-dimensional Calabi-Yau manifolds. In this paper we'll give the description of the boundary of the moduli space of special Lagrangian manifolds.

We do this by introducing special Lagrangian cones in the more general Kähler manifolds. Then we can focus on the textitalmost Calabi-Yau manifolds. We consider the behaviour of the Lagrangian manifolds near the conical singular points to classify them according to the way they are approximated from the asymptotic cone. Then we analyze their deformations in Calabi-Yau manifolds.

1 Conifold geometry. Some definitions

We formally want to introduce the categories of Riemannian manifolds which we are going to analyze and we give the necessary definitions.

Definition 1.1. Let L^m be a smooth manifold. L is a manifold with ends if the following conditions are satisfied:

- There exists compact subsets $K \subset L$ such that $S := L \setminus K$ has a finite number of connected components $S_1, ..., S_e$, that is $S = \bigcup_{i=1}^{e} S_i$.
- For every S_i there exists a compact $m-1$-manifold Σ_i without border and a diffeomorphism $\phi_i : \Sigma_i \times [1, \infty) \to \bar{S}_i$.

The S_i are called ends of L and the manifolds Σ_i are the links of L.

Definition 1.2. Let L be a manifold with ends. Let g be a Riemannian metric on L. We choose an end S_i and the corresponding link Σ_i. S_i is a conically singular (CS) end if the following conditions holds:

- Σ_i is equipped with a Riemannian metric g'. We call (θ, r) the generic point on the product manifold $C_i := \Sigma_i \times (0, \infty)$ and $\tilde{g}_i := dr^2 + r^2 g'_i$ is the conical metric on C_i.

- There exists a constant $\nu_i > 0$ and a diffemorphism $\phi_i : \Sigma_i \times (0, \epsilon] \to \bar{S}_i$ such that, when $r \to 0$ and for every $k \geq 0$,

$$|\tilde{\nabla}^k(\phi_i^* g - \tilde{g}_i)|_{\tilde{g}_i} = O(r^{\nu_i - k}),$$

  in which $\tilde{\nabla}$ is the Levi-Civita connection on C_i defined by $\tilde{g}_i$.

S_i is an asimptotically conical (AC) end if:

- $\Sigma_i$has a Riemannian metric g'_i. As in the case of the other singularities, (θ, r) is the generic point on the product manifold $C_i := \Sigma_i \times (0, \infty)$ and $\tilde{g}_i := dr^2 + r^2 g'_i$ is the conical metric on C_i.

- There exist a constant $\nu_i < 0$ and a diffeomorphism $\phi_i : \Sigma_i \times [R, \infty) \to \bar{S}_i$ such that, for $r \to \infty$ and for every $k \geq 0$,

$$|\tilde{\nabla}^k(\phi_i^* g - \tilde{g}_i)|_{\tilde{g}_i} = O(r^{\nu_i - k}),$$

  in which $\tilde{\nabla}$ is the Levi-Civita connection on C_i defined by $\tilde{g}_i$.

In the situation described here ν_i is called rate of convergence of S_i. [1]

Definition 1.3. Let $(\bar{L}, d)$ be a metric space. $\bar{L}$ is a Riemannian manifold with conical singularities (CS manifold) if it satisfies the following:

- There exist points $\{x_1, ..., x_e\} \in \bar{L}$ such that $L := \bar{L} \setminus \{x_1, ..., x_e\}$ has the structure of smooth m-manifold with ends e. We can find a real number ϵ, with $0 < 2\epsilon < \sup_{i \neq j} d(x_i, x_j)$, such that the $S_i := \{x \in L : 0 < d(x, x_i) < \epsilon\}$ are the ends of L with respect to some links Σ_i.

[1] The difference between AC and CS is that a CS manifold has isolated singularities and every singuarity defines a cone. An AC manifold tL instead converges to the cone with a parameter t which goes to 0. Both of them define the way in which, given a manifold L and a parameter t, tL converges to the cone for t which goes to 0. So the singularities modelled on cones can be studied as the limits of non-singular manifolds.

- There exists a Riemannian metric g on L which induces the distance d.
- Every end is CS with respect to g.

Every CS manifold is compact.
The points x_i are the singularities of $\bar{L}$.

Definition 1.4. Let (L, g) be a Riemannian manifold. L is a Riemannian manifold with ends asimptotically conical (AC) if the following are satisfied:

- L is a smooth manifold with ends S_i and connected links Σ_i.
- Every end S_i is AC.

Definition 1.5. Let $(\bar{L}, d)$ be a metric space. We define $\bar{L}$ to be a Riemannian CS/AC manifold if satisfies:

- There exist finite points $\{x_1, ..., x_s\}$ and a number ℓ such that $L := \bar{L} \setminus \{x_1, ..., x_s\}$ has the structure of a smooth m-manifold with $s + \ell$ ends.
- There exists a metric g on L that induces the distance d.
- There are neighbourhoods of the points x_i that has the structure of CS ends. These are called "small" ends. The remaining are the "big", that is they have AC structure.

We call Σ_0 the union of the CS links, while the ends and the links related to the AC structure will be called Σ_∞, S_∞.

Definition 1.6. Let's use the generic term conifold to call the CS,AC e CS/AC manifolds. If (L, g) is a conifold and $C := \bigcup C_i$ is the union of the related cones, equipped with the metric $\tilde{g}$, we say that (L, g) is asymptotic to $(C, \tilde{g})$.

The models of CS/AC manifolds are cones in $\mathbb{R}^n$.

Definition 1.7. A subset $\bar{\mathscr{C}} \subset \mathbb{R}^n$ is a cone if it is invariant for expansions of $\mathbb{R}^n$, that is if $t \cdot \bar{\mathscr{C}} \subseteq \bar{\mathscr{C}}$ for every $t \geq 0$. A cone of $\mathbb{R}^n$ is identified by its link $\Sigma := \bar{\mathscr{C}} \cap \mathbb{S}^{n-1}$. We introduce $\mathscr{C} := \bar{\mathscr{C}} \setminus 0$. The cone is regular se Σ is smooth. During this paper we'll consider regular cones.

2 Lagrangian conifolds. Other definitions

Let us now strengthen the hypothesis on the CS/AC manifolds by assuming that in every singularity and in every end there is a specific asymptotic cone. The definitions we are going to give will focus the attention on Lagrangian submanifolds in Kähler manifolds.

Definition 2.1. Let L^m be a smooth manifold. Given a Lagrangian immersion $\iota : L \to \mathbb{C}^m$, this with a standard structure $\tilde{J}, \tilde{\omega}$. L is an asymptotically conical special Lagrangian submanifold with rate λ if satisfies:

- There exists a compact subset $K \subset L$ such that $S := L \setminus K$ has a finite number of connected components $S_1, ..., S_e$.
- There exist Lagrangian cones $\mathscr{C}_i \subset \mathbb{C}^m$ with smooth connected links $\Sigma := \mathscr{C}_i \cap \mathbb{S}^{2m-1}$. Let $\iota_i : \Sigma_i \times (0, \infty) \to \mathbb{C}^m$ be the natural immersions which parametrize $\mathscr{C}_i$.
- We give a e-tuple of rate of convergence $\lambda = (\lambda_1, ..., \lambda_e)$ with $\lambda_i < 2$, centers $p_i \in \mathbb{C}^m$ and diffeomorphisms $\phi_i : \Sigma \times [R, \infty) \to \bar{S}_i$ for some $R > 0$ such that, for $r \to \infty$ and $k \geq 0$,

$$|\tilde{\nabla}^k(\iota \circ \phi_i - (\iota_i + p_i)| = O(r^{\lambda_i - 1 - k})$$

  with respect to the conic metric $\tilde{g}_i$ on $\mathscr{C}_i$.

The restriction $\lambda_i < 2$ guarantees that the cone is unique but is weak enough to allow submanifolds to converge to a translated copy of the cone $\mathscr{C}_i + p_i'$, or to slowly move away from the cone. [2]

Definition 2.2. Let $\bar{L}^m$ be a smooth manifold except for a finite number of singular points $\{x_1, ..., x_e\}$. We say that $\bar{L}$ is a conically singular Lagrangian submani- fold whit rate μ if satisfies:

- There exist connected open neighbourhoods S_i of x_i.
- There exist Lagrangian cones $\mathscr{C}_i \subset \mathbb{C}^m$ with smooth connected open links $\Sigma := \mathscr{C}_i \cap \mathbb{S}^{2m-1}$. Let $\iota : \Sigma_i \times (0, \infty) \to \mathbb{C}^m$ be the natural embedding that parametrizes $\mathscr{C}_i$.

[2]By the word move away here we want to show the behaviour of tL with t parameter, when t goes to 0 or ∞

- There exist a e-tuple of rates of convergence $\mu = (\mu_1, ..., \mu_e)$ with $\mu_i > 2$, centers $p_i \in \mathbb{C}^m$ and diffeomorphisms $\phi_i : \Sigma_i \times (0, \epsilon] \to \bar{S}_i \setminus \{x_i\}$ such that, for $r \to 0$ and $k \geq 0$,

$$|\tilde{\nabla}^k(\iota \circ \phi_i - (\iota_i + p_i)| = O(r^{\mu_i - 1 - k})$$

  with respect to the conical metric $\tilde{g}_i$ on $\mathscr{C}_i$. It should be noticed that $\iota(x_i) = p_i$.

It's easy to see that AC Lagrangian submanifolds, equipped with the induced metric, satisfy the definition 1.4 with $\nu_i = \lambda_i - 2$. The same thing happens for the CS Lagrangian submanifolds.

Definition 2.3. Let $\bar{L}^m$ be a smooth manifold except for a finite number of singular points $\{x_1, ..., x_s\}$and with ℓ ends. Let's consider a continuous map $\iota : \bar{L} \to \mathbb{C}^m$ which shrinks to a smooth Lagrangian embedding of $L := \bar{L} \setminus \{x_1, ..., x_s\}$. We say that $\bar{L}$ (or L) is a CS/AC Lagrangian submanifold with rates (μ, λ) if in a neighbourhood of the points x_i it has the structure of CS manifold with rates μ_i eand in a neighbourhood of the remaining ends it has the structure of AC manifold with rates λ_i. Let's use the generic term Lagrangian conifold to define CS,AC or CS/AC Lagrangian submanifolds.

Now we can generalize the definitions of CS Lagrangian submanifolds to Kähler The standard complex structure on $\mathbb{C}^m$ ȟere is called $\tilde{J}, \tilde{\omega}$.

Definition 2.4. Let (M^{2m}, J, ω) be a Kähler manifold and $\bar{L}^m$ a smooth manifold except for a finite number of singular points $\{x_1, ..., x_e\}$. Given a continuous map $\iota : \bar{L} \to M$ which shrinks to smooth Lagrangian embeddings of $L := \bar{L} \setminus \{x_1, ..., x_e\}$. $\bar{L}$ (or L) is a Lagrangian submanifold with conical singularities (CS Lagrangian submanifold) if it satisfies the following conditions:

- There exist isomorphisms $\nu_i : \mathbb{C}^m \to T_{\iota(x_i)}M$ such that $\nu_i^* \omega = \tilde{\omega}$ e $\nu_i^* J = \tilde{J}$. According to Darboux Theorem, there exists an open ball B_R in $\mathbb{C}^m$ (with small radius R) and diffeomorphisms $\Upsilon_i : B_R \to M$ such that $\Upsilon(0) = \iota(x_i), d\Upsilon_i(0) = \nu_i$ and $\Upsilon_i^* \omega = \tilde{\omega}$.

- There exist open neighbourhoods S_i of x_i in $\bar{L}$. If we assume that the S_i are small, then the compositions

$$\Upsilon_i^{-1} \circ \iota : S_i \to B_R$$

are well-definied.

Moreover, there exist Lagrangian cones $\mathscr{C}_i \subset \mathbb{C}^m$ with smooth connected links $\Sigma_i := \mathscr{C}_i \cap \mathbb{S}^{2m-1}$. Let $\iota_i : \Sigma_i \times (0, \infty) \to \mathbb{C}^m$ be the natural embedding, which parameterizes $\mathscr{C}_i$.

- We give an e-tuple of rates of convergence $\mu = (\mu_i, ..., \mu_e)$ with $\mu_i \in (2, 3)$ and diffeomorphisms $\phi_i : \Sigma_i \times (0, \epsilon] \to \bar{S}_i \setminus \{x_i\}$ such that, when $r \to 0$ and for every $k \geq 0$,

$$|\tilde{\nabla}^k (\Upsilon_i^{-1} \circ \iota \circ \phi_i - \iota_i)| = O(r^{\mu_i - 1 - k}) \tag{2.1}$$

with respect to the conic metric $\tilde{g}_i$ on $\mathscr{C}_i$.

We call x_i the singularities of $\bar{L}$ and v_i le identifications.

In the remaining part of this paper we'll study only CS Lagrangian submani- folds.

3 Deformation of Lagrangian conifolds: Lagrangian neighbourhood theorem

We now analyze how to parameterize the deformations of a Lagrangian conifold $L \subset M$.Since the Lagrangian condition is invariant under reparameterizations of L, we'll work with non-parameterized submanifolds in order to simplify the notation; to say it in other words, we'll work with equivalence classes of embed- ded submanifolds, in which two embeddings are equivalent if they differs for a reparameterization. Then, to reparameterize the deformations of L, we need only the theorem of the Lagrangian neighbourhood. Before we state it, let's do some preparatory considerations.

Let NL be the normal bundle. By using the tubular neighbourhood theorem we define a natural injection

$$\Lambda^0(NL) \to Imm(L, M)/Diff(L).$$

n the tipical situations, this defines a local omeomorphism between the topologies of these spaces.

To study the theory of Lagrangian neighbourhood we have to build a linear algebric structure.

Let W be a real vector space of finite dimension. Then $W \oplus W^*$ admits a canonical symplectic structure $\hat{\omega}$ defined by

$$\hat{\omega}(w_i + \alpha_1, w_1 + \alpha_2) := \alpha_2(w_1) - \alpha_1(w_2). \tag{3.1}$$

If $(V, \omega)$is a symplectic vector space and if $W \subset V$ is a Lagrangian subspace, we can choose $Z \subset V$ such that $V = W \oplus Z$. One can easily verify that the restriction of ω to Z defines an isomorphism

$$\omega|_Z : Z \to W^*, \quad z \mapsto \omega(z, \cdot) \tag{3.2}$$

and, by using this isomorphism, we can build an isomorphism $\gamma : (W \oplus W^*, \hat{\omega}) \simeq (V, \omega)$. Moreover, γ is uniquely defined if we impose that it is equal to the identity on W. By adding this condition, γ is defined in an unique way by the choice of Z.

We can do the same construction even for the manifolds. It will be enough to notice that, given a manifold L, the cotangent bundle T^*L admits a canonical symplectic structure $\hat{\omega}$. Let's now consider the tautologic 1-form on T^*L defined as

$$\hat{\lambda}[\alpha](v) := \alpha(\pi_*(v)), \tag{3.3}$$

in which $\pi : T^*L \to L$ is the natural projection. Then $\hat{\omega} := -d\hat{\lambda}$. Let's remember that the section of T^*L is a 1-form α on L. The graph $\Gamma(\alpha)$ is Lagrangian if and only if α is closed. The null section $L \subset T^*L$ is Lagrangian. Moreover, every fiber $\pi^{-1}(p) = T^*_pL$ is a Lagrangian submanifold. Every 1-form α defines a transition map

$$\tau_\alpha : T^*L \to T^*L, \quad \tau_\alpha(x, \eta) := (x, \alpha(x) + \eta). \tag{3.4}$$

If α is closed, this is a symplectomorphism.

Let's now show the Lagrangian neighbourhood Theorem

Theorem 3.1. Let (M, ω) be a symplectic manifold and $L \subset M$ a compact Lagrangian submanifold. Then there exist a neighbourhood $\mathscr{N}(L_0) \subset T^*L$ of the null section, a neighbourhood $V \subset M$ of L, and a diffeomorphism $\phi : \mathscr{N}(L_0) \to V$ such that

$$\phi * \omega = -d\alpha \quad \phi|_L = id, \tag{3.5}$$

in which α is the canonical 1-form on T^*L

Proof. The proof is based on the fact that the normal bundle of L to M is isomorphic to the tangent bundle. To define an explicit isomorphism, we can use the complex structure J on the tangent bundle TM. The subspace $JTL \subset TM$ is the orthogonal completion of TL with respect to the metric g_J induced by J, and it's a Lagrangian subspace of (TM, ω). Let $\phi : TL \to TL$ be the isomorphism induced by g_J:

$$g_J(\psi(v^*), v) = v^*(v) \tag{3.6}$$

for $v \in TL$. We can now consider the map $\phi : T^*L \to M$ given by the exponential map of the metric g_J:

$$\phi(q, v^*) = exp_q(J_q\psi_q(v^*)).$$

Then, for $v = (v_0, v_1^*) \in T_qL \oplus T_q^*L = T_{(q,0}T^*L$, we have

$$d\phi_{q,0}(v) = v_0 + J_q\psi_q(v_1^*),$$

e quindi

$$\begin{aligned} \phi^*\omega_{(q,0)}(v, w) &= \omega_q(v_0 + J_q\psi_q(v_1^*), w_0 + J_q\psi_q(w_1^*)) \\ &= \omega_q(v_0, J_q\psi_q(w_1^*)) - \omega_q(w_0, J_q\psi_q(v_1^*)) \\ &= g_J(v_0, \psi_q(w_1^*)) - g_J(w_0, \psi_q(v_1^*)) \\ &= w_1^*(v_0) - v_1^*(w_0) \\ &= -d\lambda_{(q,0)}(v, w). \end{aligned}$$

This shows that the 2-form $\phi^*\omega \in \Omega^2(T^*L)$ corresponds to the canonical form $-d\lambda$ on the null section. If two closed 2-forms ω_0, ω_1 on a submanifold $Q \subset M$ are identical and not degenerate on TM, then there exist neighbourhoods $\mathcal{N}_0, \mathcal{N}_1$ of Q and a diffeomorphism $\psi : \mathcal{N}_0 \to \mathcal{N}_1$ such that

$$\psi|_Q = id \quad \psi^*\omega_1 = \omega_0.$$

By applying it to our case we can obtain the proof of the theorem. □

4 Deformations of Lagrangian conifolds

Deformations of Lagrangian cones in $\mathbb{C}^m$

Let$\mathscr{C}$ be a Lagrangian cone in $\mathbb{C}^m$ with link (Σ, g') and conical metric $\tilde{g}$.We want to give a deformation theory similar to the one we have in the case of smooth deformations. For this manifold, we give a correspondance between closed 1- form in$\mathscr{C}^\infty_{(\mu-1,\lambda-1)}(\Lambda^1)$ and Lagrangian deformations of $\mathscr{C}$ with rates (μ, λ).

Let θ be the generic point of Σ. We identify $\Sigma \times (0, \infty)$ with $\mathscr{C}$ using the immersion:

$$\iota : \Sigma \times (0, \infty) \to \mathbb{C}^m, \quad (\theta, r) \mapsto r\theta. \tag{4.1}$$

Observation 4.1. Let $\theta(t)$ be a curve in Σ such that $\theta(0) = \theta$. Let $r(t)$ be a curve in $\mathbb{R}^+$ such that $r(0) = r$. By differentiating ι in the point (θ, r), we obtain identifications

$$\begin{aligned} &\iota_* : T_\theta\Sigma \oplus \mathbb{R} \to T_{r\theta}\mathscr{C} \subset \mathbb{C}^m \\ &(\theta'(0), r'(0)) \mapsto d/dt(r(t)\theta(t))|_{t=0} = r'(0)\theta + r\theta'(0) \in \mathbb{C}^m \end{aligned} \tag{4.2}$$

This give the general equation $\iota_*|_{\theta,r}(v, a) = a\theta + rv$.

We can now give the identification $\Psi_{\mathscr{C}}$ of $T^*\mathscr{C}$ with $\mathbb{C}^m$ as follows.

The metric $\tilde{g}$ gives an identification

$$T^*C \to T\mathscr{C}, \quad (\theta, r, \alpha_1 + \alpha_2 dr) \mapsto (\theta, r, r^{-2}A_1 + \alpha_2\partial r), \tag{4.3}$$

in which $g'(A_1, \cdot) = \alpha_1$. So, for the Observation 4.1, the correspondant vector in $\mathbb{C}^m$ is $\iota_*(r^{-2}A_1 + \alpha_2\partial r) = \alpha_2\theta + r^{-1}A_1$. Moreover, the equation (4.3) defines an isometry of vector bundles on $\mathscr{C}$. Let $\tilde{\nabla}$ be the standard connection on the tangent bundle of $\mathbb{C}^m$. Since $\mathscr{C}$ has the induced metric, the Levi-Civita connection on $T\mathscr{C}$ corresponds to the tangent projection $\tilde{\nabla}^T$. Let's assume that $T^*\mathscr{C}$ has the induced Levi-Civita connection. Then the equation (4.3) defines an isomorphism between the two connections.

In the Lagrangian cone $\mathscr{C}$ the complex structure gives an identification:

$$\tilde{J} : T\mathscr{C} \simeq N\mathscr{C}. \tag{4.4}$$

The perpendicular component of $\tilde{\nabla}^\perp$ defines a connection on $N\mathscr{C}$. Since $\mathbb{C}^m$ is Kähler, $\tilde{\nabla}\tilde{J} = \tilde{J}\tilde{\nabla}$. Quindi $\tilde{\nabla}^\perp\tilde{J} = \tilde{J}\tilde{\nabla}^\perp$. The equation (4.4) defines an isomorphism between the two connections.

The Riemannian tubular neighbourhood gives another explicit identification

$$N\mathscr{C} \to \mathbb{C}^m, \quad v \in N_{r\theta}\mathscr{C} \mapsto r\theta + v. \tag{4.5}$$

Now let's put together and we obtain the requested identification

$$\Psi_{\mathscr{C}} : \mathscr{U} \subset T^*\mathscr{C} \to \mathbb{C}^m, \quad (\theta, r, \alpha_1 + \alpha_2 dr) \mapsto r\theta + \tilde{J}(\alpha_2\theta + r^{-1}A_1). \tag{4.6}$$

Now, if we consider α 1-form on $\mathscr{C}$, then by the identifications introduced before, $\Psi_{\mathscr{C}} \circ \alpha - \iota \simeq \alpha$. This shows that, if $\alpha \in \mathscr{C}^{\infty}_{(\mu-1,\lambda-1)}(\mathscr{U})$ for some $\mu > 2, \lambda < 2$, then $\Psi_{\mathscr{C}} \circ \alpha$ is a CS/AC submanifold in $\mathbb{C}^m$ asymptotic to $\mathscr{C}$ with rates (μ, λ).

We see in addition that

$$\Psi_{\mathscr{C}}(\theta, tr, t^2\alpha_1 + t\alpha_2 dr) = t\Psi_{\mathscr{C}}(\theta, r, \alpha_1 + \alpha_2 dr). \tag{4.7}$$

Deformation of CS/AC Lagrangian submanifolds in $\mathbb{C}^m$

Let $\iota : L \to \mathbb{C}^m$ be an AC Lagrangian submanifold with rate λ, centers p_i and ends S_i. Using the notation introduced before, the map $\Phi_{\mathscr{C}_i} + p_i : t^*\mathscr{C}_i \to \mathbb{C}^m$ identifies $\iota(S_i) \subset \mathbb{C}^m$ with the graph $\Gamma(\alpha_i)$ of some closed 1-forms α_i. This defines a construction of a coordinate system ϕ_i by imposing the relation

$$\phi_i : \mathscr{C} \to S_i, \ \ \iota \circ \phi_i = \Phi_{\mathscr{C}_i} \circ \alpha_i.$$

Let $(d\phi_i)^* : T^*S_i \to T^*\mathscr{C}_i$ be the identification between cotangent bundles, that identifies the null section $\mathscr{C}_i$ with the null section S_i.

We can now use the symplectomorphism defined in (3.4) to join these identifications and to obtain a symplectomorphism

$$\Phi_{S_i} : \mathscr{U}_i \subset T^*S_i \to \mathbb{C}^m, \ \ \Phi_{S_i} := \Phi_{\mathscr{C}_i} \circ \tau_{\alpha_i} \circ (d\phi_i)^* + p_i \tag{4.8}$$

which shrinks to the identity on S_i. These maps give a Lagrangian neighbourhood for every end of L. By using these maps, we obtain a symplectomorphism

$$\Phi_L : \mathscr{U} \subset T^*L \to \mathbb{C}^m \tag{4.9}$$

that shrinks to the identity on L and allows us to parameterize the AC deformations of L with rate λ in terms of closed 1-forms in the space $\mathscr{C}^{\infty}_{\lambda-1}(\mathscr{U})$.

Deformations with moving singularities

Until now we have seen that the deformations of Lagrangian submanifolds have been considered by taking the singular points fixed. We now want to deform these manifolds by moving the point with the space. In the same way, we want the Lagrangian cones $\mathscr{C}_i$ to be allowed to rotate in $\mathbb{C}^m$.

Let's now consider the case of CS Lagrangian submanifolds $\iota : L \to M$ with singularities$\{x_1, ..., x_s\}$ and identifications $v_i$defined as follows. These construction has been donw by Joyce.(see [?] and [?]

We define

$$P := \{(p, v) : p \in M, v : \mathbb{C}^m \to T_pM \text{ tale che } v^*\omega = \tilde{\omega}, v^*J = \tilde{J}\}. \qquad (4.10)$$

P is an $U(m)$principal bundle on M with the action

$$U(m) \times P \to P, \quad M \cdot (p, v) := (p, v \circ M^{-1}). \qquad (4.11)$$

P is a smooth manifold of dimension $m^2 + 2m$.

We now have to use a copy of P to parameterize the position of every singular point $p_i = \iota(x_i) \in M$ and the direction of the correspondant cone $\mathscr{C}_i \subset \mathbb{C}^m$: the action of the group will allow the cone to rotate but will mantain the singular point fixed. Since we are studying the little deformations of L, we analyze an open neighbourhood of the couple $(p_i, v_i) \in P$. $\mathscr{C}_i$ in general has a symmetry group $G_i \subset U(m)$, that is by $G_i$the cone remain fixed.

To remove unnecessary parameter, we must focus on a part of the neighbourhood, specifically on a smooth submanifold transversal to the orbit of G_i. We call this little submanifold ε_i: it's a subset of P that contains (p_i, v_i) and has dimension $m^2 + 2m - dim(G_i)$. Now we set $\varepsilon = \varepsilon_1 \times ... \times \varepsilon_s$. The point $e := ((p_1, v_1), ..., (p_s, v_s)) \in \varepsilon$ is the starting point.

We now want to extend (L, ι) to a family of Lagrangian submanifolds $(L, \iota_{\tilde{e}}$, parameterized by $\tilde{e} = ((\tilde{p}_1, \tilde{v_1}), ..., (\tilde{p}_s, \tilde{v_s})) \in \varepsilon$. Every $(L, \iota_{\tilde{e}})$ must satisfy $\iota_{\tilde{e}}(p_i) = \tilde{p}_i$ and admit identifications $\tilde{v}_i$ and cones $\mathscr{C}_i$. We want also that $\iota_e = \iota$ globally and that $\iota_{\tilde{e}} = \iota$ out of a neighbourhood of the singularities.

The construction of this family is immediate: we use the maps Υ_i and all the construction is only a choice of appropriate family of symplectomorphisms with compact support of $\mathbb{C}^m$.

We choose an open neighbourhood $\mathscr{U} \subset T^*L$ and immersions $\Phi_L^{\tilde{e}} : \mathscr{U} \to M$ which, far from singularities, correspond to the embeddings Φ_L introduced before. The final result is that the moduli space of CS Lagrangian deformations of L with rate μ and moving singularities can be parameterized by couples $(\tilde{e}, \alpha)$ in which $\tilde{e} \in \varepsilon$ and α is a closed 1-form of L which belongs to the space $\mathscr{C}^{\infty}_{\mu-1}(\mathscr{U})$.

Special Lagrangian cones

We define AC,CS and CS/AC Lagrangian submanifolds exactly as in the definition already given for the Lagrangian manifolds, adding the condition that these will be special. The cones $\mathscr{C}_i$ will be special Lagrangian in $\mathbb{C}^m$. To define CS Lagrangian submanifolds in a Calabi-Yau manifold M we have to add to the definition given above the requirement that $v_i^*\Omega = \tilde{\Omega}$. We use the term special Lagrangian conifold for these manifolds.

5 A deformation problem

If $\iota : L \to M$is a special Lagrangian conifold we can refine the analysis and study the special Lagrangian deformations of L. Since the Lagrangian condition is invariant for reparameterization, if L is smooth and compact, the moduli space $\mathscr{M}_L$ of special Lagrangian deformations is the connected component containing L of the subset of special Lagrangian submanifolds $Lag(L, M)/Diff(L)$, in which $Lag(L, M)$are the Lagrangian submanifolds of L in M. If L is an AC,CS or CS/AC Lagrangian submanifold, we obtain the moduli space of special Lagrangian de- formations by simply considering the closed 1-forms on L that satisfies the same conditions of growth or decay. The purpose is to show that the moduli space of special Lagrangian conifolds admits a smooth natural structure with respect to whom the conifolds are finite-dimensional manifolds. To achieve this result, it's enough to show that $\mathscr{M}_L$ is locally the locus of the zeroes of some smooth map F defined on the space of the 1-forms $\mathscr{C}^\infty(\mathscr{U})$.We use then the theorem of implicit functions to prove that this set of zeroes is smooth.

Let's now give the definitions on the smooth compact special Lagrangian man- ifolds, then we'll extend this to the conifolds. Let's start with a result by Souriau (in [?]).

Theorem 5.1. Let (M, ω) be a symplectic manifold. Let $L \subset M$ be a smooth Lagrangian manifold. Then, tehere exists a neighbourhood $\mathscr{U}$ of the null section of L in its cotangent bundle T^*L and an immersion $\Phi_L : \mathscr{U} \to M$, such that $\Phi_{L|L} = Id : L \to L$ e $\Phi_L^*\omega = \hat{\omega}$.

Here we omit the proof, that can be found in Weinstein ([?]).

Let $L \subset M$ be a smooth special Lagrangian submanifold with metric induced by g. Let's consider the pull-back $\Phi_L^*(Im\Omega)$, defined on $\mathscr{U}$, with Φ defined as in 5.1. Given a closed form $\alpha \in \mathscr{C}^\infty(\mathscr{U})$, the $\Gamma(\alpha)$ is the submanifold in $\mathscr{U}$defined by its graph. It's diffeomorphic to L with the projection $\pi : T^*L \to L$. The pull-back shrinks to a m-form $\Phi^*|_L(Im\Omega)|_{\Gamma(\alpha)}$ on $\Gamma(\alpha)$. $\Gamma(\alpha)$ is special Lagrangian if and only if this form vanishes. Let's now pull-back this form to L with α, obtaining a real m-form on L: so $\Gamma(\alpha)$ is special Lagrangian if this form vanishes on L. Lastly, let $\star$ be the Hodge operator defined on L by g and by the orientation. By using it, we can reduce every m-form a function.

Let $\mathscr{D}_L$ be the space of the closed 1-forms on L which have the graph in $\mathscr{U}$. We define the map F:

$$F : \mathscr{D}_L \to \mathscr{C}^\infty(L), \quad \alpha \mapsto \star(\alpha^*(\Phi_L^* Im\Omega) = \star((\Phi_L \circ \alpha)^* Im\Omega). \tag{5.1}$$

Proposition 5.2. The non-linear map F has the following properties:

- The set $F^{-1}(0)$ parameterizes the space of the special Lagrangian deformations of L that are C^1-close to L.
- For every $\alpha \in \mathscr{D}_L$, $\int_L F(\alpha) vol_g = 0$.
- The linearization $dF[0]$ of F in 0 corresponds to the operator d^*, so:

$$dF[0](\alpha) = d^*\alpha \tag{5.2}$$

Proof. We use the notations introduced before in the paper. To simplify the notation, we identify $\mathscr{U}$ with its image in M by Φ_L. We write:

$$F(\alpha) = \star(\pi_*(Im\Omega|_{\Gamma(\alpha)}). \tag{5.3}$$

We identify L with the null section of T^*L. he first property follows from the definition of $F$and from the results on the smooth manifolds already considered. Precisely, the first property says that, by the compositions with Φ_L, $F^{-1}(0)$ corre- sponds to the set of special Lagrangian submanifolds that admits a parameteriza- tion which is con l'insieme C^1 -close to some parameterization of$L$.

To prove the second point, we note that $\int_L F(\alpha) vol_g = \int_{\Gamma(\alpha)} Im\Omega$. The fact that Ω is closed implies that $Im\Omega$ is closed. Moreover, the submanifold $\Gamma(\alpha)$ is homologous to the null section of L. So

$$\int_{\Gamma(\alpha)} Im\Omega = \int_L Im\Omega = 0,$$

because L is special Lagrangian. The fact that it is smooth is clear from the definition.

To prove equation (5.3), we fix $\alpha \in \Lambda^1(L)$ and let v be the normal vector field on L caused by having imposed $\alpha(\cdot) \equiv \omega(v, \cdot)$. We can now extend

v to a global vector field v on M. Let phi_S be a one-parameter family of diffeomorphisms of M such that $d/ds(\phi_S(x))|_{s=0} = v(x)$.

Then the two one-parameter families of m-forms on L, $(s\alpha)^*(Im\Omega) = \pi_*(Im\Omega|_{\Gamma(s\alpha)})$ e $(\phi_S Im\Omega)|_L$, are the same at the first order. As a matter of fact, the calculation of Lie derivative shows

$$\begin{aligned} sF[0](\alpha)vol_g &= d/ds(F(s\alpha)vol_g)|_{s=0} \\ &= d/ds(\phi_S^* Im\Omega)|_{L;s=0} \\ &= (\mathrm{L}_v Im\Omega)|_L = (di_v Im\Omega)|_L, \end{aligned}$$

in which in the last equality we use the Cartan formula $\mathrm{L}_v = di_v + i_v d$ and the fact that $Im\Omega$ is closed.

Now we say that $(i_v Im\Omega)|_L = - \star \alpha$ on L. This is a statement of linear algebra that can be easily proved. We can assume that v is a unit vector in that point.

We fixed $x \in L$ and an isomorphism $T_x M \simeq \mathbb{C}^m$ and doing so we identify the Calabi-Yau structures on $T_x M$ with the standard structures on $\mathbb{C}^m$. This map identifies $T_x L$ with a special Lagrangian plane Γ in $\mathbb{C}^m$. We consider the action of $SU(m)$ on the Grassmanian of m-planes in $\mathbb{C}^m$, then $SU(m)$ acts transitively on the subset of the special Lagrangian planes of dimension $m$and the isotropy subgroup corresponding to the distinct special Lagrangian planes in $\mathbb{R}^m := span\{\partial x^1, ..., \partial x^m\}$ is $SO(m) \subset SU(m)$. In other words, the set of the special Lagrangian planes in $\mathbb{C}^m$ corresponds to the homogeneus spaces $SU(m)/SO(m)$. Let's assume $\Gamma = \mathbb{R}^m$. Unless rotations in $SO(m)$, we assume that $v(x) = \partial y^1$. We can now write

$$Im\Omega = dy^1 \wedge dx^2 \wedge ... \wedge dx^m + (...).$$

It follows that $(i_v Im\Omega)|_{\mathbb{R}^m} = dx^2 \wedge ... \wedge dx^m$. On the other side $\alpha = -dx^1$, that proves the last point of the theorem. □

Extension to special Lagrangian cones

After obtaining some basic results on smooth manifolds, we can generalize them to special Lagrangian cones in $\mathbb{C}^m$. With the same notation used until now, we define:

- $\mathscr{C}$ is a special Lagrangian cone, with conical metric $\tilde{g}$ and orientation;

- $\Phi_{\mathscr{C}} : \mathscr{U} \to \mathbb{C}^m$ as defined above;

- μ, λ such that $\mu > 2, \lambda < 2$;

- $\mathscr{D}_{\mathscr{C}}$ the space of the closed 1-forms in $\mathscr{C}^{\infty}_{\mu-1,\lambda-1}(\Lambda^1)$ the graph of which is in $\mathscr{U}$.

- Given $\alpha \in \mathscr{D}_{\mathscr{C}}$, $F(\alpha)$ sia is like in equation (5.1)

Proposition 5.3. The non-linear map F has the following properties:

- The set $F^{-1}(0)$ parameterizes the space of all the special Lagrangian deformations of $\mathscr{C}$ that are vC^1-close to L and asymptotic to $\mathscr{C}$ with rates (μ, λ).

- F is a well defined smooth map

$$F : \mathscr{D}_{\mathscr{C}} \to \mathscr{C}^{\infty}_{(\mu-2,lambda-2)}(\mathscr{C}).$$

  In particular, for every $\alpha \in \mathscr{D}_{\mathscr{C}}$, $F(\alpha) \in \mathscr{C}^{\infty}_{(\mu-2,\lambda-2)}(\mathscr{C})$.

- The linearization corresponds to the operator d^*, that is

$$dF[0](\alpha) = d^*\alpha. \tag{5.4}$$

Proof. The first point follows immediately from the definition of F.

For the second point, let's consider:

$$\begin{aligned} F(\alpha) &= \star\,(\alpha^*(\Phi^*_{\mathscr{C}} Im\tilde{\Omega})) = Im\tilde{\Omega}((\Phi_{\mathscr{C}} \circ \alpha)_*(e_1), ..., (\Phi_{\mathscr{C}} \circ \alpha)_*(e_m)) \\ &= Im\tilde{\Omega}((\Psi_{\mathscr{C}} \circ \alpha)_*(e_1) + (R \circ \alpha)_*(e_1), ..., (\Psi_{\mathscr{C}} \circ \alpha)_*(e_m) + (R \circ \alpha)_*(e_m)) \\ &= Im\tilde{\Omega}((\Psi_{\mathscr{C}} \circ \alpha)_*(e_1), ..., (\Psi_{\mathscr{C}} \circ \alpha)_*(e_m) + ..., \end{aligned}$$

in which e_i is a local orthonormal $\tilde{g}$-base of $T\mathscr{C}$. When $r \to \infty$, the first term is of the form $Im\tilde{\Omega}(e_1, ..., e_m) + O(r^{\lambda-2}$. Since $\mathscr{C}$ is a special Lagrangian cone, the first term vanishes and only the term $O(r^{\lambda-2})$ remains. For the reasons we have seen before, the remaining terms in $F(\alpha)$ are of the form $O(r^{2\lambda-4})$. We apply analogous methods for $r \to 0$, and we have $F(\alpha) \in C^0_{(\mu-2,\lambda-2)}(\mathscr{C})$.

To study the derivatives of $F(\alpha)$, we put on $\mathscr{U}$ the metric and the Levi-Civita connnection ∇ pulled back from $\mathbb{C}^m$ with $\Phi_{\mathscr{C}}$, so we have $\nabla(\Phi^*_{\mathscr{C}} Im\tilde{\Omega}) = \Phi_{\mathscr{C}}(\tilde{\nabla} Im\tilde{\Omega}) = 0$. Let g be the metric induced on $\Gamma(\alpha)$. Then $\mathscr{C}$ can be equipped both with the metric $\tilde{g}$ and the connection $\tilde{\nabla}$, and the metric $\alpha^* g$

and the induced connection ∇. If $\alpha^* g$ is asymptotic to $\tilde{g}$, we have that the tensor $A := \nabla - \tilde{\nabla}$ satisfies to $|A| = O(r^{\lambda-3})$, when $r \to \infty$. Note that:

$$F(\alpha)vol_{\tilde{g}} = (\Phi_{\mathscr{C}} \circ \alpha)^* Im\tilde{\Omega}$$

thus, considering the derivatives,

$$\nabla(F(\alpha)vol_{\tilde{g}}) = \nabla((\Phi_{\mathscr{C}} \circ \alpha)^* Im\tilde{\Omega}) = (\Phi_{\mathscr{C}} \circ \alpha)^*(\tilde{\nabla} Im\tilde{\Omega}) = 0.$$

It follows

$$|(\nabla F(\alpha)) \oplus vol_{\tilde{g}}| = |F(\alpha) \cdot \nabla(vol_{\tilde{g}})| = O(r^{\lambda-2})|\nabla(vol_{\tilde{g}}|.$$

Let $vol_{\tilde{g}} = e_1^* \oplus ... \oplus e_m^*$, then $\nabla(vol_{\tilde{g}}) = \nabla e_1^* \oplus ... \oplus e_m^* + e_1^* \oplus ... \oplus \nabla e_m^*$. We assume that $\tilde{\nabla} e_i^* = 0$. Then $\nabla e_i^* = (\nabla - \tilde{\nabla})e_i^* = Ae_i^*$, from which $|\nabla(vol_{\tilde{g}})| = O(r^{\lambda-3})$. This shows that $F(\alpha) \in C^1_{(\mu-2,\lambda-2)}(\mathscr{C})$. Other calculation of the same type applied to the higher order derivatives shows that $F(\alpha) \in \mathscr{C}^{\infty}_{(\mu-2,\lambda-2)}(\mathscr{C})$. It's clear that F is smooth.

The third point can be proved as in the previous theorem. □

We avoid to extend the problem to the CS/AC Lagrangian submanifolds, because for these manifolds we can use the same technique used in the cases seen up until now, except for the fact that in this case we have to modify some definitions to understand how one can parameterize the special Lagrangian deformations of Lin which singularities move in the general space. Starting from the definition of CS/AC special Lagrangian submanifolds given in this chapter, we can build a principal $SU(m)$-bundle on M and repeat the method used for the case already analyzed, making the right modification.

For a study of this case one can refer to [?]

6 Some results on Laplace operator on conifolds

Let's talk about some analytical results on Laplace operator on confifolds.

Definition 6.1. Let (Σ, g')be a compact Riemannian manifold. Let's consider the cone $\mathscr{C} := \Sigma \times (0, \infty)$ equipped with the conical metric $\tilde{g} := dr^2 + r^2 g'$. Let $\Delta_{\tilde{g}}$ be the corresponding Laplace operator.

For every component (Σ_j, g'_j) of (Σ, g') and every $\gamma \in \mathbb{R}$, let's consider the space of the homogeneus function

$$V^j_\gamma := \{r^\gamma \sigma(\theta) : \Delta_{\tilde{g}}(r^\gamma \sigma) = 0\}. \tag{6.1}$$

Let $m^j := dim(V^j_\gamma)$. We show that $m^j_\gamma > 0$ if and only if γ satisfies

$$\gamma = \frac{(2-m) \pm \sqrt{(2-m)^2 + 4e^j_n}}{2}, \tag{6.2}$$

for some eigenvalues e^j_n of $\Delta_{g'_j}$ on Σ_j. Given the weight $\gamma \in \mathbb{R}^e$, we put $m(\gamma) := \sum_{j=1}^e m^j(\gamma_j)$. Let $\mathscr{D} \subset \mathbb{R}^e$ be the set of the weights γ such that $m(\gamma) > 0$. We call these the exceptional weights of $\Delta_{\tilde{g}}$. [3]

Definition 6.2. Let X and Y be two Banach spaces and let $T : X \to Y$ be a bounded linear operator. T is called Fredholm operator if the following holds:

- the kernel of T is finite-dimensional;
- the rank of T is closed;
- the corank of T is finite-dimensional.

If T is Fredholm we define the index of T as $dim(ker(T)) - dim(Coker(T))$.

Let (L, g) be a conifold. Let's assume (L, g) to be asymptotic to the cone $(\mathscr{C}, \tilde{g})$ in the sense specified in the definitions given above. Intuitively, the fact that g is asymptotic to $\tilde{g}$ implies that the Laplace operator Δ_g is asymptotic to $\Delta_{\tilde{g}}$. By applying the Definition 6.1 to $\mathscr{C}$ we define the weights $\mathscr{D} \subseteq \mathbb{R}^e$: these are the exceptional weights of Δ_g.

Definition 6.3. Given a metric couple (E, ∇), a vector $\beta = (\beta_1, ..., \beta_e) \in \mathbb{R}^e$, we define the weighted Sobolev spaces as:

$$W^p_{k,\beta}(E) := \text{ the completion of Banach spaces } \{\sigma \in \mathscr{C}^\infty(E) : \|\sigma\|_{W^p_{k,\beta}} < \infty\}, \tag{6.3}$$

with the norm $\|\sigma\|_{W^p_{k,\beta}} := \{\sum_{j=0}^k \int_L |\rho^{-\beta+j}\nabla^j\sigma|^p \rho^{-m} vol_g)^{1/p}$, in which ρ is the radius function already defined.

The weighted spaces of section C^k are defined by

$$C^k_\beta(E) := \{\sigma \in C^k(E) : \|\sigma\|_{C^k_\beta} < \infty\}, \tag{6.4}$$

where we use the norm $\|\sigma\|_{C^k_\beta} := \sum_{j=0}^k sup_{x\in L}|\rho^{-\beta+j}\nabla^j\sigma|$. These also are Banach spaces.

[3] 3For a complete definition of poids, see [?]

Let's give now a result related to weighted spaces.

Theorem 6.4. Let (L, g) be a conifold with e ends. Let $\mathscr{D}$ be the exceptional weights of ∇_g. Then $\mathscr{D}$ is a discrete subset of $\mathbb{R}^e$ and the Laplace operator

$$\Delta_g : W^p_{k,\beta}(L) \to W^p_{k-2,\beta-2}(L)$$

is Fredholm if and only if $\beta \notin \mathscr{D}$.

7 Moduli spaces of special Lagrangian conifolds

Let's remember the Implicit functions theorem.

Theorem 7.1. Let $F : E_1 \to E_2$ be a smooth map between Banach spaces such that $F(0) = 0$. Let's assume that $P := dF[0]$ is surjective and $Ker(P)$ admits a closed completion Z, so that $E_1 = Ker(P) \oplus Z$. Then, there exists a smooth map $\Phi : Ker(P) \to Z$ such that $F^{-1}(0)$ locally coincides with the graph $\Gamma(\Phi)$ of Φ. In particular, $F^{-1}(0)$ locally is a smooth Banach submanifold of E_1.

So we have the following result

Proposition 7.2. Let $F : E_1 \to E_2$ be a smooth map between Banach spacese such that $F(0) = 0$. Let's assume that $P := df[0]$ is Fredholm. We put $\mathscr{I} = Ker(P)$ and choose Z such that $E_1 = \mathscr{I} \oplus Z$. Let $\mathscr{O}$ be a finite-dimensional subspace of E_2 such that $E_2 = \mathscr{O} \oplus Im(P)$. Let's define:

$$G : \mathscr{O} \oplus E_1 \to E_2, \quad (\gamma, e) \mapsto \gamma + F(e).$$

We identify E_1 with $(0, E_1) \subset \mathscr{O} \oplus E_1$. Then:

- The map $dG[0] = Id \oplus P$ is surjective and $Ker(dG[0]) = Ker(P)$.Then, for the Implicit function theorem, there exists $\Phi : \mathscr{I} \to \mathscr{O} \oplus Z$ such that $G^{-1}(0) = \Gamma(\Phi)$.

- $F{-}1(0) = \{(i, \Phi(i)) : \Phi(i) \in Z\} = \{(i, \Phi(i)) : \pi_{\mathscr{O}} \circ \Phi(i) = 0\}$ in which $\pi_{\mathscr{O}} : \mathscr{O} \oplus Z \to \mathscr{O}$ is the standard projection map.

- Let $\pi_{\mathscr{I}} : \mathscr{I} \oplus Z \to \mathscr{I}$ be the standard projection. Then $\pi_{\mathscr{I}}$ is a continuous open map and shrinks to an omeomorphism

$$\pi_{\mathscr{I}} : F^{-1}(0) \to (\pi_{\mathscr{O}} \circ \Phi)^{-1}(0)$$

between $F^{-1}(0)$ and the zero set [4] of the smooth map $\pi_{\mathcal{O}} \circ \Phi : \mathcal{I} \to \mathcal{O}$, che which is defined between finite-dimensional spaces.

Now we have all the tools to prove smoothness results in the special Lagrangian moduli space by extending McLean theorem.

We'll do the following steps. We can view the moduli space as the zero set of a map F. The next step is to use the regularity to show that is one can consider in the same way the zero set of another map $\tilde{F}$. The domain of $\tilde{F}$ is of the form $K \times X^p_{k,(\mu,\lambda)}(L)$ in which K is a finite-dimensional vector space. So, the differential $d\tilde{F}$ is a finite-dimensional pertubation of the Laplace operator Δ_g. The second step is to analyze the linearized operator, by showing that, under the right condition, it is surjective. The third step is to identify the ker of $d\tilde{F}[0]$ and to apply the implicit function Theorem.

We want to study the problem on the AC special Lagrangian, for the CS and the CS/AC, refer to [?], [?] e [?]

AC Special Lagrangian submanifolds

An extension of McLean theorem to the AC special Lagrangian submanifolds has been poved by Pacini and Marshall. Let's give a version of the proof, starting by the regularity theorems proved by Joyce (in [?]).

Lemma 7.3. Let $\mathcal{C}$ be a special Lagrangian cone in $\mathbb{C}^m$, with a metric $\tilde{g}$ and an orientation. Let's define $\Phi_{\mathcal{C}} : \mathcal{U} \to \mathbb{C}^m$ and the map F as introduced before. We fix $\mu > 2$ and $\lambda < 2$, with $\lambda \neq 0$. We consider a closed 1-fom $\alpha \in C^1_{(\mu-1,\lambda-1)}(\mathcal{U})$ which satisfies $F(\alpha) = 0$. We write $\alpha = \alpha' + dA$ in which α' has compact support on little ends and translation invariant on the big ones, and $A \in C^1_{(\mu,\lambda)}(L)$. Then α' is smooth and $A \in \mathcal{C}^\infty_{(\mu,\lambda)}(L)$, so $\alpha \in \mathcal{C}^\infty_{(\mu-1,\lambda-1)}(\mathcal{U})$.

[5]. Starting from this lemma and from the regularity which descends from it, we can prove the results we have anticipated:

Theorem 7.4. Let L be a special Lagrangian submanifold of $\mathbb{C}^m$ with rate λ. Let $\mathcal{M}_L$ be the moduli space of the special Lagrangian deformations of L with rate λ. Let's consider the operator

$$\Delta_g : W_{k,\lambda}(L) \to W^p_{k-2,\lambda-2}(L). \tag{7.1}$$

[4]Given an open set U, the zero set of a function F is the set $Z := \{z \in U : f(z) = 0\}$

[5]for the proof of the lemma, see Joyce [?]

- If $\lambda \in (0, 2)$ is a non-exceptional weights for Δ_g, then $\mathscr{M}_L$ is a smooth manifolds of dimension
- Se $\lambda \in (2-m, 0)$, then $\mathscr{M}_L$ is a smooth manifold of dimension $b^1_{\mathscr{C}}(L)$.

Proof. We choose $k \geq 3$ and $p > m$ such that $W^p_{(k-1,\lambda-1)}(\Lambda^1) \subset C^1_{\lambda-1}(\Lambda^1)$. Let $\mathscr{D}_L$ be the space of the closed 1-forms in $W^p_{(k-1,\lambda-1)}(\Lambda^1)$ that have the graph $\Gamma(\alpha)$ in $\mathscr{U}$. Let's consider the map

$$F : \mathscr{D}_L \to W^p_{(k-2,\lambda-2)}(L), \quad \alpha \mapsto \star(\pi_*((\Phi_L^* Im\tilde{\Omega})|_{\Gamma(\alpha)})). \tag{7.2}$$

We assume $\lambda < 2$. Since $W^p_{(k-2,\lambda-2)}$is closed by multiplication, we have that $F$is a smooth map between Banach spaces that is locally well defined, with differential $dF[0](\alpha) = d^*\alpha$. We assume $F(\alpha) = 0$. For the lemma 7.3, $\alpha \in \mathscr{C}^\infty_{\lambda-1}(\Lambda^1)$, so $F^{-1}(0)$ is locally omeomorphic, using the Φ_L already introduced, to $\mathscr{M}_L$.

We now assume $\lambda \in (0, 2)$. Every $\alpha \in F^{-1}(0)$ is of the form $\alpha = \beta + df$, for $\beta \in H$ (by using H we point to a finite dimensional vector space of closed 1-forms on L) and $f \in \mathscr{C}^\infty_\lambda(L)$. Let's define $\tilde{\mathscr{D}}_L$ as the space of the couples (β, f) in $H \times W^p_{(k,\lambda)}(L)$ such that $\alpha := \beta + df \in \mathscr{D}_L$. From here we have that $\tilde{\mathscr{D}}_L$is an open neighbourhood of the origin.

Then $\tilde{\mathscr{D}}_L$ is the domain of a smooth map between Banach spaces locally defined:

$$\tilde{F} : H \times W^p_{(k,\lambda)}(L) \to W^p_{(k-2,\lambda-2)}(L), \quad \tilde{F}(\beta, f) := F(\beta + df) \tag{7.3}$$

with $d\tilde{F}[0](\beta, f) = d^*\beta + \Delta_g f$ and translation invariant in $\mathbb{R}$. Let $\tilde{F}(\beta, f) = 0$. From the lemma 7.3 we have that $f \in \mathscr{C}^\infty_\lambda(L)$. This shows that $\mathscr{M}_L$ is locally omeomorphic, using Φ_L, to the quotient space $\tilde{F}^{-1}(0)/\mathbb{R}$.

To finish, we only have to show that $\tilde{F}^{-1}(0)$ is smooth. To do this, we have to assume that $\lambda$is not exceptional. In this case, in view of the things said for the Laplacian operator, 7.1 is surjective, so $d\tilde{F}[0]$is surjective. Let β_i be a base of H. For every β_i, the equation $d\tilde{F}[0](\beta_i, f) = 0$ admits a solution f_i. Other solutions are given by the couples $\beta = 0, f \in Ker(\Delta_g)$. Let's notice that these ones gives a basis for the kernel of $d\tilde{F}[0]$. By applying the implicit function Theorem we can conclude saying that $\tilde{F}^{-1}(0)$ is smooth of dimension $dim(H \oplus Ker(\Delta_g))$. So, $\mathscr{M}_L$ is smooth and has the asserted dimension.

Let's finally assume that $\lambda \in (2-m, 0)$. In this case, we can write every $\alpha \in F^{-1}(0)$ as $\alpha = \beta + dv + df$, in which $\beta \in \tilde{H}$ ($\tilde{H}$ and here the spaces of translation invariant 1-forms), $dv \in dE$.We have called E the vector space spanned by the smooth functions f_i on L, such that $f_i \equiv 1$ on the end S_i and $f_i \equiv 0$ on the other ends. Moreover, $\sum f_i \equiv 1$. df is an element of $d(\mathscr{C}^\infty_\lambda(L))$.

We can use the regularity of the lemma 7.3,as done before, to prove that $\mathscr{M}_L$ is locally omeomorphic to the quotient space $\tilde{F}^{-1}(0)/\mathbb{R}$ for the map

$$\tilde{F} : \tilde{H} \times E \times W^p_{k,\lambda}(L) \to W^p_{k-2,\lambda-2}(L), \quad \tilde{F}(\beta, v, f) = F(\beta + dv + df). \tag{7.4}$$

We can conclude that $\tilde{F}^{-1}(0)$ is smooth of dimension $dim(\tilde{H} \oplus E)$. □