# The “Fair” Bet Paradox

Most people would consider odds of $2 for every $1 bet on correctly calling a tossed coin a “fair bet”. On average, you’d expect a gain of ½ (amount bet) + ½ (-amount bet) or $0 per toss. Simple.

But, believe it or not, there are ***ways*** of betting that can dramatically turn a “fair bet” into an unfair one.

To see this, consider two punters – **Alice and Bob**.

Alice only ever bets $1 per toss, always on “heads”.
Bob also always bets “heads” but instead of betting $1 per toss, he bets ***half his current holdings.***

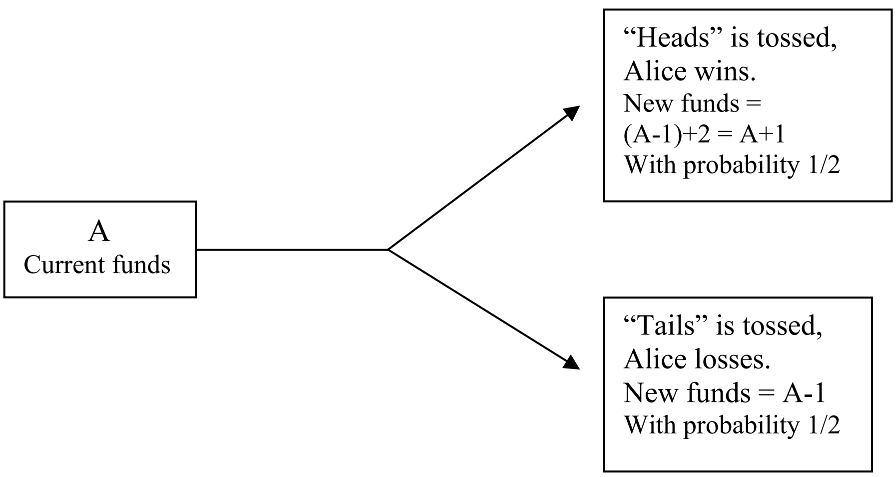
Now, imagine they both start with $1000. Both should neither win nor lose on average in the long-term, right?

Well, guess what…

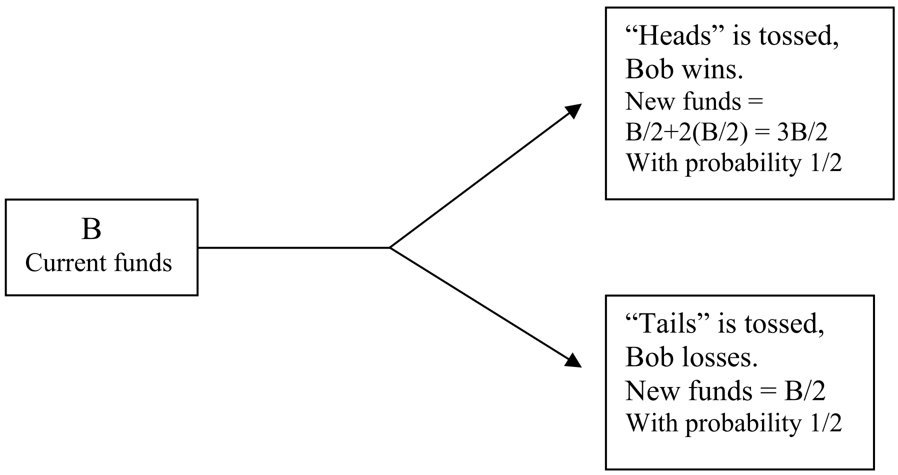
You’ll find Alice’s funds tend to fluctuate around $1000 in a “drunkard’s walk” fashion – as expected.

But Bob tends to quickly go broke! Amazing.

To see why, consider what happens on each toss. Let **A** represent Alice’s current funds. Then her fortunes change each toss as follows:




Now consider Bob. Let B represent his current funds. Then his fortunes change each toss as follows:



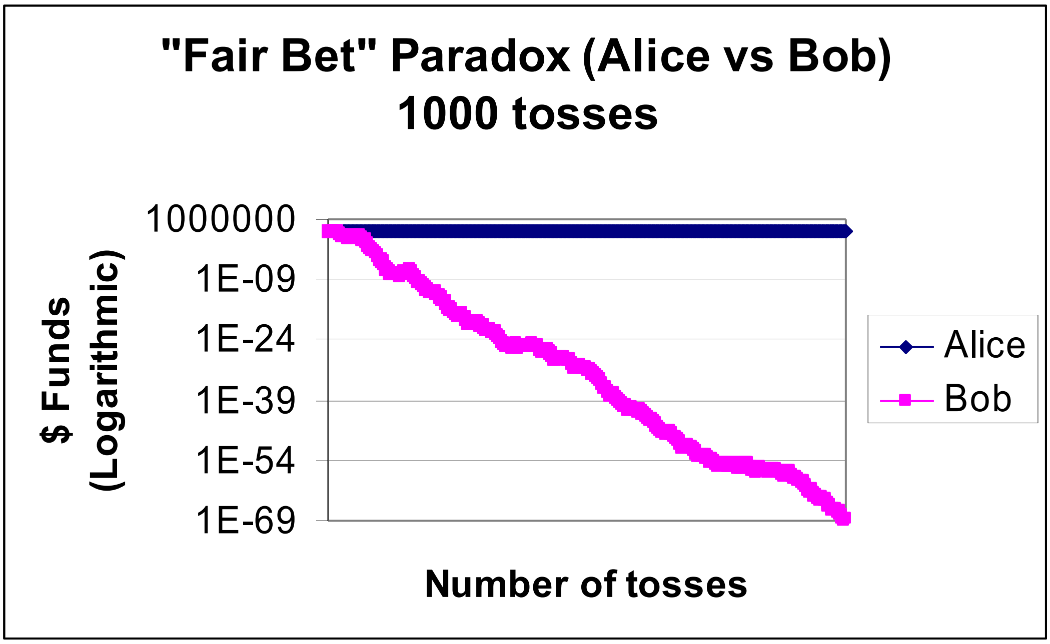

So, with probability of ½, his funds either go to 1.5 times or 0.5 times what they are. A moments consideration should indicate that the continued application of such an action quickly tends to zero.

Hard to believe? Join the club. I had to do some simulations using the following program to convince myself.

**Program**

| | **Description** |
|---|---|
| 10 A=1000 : B=1000 | Alice (A) and Bob (B) start with $1000. |
| 20 R = ran# | R is a random number between 0 and 1. |
| 30 If R<0.5 then A=A+1 and B=1.5*B | “Heads” is tossed: Alice and Bob win. |
| 40 If R>0.5 then A=A-1 and B=0.5*B | “Tails” is tossed: Alice and Bob lose. |
| 50 Print A, B : Goto 20 | Print current funds then toss next coin. |

A typical output of the above program (using Microsoft Excel) goes:




Alice's funds fluctuate around $1000, while poor old Bob goes broke. This is despite the same odds and the same choice of "heads". Crazy, right? But true.

Bob's betting strategy turns a "fair bet" into an "unfair bet". A similar result applies if Bob bets a different proportion of funds – like a quarter or one-tenth, etc. The principle can also be translated into other realms like horse racing and even economics (funds management, etc). Thus, a **"fair" game** of chance consists not just of **"fair" odds** but also an **appropriate betting strategy** as well. Most unexpected.

In the long-term Bob's funds tend to decrease by (roughly) a factor of sqr((1/2)*(3/2)) =0.8660… each toss. This factor is a **discrete** **multigral** (see the website www.geocities.com/multigrals2000 for more on multigrals).

Now think – How many punters (and funds managers, etc) adopt something like Bob's "bet a portion of your funds" strategy either consciously or unconsciously and thereby rob themselves even more than would normally be the case? It's a frightful thought.

There are ways a group of punters using Bob's damaging betting strategy can ***partially*** reduce the rate at which they go broke, using something called the Dunham Effect. Lack of space forbids discussion of this.

So, was that a surprise? It certainly surprised me. Hopefully you will examine whether you are unconsciously robbing yourself next time you punt. Remember – if you must bet, small **fixed** amounts not a proportion of your current funds is the way to go.

## Random Betting

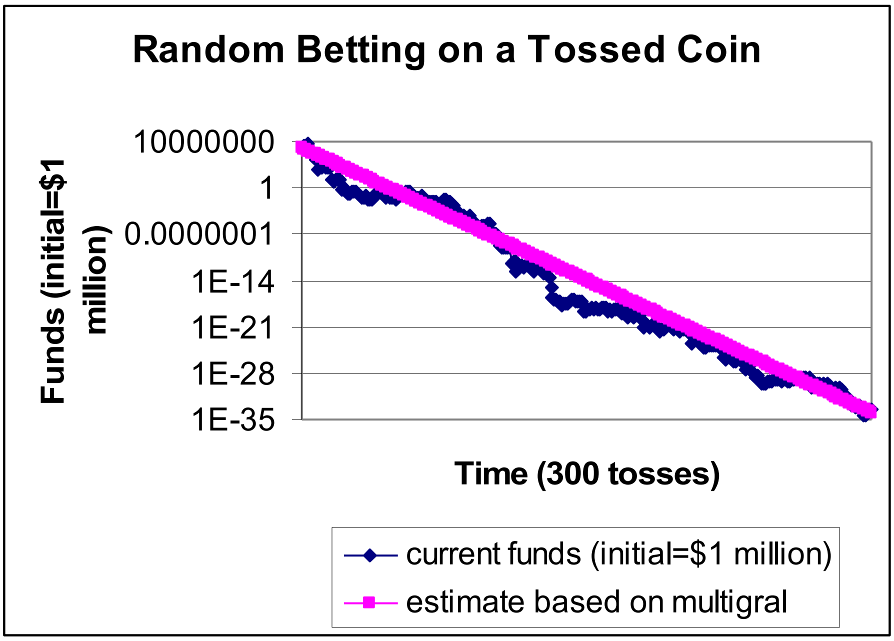
Consider the case where the amount bet is a uniformly random fraction between 0 and 1 of current funds. What happens then? Below is the associated multigral for such a betting scheme. Notice the end result in the median is worse than for betting half your current funds (as above).

$$A_n \to \left[\prod_0^1 (1+x)^{dx/2} * \prod_0^1 (1-x)^{dx/2}\right]^n * A_0 \text{ in the median}$$

$$\to \left[\prod_1^2 x^{dx/2} * \prod_0^1 x^{dx/2}\right]^n * A_0 = \left[\prod_0^2 x^{dx/2}\right]^n * A_0$$

$$\to \left[e^{(1/2)(x\ln(x)-x))_0^2}\right]^n * A_0 \to (0.735788824...)^n * A_0 \text{as n} \to \infty$$

Simulations supports the above multigral.




It is relatively easy to genealise the above multigral expression for other types of stochastic betting, uneven probabilities from a biased coin, and “unfair” returns on a correct choice.

<u>References</u>:

1. Feller, William:

"**Introduction to Probability Theory and its Applications**" Vol 1 Wiley 1950: chapter 8 section 2 'Systems of Gambling' p185 and chapter 10 section 3 'The Theory of Fair Games' p233

2. www.geocities.com/multigrals2000 - a somewhat scrappy website on multigrals in general – download some word docs.

3 http://www.probabilitytheory.info/topics/fair_unfair_games_prospect_ruin.htm

4. http://en.wikipedia.org/wiki/Expected_value

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**Supplementary material:**

The material below, taken from the web, shows the way most people, even mathematicians, think about "fair" bets/games. You have been warned!

**Extract from** :

http://www.probabilitytheory.info/topics/fair_unfair_games_prospect_ruin.htm

# Fair or unfair games - "Expectation"

One of the most valuable uses to which a gambler can put his knowledge of probabilities is to decide whether a game or proposition is fair, or equitable. To do this a gambler must calculate his 'expectation'. A gambler's expectation is the amount he stands to win multiplied by the probability of his winning it. A game is a fair game if the gambler's expectation equals his stake. If a gambler is offered 10 units each time he tosses a head with a true coin, and pays 5 units for each toss is this a fair game? The gambler's expectation is 10 units multiplied by the probability of throwing a head, which is 1/2. His expectation is 10 units = 5 units, which is what he stakes on the game, so the game is fair.

Suppose a gambler stakes 2 units on the throw of a die. On throws of 1, 2, 3 and 6 he is paid the number of units shown, on the die. If he throws 4 or 5 he loses Is this fair? This can be calculated as above. The probability of throwing any number is 1/6, so his expectation is 6/6 on number 6, 3/6 on number 3, 2/6 on number 2 and 1/6 on number 1.

His total expectation is therefore 6/6+3/6+2/6+1/6, which equals 2, the stake for a throw, so the game is fair.

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## Notice: no mention of the way you bet affecting outcomes.

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# Expected value

From Wikipedia, the free encyclopedia.

In probability theory (and especially gambling), the **expected value** (or **mathematical expectation**) of a random variable is the sum of the probability of each possible outcome of the experiment multiplied by its payoff ("value"). Thus, it represents the average amount one "expects" to win per bet if bets with identical odds are repeated many times. Note that the value itself may not be expected in the general sense, it may be unlikely or even impossible.

For example, an American roulette wheel has 38 equally possible outcomes. A bet placed on a single number pays 35-to-1 (this means that you are paid 35 times your bet and your bet is returned, so you get 36 times your bet). So the expected value of the profit resulting from a $1 bet on a single number is, considering all 38 possible outcomes: $(-\$1 \times \frac{37}{38}) + (\$35 \times \frac{1}{38})$, which is about -$0.0526. Therefore one expects, on average, to lose over five cents for every dollar bet.