Improvement on Stirling's Formula for n! using Product Integrals

(The following is inspired by Tyler Neylon's use of Product Integrals for deriving Stirling's Formula-like expressions).

Consider:

i)

$$\prod_{a,x\in R}^{b} f(x)^{pr(x)dx} = e\uparrow\int_a^b \ln(f(x))pr(x)dx \le \int_a^b f(x)pr(x)dx$$

That is, the geometric mean of a continuous distribution over an interval (a,b) is < or = its average.

Now consider f(x)=x with b=a+1 and pr(x)=1 for all unitary subintervals, then

ii)

$$\prod_{0.5,x\in R}^{1.5} x^{dx} * \prod_{1.5,x\in R}^{2.5} x^{dx} * \prod_{2.5,x\in R}^{3.5} x^{dx} * ... * \prod_{n-0.5,x\in R}^{n+0.5} x^{dx} < \int_{0.5}^{1.5} xdx * \int_{1.5}^{2.5} xdx * ... * \int_{n-0.5}^{n+0.5} xdx$$

And so LHS of ii) is:

iii)

$$LHS = \prod_{0.5,x\in R}^{n+0.5} x^{dx} = e\uparrow\int_{0.5}^{n+0.5} \ln(x)dx = \sqrt{2n+1}\left(\frac{n+0.5}{e}\right)^n$$

And since RHS of ii) = 1*2*3*…*n=n! , you get the lower bound below of

iv)

$$\sqrt{2n+1}\left(\frac{n+0.5}{e}\right)^n < n!$$

For an upper bound, shift the interval of "multigration" 0.05 units to the right giving (by a similar argument):

v)

$$\prod_{0.55,x\in R}^{1.55} x^{dx} * \prod_{1.55,x\in R}^{2.55} x^{dx} * \prod_{2.55,x\in R}^{3.55} x^{dx} * ... * \prod_{n-0.45,x\in R}^{n+0.55} x^{dx} > n!$$

This gives the upper bound of:

vi) $$n! < \left(\frac{n+0.55}{0.55}\right)^{0.55}\left(\frac{n+0.55}{e}\right)^{n}$$

Thus combining with iv) gives:

vii)
$$\sqrt{2n+1}\left(\frac{n+0.5}{e}\right)^{n} < n! < \left(\frac{n+0.55}{0.55}\right)^{0.55}\left(\frac{n+0.55}{e}\right)^{n}$$

The table below shows a comparison with n!

n	lower bound	n!	upper bound
1	0.955778825	1	1.008138432
2	1.891368082	2	2.045921739
3	5.647675135	6	6.21194681
4	22.53167002	24	25.09426315
5	112.4701775	120	126.5185388
6	674.0388432	720	764.5645267
7	4714.255321	5040	5385.656275
8	37689.46834	40320	43326.6336
9	339030.6099	362880	391906.4239
10	3388892.706	3628800	3937033.869
11	37264977.3	39916800	43489372.13
12	447050286.6	479001600	523896499.9
13	5810220412	6227020800	6835176069
14	81325788666	87178291200	96013947810
15	1.21966E+12	1.30767E+12	1.44474E+12
16	1.95114E+13	2.09228E+13	2.31844E+13
17	3.31646E+14	3.55687E+14	3.95237E+14
18	5.96886E+15	6.40237E+15	7.13312E+15

On closer inspection, most the difference between n factorial and these approximations appears to arise from the first few terms.

So, if we could avoid this, we'd get closer approximations. This can done using the result below:

viii)

$$For\ \text{n>m:}\quad n! \geq m!\left(\frac{e}{m+0.5}\right)^{(m+0.5)} * \left(\frac{n+0.5}{e}\right)^{(n+0.5)}$$

For example: with m=3 then for n ≥ 3,

$$\text{n}! \geq 6\left(\frac{e}{3.5}\right)^{(3.5)} * \left(\frac{n+0.5}{e}\right)^{(n+0.5)}$$

Define Q(n,m) as:

ix)

$$\text{Q(n,m)=m!}\left(\frac{e}{m+0.5}\right)^{(m+0.5)} * \left(\frac{n+0.5}{e}\right)^{(n+0.5)}$$

Thus, for example, we'd have:

x)

$$\text{Q(n,3)=6}\left(\frac{e}{3.5}\right)^{(3.5)} * \left(\frac{n+0.5}{e}\right)^{(n+0.5)}$$

This gives output below:

n	n!	Q(n,3)	Stirling's Formula
3	6	6	5.836209591
4	24	23.93728691	23.50617513
5	120	119.486523	118.019168
6	720	716.0881181	710.0781846
7	5040	5008.349675	4980.395832
8	40320	40040.69013	39902.39545
9	362880	360180.7134	359536.8728
10	3628800	3600305.568	3598695.619
11	39916800	39589717.61	39615625.05
12	479001600	474939095.3	475687486.5
13	6227020800	6172685510	6187239475
14	87178291200	86399220971	86661001741
15	1.30767E+12	1.29575E+12	1.30043E+12
16	2.09228E+13	2.07286E+13	2.08141E+13
17	3.55687E+14	3.52335E+14	3.53948E+14
18	6.40237E+15	6.34122E+15	6.3728E+15

19	1.21645E+17	1.20469E+17	1.21113E+17
20	2.4329E+18	2.40913E+18	2.42279E+18
21	5.10909E+19	5.0587E+19	5.08886E+19
22	1.124E+21	1.11282E+21	1.11975E+21
23	2.5852E+22	2.55928E+22	2.57585E+22

Notice the reasonable approximation by Q(n,3), whose closeness to n! is overtaken by Stirling's formula for higher n.

Using higher m for Q(n,m) gives a closer approximation to n! for higher n – see Q(n,6) below.

n	n!	Q(n,6)	Stirling's Formula
3	6	6.032777099	5.836209591
4	24	24.06805272	23.50617513
5	120	120.1392599	118.019168
6	720	720	710.0781846
7	5040	5035.709537	4980.395832
8	40320	40259.42641	39902.39545
9	362880	362148.3266	359536.8728
10	3628800	3619973.496	3598695.619
11	39916800	39805990.3	39615625.05
12	4.79E+08	477533616.3	475687486.5
13	6.23E+09	6206405964	6187239475
14	8.72E+10	86871206947	86661001741
15	1.31E+12	1.30283E+12	1.30043E+12
16	2.09E+13	2.08418E+13	2.08141E+13
17	3.56E+14	3.5426E+14	3.53948E+14
18	6.4E+15	6.37586E+15	6.3728E+15
19	1.22E+17	1.21127E+17	1.21113E+17
20	2.43E+18	2.4223E+18	2.42279E+18
21	5.11E+19	5.08634E+19	5.08886E+19
22	1.12E+21	1.1189E+21	1.11975E+21
23	2.59E+22	2.57326E+22	2.57585E+22

So what value of m gives the best approximation?

Looking at

$$m!\left(\frac{e}{m+0.5}\right)^{(m+0.5)}$$

in expression viii) shows that it tends to sqr(2*pi) as m tends to infinity. (This is easy to show using Stirling's Formula).

This makes the expression

xi)

$$n! \approx \sqrt{2\pi}\left(\frac{n+0.5}{e}\right)^{n+0.5}$$

a worthwhile candidate for approximating n!

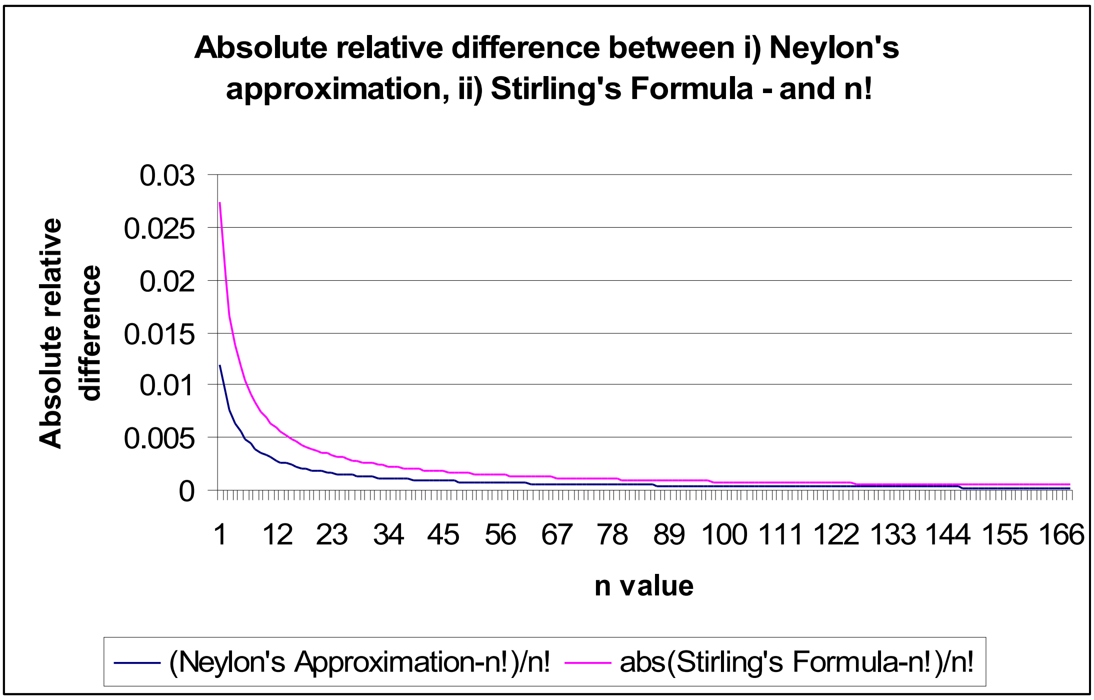
Call this Neylon's Approximation.

The table below shows some values.

n	n!	Neylon's Approximation	Stirling's Formula
3	6	6.07151962	5.836209591
4	24	24.22261785	23.50617513
5	120	120.9107948	118.019168
6	720	724.6238431	710.0781846
7	5040	5068.048885	4980.395832
8	40320	40517.97261	39902.39545
9	362880	364474.0447	359536.8728
10	3628800	3643220.982	3598695.619
11	39916800	40061624.54	39615625.05
12	4.79E+08	480600339.2	475687486.5
13	6.23E+09	6246263530	6187239475
14	8.72E+10	87429094208	86661001741
15	1.31E+12	1.31119E+12	1.30043E+12
16	2.09E+13	2.09757E+13	2.08141E+13
17	3.56E+14	3.56535E+14	3.53948E+14
18	6.4E+15	6.41681E+15	6.3728E+15
19	1.22E+17	1.21905E+17	1.21113E+17
20	2.43E+18	2.43785E+18	2.42279E+18
21	5.11E+19	5.119E+19	5.08886E+19
22	1.12E+21	1.12608E+21	1.11975E+21
23	2.59E+22	2.58979E+22	2.57585E+22

Notice the improvement over Stirling's Formula which applies for all n.

The graph below shows the absolute relative difference between Neylon's Approximation, Stirlings Formula and n!



Further improvements can be obtained. Consider a modified version of Neylon's Approximation

xii)

$$n! \approx \sqrt{2\pi}\left(\frac{n+0.5-\dfrac{k}{n}}{e}\right)^{n+0.5}$$

for a given fixed k

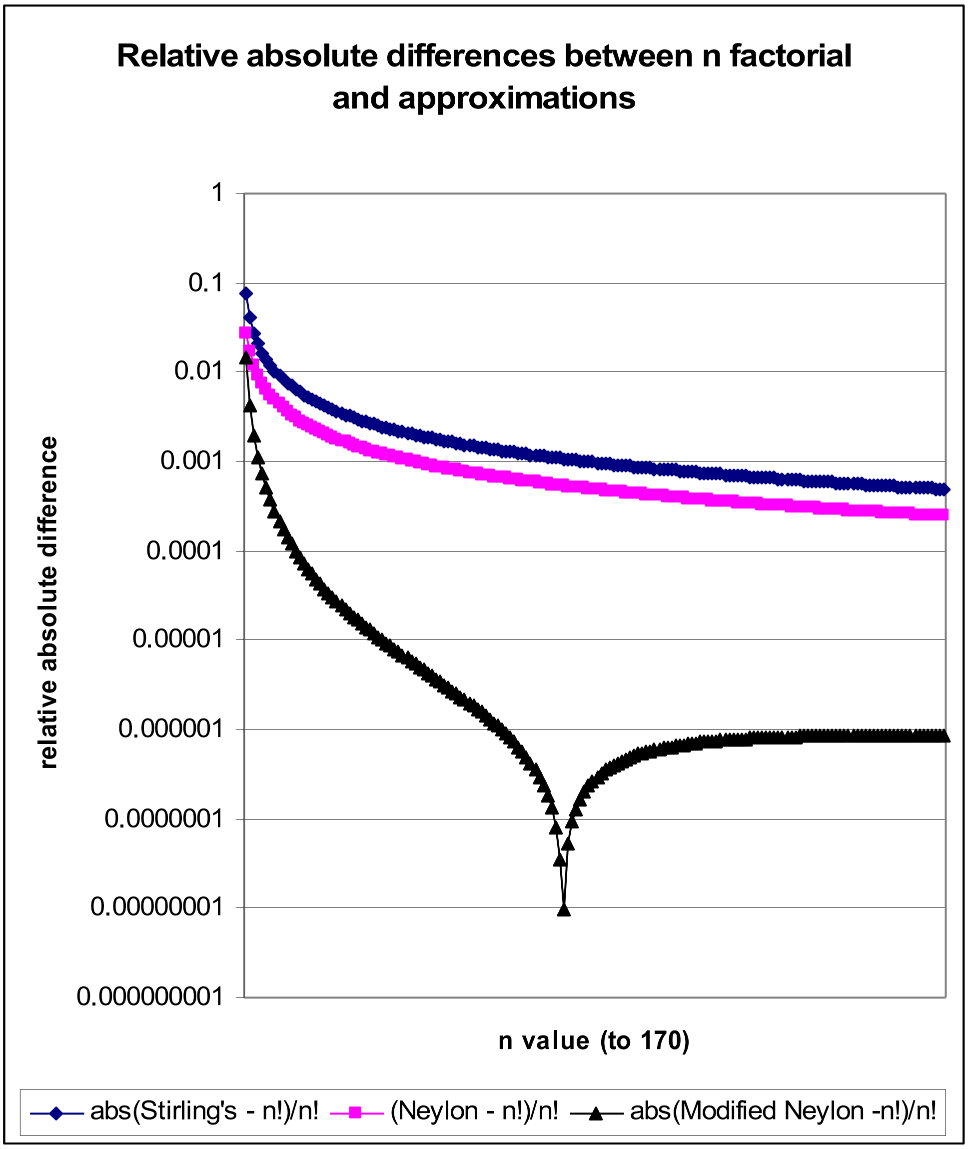
For demonstration purposes, try k=0.0414.

xiii)

$$n! \approx \sqrt{2\pi}\left(\frac{n+0.5-k/n}{e}\right)^{n+0.5}$$

for k = 0.0414

Below is a graph of the relative absolute difference between three n! approximations (Stirling's, Neylon's and Neylon's Modified for k=0.0414) and n!



Notice the closer approximation of n! of Neylon's Modified to both its unmodified cousin and Stirling's Formula.

The question is what value of k gives the best approximation?

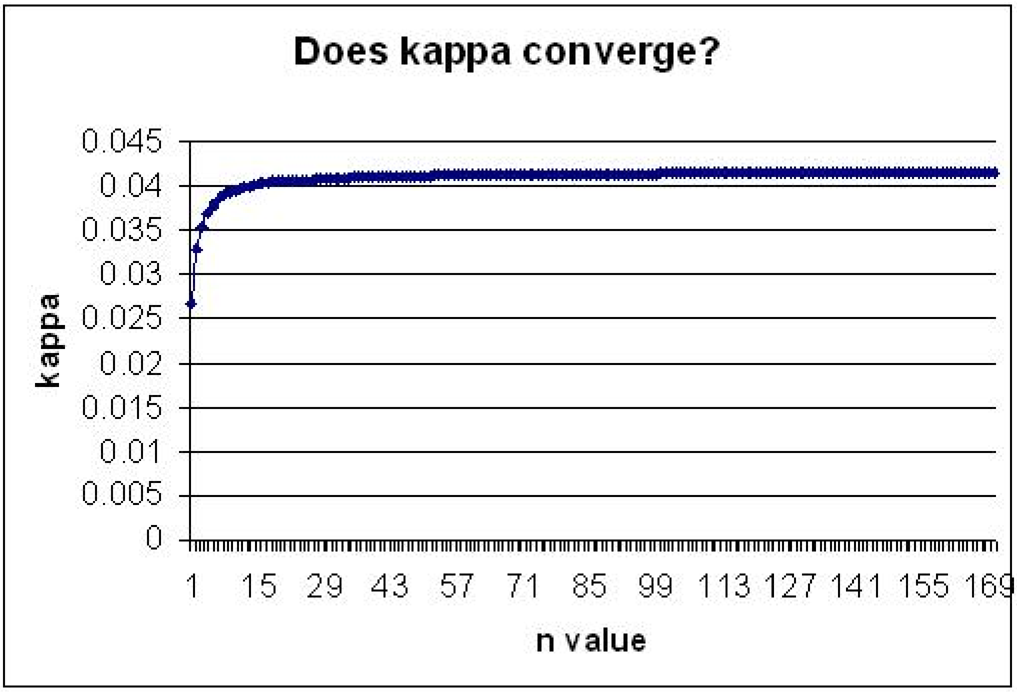
Consider the expression for kappa below.

$$\kappa = \eta * (\eta + 0.5 - e * (\frac{\eta!}{\sqrt{2\pi}})^{\frac{1}{\eta+0.5}}) = ?$$

$$\text{as } \eta \rightarrow \infty$$

Choosing a sufficiently large eta value in expression x) would give a k value that shifts the “dip” in the above graph “off-scale” to the right.

Graphing the lower of expression of x) for eta from 1 to 170 suggests kappa may converge for large eta.



So, does kappa converge?

n	kappa	n	kappa
1	0.026892211	86	0.041425381
2	0.032927464	87	0.041428143
3	0.035488253	88	0.041430843
4	0.036894091	89	0.041433483
5	0.037780513	90	0.041436064
6	0.038389919	91	0.041438588
7	0.038834413	92	0.041441058
8	0.039172874	93	0.041443475
9	0.03943916	94	0.041445841
10	0.039654114	95	0.041448157
11	0.039831263	96	0.041450425
12	0.039979768	97	0.041452646
13	0.040106053	98	0.041454822
14	0.040214756	99	0.041456955
15	0.040309308	100	0.041459045
16	0.040392303	101	0.041461094
17	0.040465736	102	0.041463102

18	0.04053117	103	0.041465072
19	0.040589844	104	0.041467004
20	0.040642754	105	0.041468899
21	0.040690708	106	0.041470759
22	0.040734373	107	0.041472584
23	0.040774299	108	0.041474375
24	0.040810946	109	0.041476134
25	0.040844702	110	0.041477861
26	0.040875896	111	0.041479556
27	0.04090481	112	0.041481222
28	0.040931684	113	0.041482858
29	0.040956727	114	0.041484466
30	0.04098012	115	0.041486045
31	0.04100202	116	0.041487598
32	0.041022567	117	0.041489124
33	0.041041881	118	0.041490624
34	0.041060071	119	0.041492099
35	0.041077232	120	0.04149355
36	0.041093449	121	0.041494976
37	0.041108797	122	0.04149638
38	0.041123345	123	0.04149776
39	0.041137153	124	0.041499119
40	0.041150277	125	0.041500455
41	0.041162767	126	0.041501771
42	0.041174666	127	0.041503066
43	0.041186017	128	0.041504341
44	0.041196855	129	0.041505596
45	0.041207216	130	0.041506831
46	0.041217129	131	0.041508048
47	0.041226624	132	0.041509247
48	0.041235726	133	0.041510427
49	0.041244459	134	0.04151159
50	0.041252845	135	0.041512736
51	0.041260904	136	0.041513865
52	0.041268656	137	0.041514978
53	0.041276117	138	0.041516074
54	0.041283303	139	0.041517155
55	0.04129023	140	0.04151822
56	0.041296911	141	0.04151927
57	0.041303359	142	0.041520306
58	0.041309586	143	0.041521327
59	0.041315603	144	0.041522334
60	0.04132142	145	0.041523327
61	0.041327048	146	0.041524306
62	0.041332496	147	0.041525273
63	0.041337771	148	0.041526226
64	0.041342883	149	0.041527166
65	0.041347838	150	0.041528094
66	0.041352643	151	0.04152901
67	0.041357306	152	0.041529913
68	0.041361833	153	0.041530805
69	0.041366229	154	0.041531685
70	0.041370499	155	0.041532554
71	0.041374651	156	0.041533412
72	0.041378687	157	0.041534259
73	0.041382614	158	0.041535095
74	0.041386434	159	0.041535921
75	0.041390154	160	0.041536736

76	0.041393776	161	0.041537542
77	0.041397304	162	0.041538337
78	0.041400742	163	0.041539123
79	0.041404094	164	0.041539899
80	0.041407362	165	0.041540665
81	0.041410549	166	0.041541423
82	0.04141366	167	0.041542171
83	0.041416695	168	0.041542911
84	0.041419659	169	0.041543642
85	0.041422553	170	0.041544364

If it does, then replacing k in xiii) with this kappa value should provide a closer approximation. Whether kappa converges is still an open question.