The internal structure of natural numbers and one method for the definition of large prime numbers

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Abstract

It holds that every product of natural numbers can also be written as a sum. The inverse does not hold when 1 is excluded from the product. For this reason, the investigation of natural numbers should be done through their sum and not through their product. Such an investigation is presented in the present article. We prove that primes play the same role for odd numbers as the powers of 2 for even numbers, and vice versa. The following theorem is proven: "Every natural number, except for 0 and 1, can be uniquely written as a linear combination of consecutive powers of 2 with the coefficients of the linear combination being -1 or +1." This theorem reveals a set of symmetries in the internal order of natural numbers which cannot be derived when studying natural numbers on the basis of the product. From such a symmetry a method for identifying large prime numbers is derived.

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1 Introduction

It holds that every product of natural numbers can also be written as a sum. The inverse (i.e. each sum of natural numbers can be written as a product) does not hold when 1 is excluded from the product. This is due to prime numbers p which can be written as a product only in the form of $p = 1 \cdot p$. For this reason, the investigation of natural numbers should be done through their sum and not through their product. Such an investigation is presented in the present article.

We prove that each natural number can be written as a sum of three or more consecutive natural numbers except of the powers of 2 and the prime numbers. Each power of 2 and each prime number cannot be written as a sum of three or more consecutive natural numbers. Primes play the same role for odd numbers as the powers of 2 for even numbers, and vice versa.

We prove a theorem which is analogous to the fundamental theorem of arithmetic, when we study the positive integers with respect to addition: ‘’Every natural number, with the exception of 0 and 1, can be written in a unique way as a linear combination of consecutive powers of 2, with the coefficients of the linear combination being -1 or +1.’’ This theorem reveals a set of symmetries in the internal order of natural numbers which cannot be derived when studying natural numbers on the basis of the product. From such a symmetry a method for identifying large prime numbers is derived.

2 The sequence $\mu(k,n)$

We consider the sequence of natural numbers

$$\begin{aligned}&\mu(k,n)=k+(k+1)+(k+2)+...+(k+n)=\frac{(n+1)(2k+n)}{2}\\&k\in\mathbb{N}^*=\{1,2,3,...\}\\&n\in A=\{2,3,4,...\}\end{aligned}. \tag{2.1}$$

For the sequence $\mu(k,n)$ the following theorem holds:

Theorem 2.1. *For the sequence* $\mu(k,n)$ *the following hold*:

1. $\mu(k,n)\in\mathbb{N}^*$.

2. *No element of the sequence is a prime number.*

3. *No element of the sequence is a power of* 2.

4. *The range of the sequence is all natural numbers that are not primes and are not powers of* 2.

Proof. 1. $\mu(k,n)\in\mathbb{N}^*$ as a sum of natural numbers.

2. $n\in A=\{2,3,4,...\}$ and therefore it holds that

$$\begin{aligned}&n\geq 2\\&n+1\geq 3\end{aligned}.$$

Also we have that

$$2k+n\geq 4$$

$$\frac{2k+n}{2}\geq\frac{3}{2}>1$$

since $k \in \mathbb{N}^*$ and $n \in A = \{2,3,4,...\}$. Thus, the product

$$\frac{(n+1)(2k+n)}{2} = \mu(k,n)$$

is always a product of two natural numbers different than 1, thus the natural number $\mu(k,n)$ cannot be prime.

3. Let that the natural number

$$\mu(k,n) = \frac{(n+1)(2k+n)}{2}$$

is a power of 2. Then, it exists $\lambda \in \mathbb{N}$ such as

$$\frac{(n+1)(2k+n)}{2} = 2^{\lambda}$$

and equivalently

$$(n+1)(2k+n) = 2^{\lambda+1}. \qquad (2.2)$$

Equation (2.2) can hold if and only if there exist $\lambda_1, \lambda_2 \in \mathbb{N}$ such as

$$n+1 = 2^{\lambda_1} \wedge 2k+n = 2^{\lambda_2}$$

and equivalently

$$\left.\begin{array}{l} n = 2^{\lambda_1} - 1 \\ n = 2^{\lambda_2} - 2k \end{array}\right\}. \qquad (2.3)$$

We eliminate n from equations (2.3) and we obtain

$$2^{\lambda_1} - 1 = 2^{\lambda_2} - 2k$$

$$2k - 1 = 2^{\lambda_2} - 2^{\lambda_1},$$

which is impossible since the first part of the equation is an odd number and the second part is an even number. Thus, the range of the sequence $\mu(k,n)$ does not include the powers of 2.

4. We now prove that the range of the sequence $\mu(k,n)$ includes all natural numbers that are not primes and are not powers of 2. Let a random natural number N which is not a prime nor a power of 2. Then, N can be written in the form

$$N = \chi\psi$$

where at least one of the χ, ψ is an odd number ≥ 3. Let χ be an odd number ≥ 3. We will prove that there are always exist $k \in \mathbb{N}$ and $n \in A = \{2, 3, 4, ...\}$ such as

$N = \chi \cdot \psi = \mu(k, n)$.

We consider the following two pairs of k and n:

$$\begin{aligned} &\chi \leq 2\psi - 1, \chi, \psi \in \mathbb{N} \\ &k = k_1 = \frac{2\psi + 1 - \chi}{2} \\ &n = n_1 = \chi - 1 \end{aligned} \tag{2.4}$$

$$\begin{aligned} &\chi \geq 2\psi + 1, \chi, \psi \in \mathbb{N} \\ &k = k_2 = \frac{\chi + 1 - 2\psi}{2} \\ &n = n_2 = 2\psi - 1 \end{aligned} \quad . \tag{2.5}$$

For every $\chi, \psi \in \mathbb{N}$ it holds either the inequality $\chi \leq 2\psi - 1$ or the inequality $\chi \geq 2\psi + 1$. Thus, for each pair of naturals (χ, ψ), where χ is odd, at least one of the pairs (k_1, n_1), (k_2, n_2) of equations (2.4), (2.5) is defined. We now prove that when the natural number k_1 of equation (2.4) is $k_1 = 0$ then the natural number k_2 of equation (2.5) is $k_2 = 1$ and additionally it holds that $n_2 > 2$. For $k_1 = 0$ from equations (2.4) we take

$\chi = 2\psi + 1$

and from equations (2.5) we have that

$$\begin{aligned} &k_2 = \frac{(2\psi + 1) + 1 - 2\psi}{2} = 1 \\ &n_2 = 2\psi - 1 \end{aligned}$$

and because $\psi \geq 2$ we obtain

$$\begin{aligned} &k_2 = 1 \\ &n_2 = 2\psi - 1 \geq 3 > 2 \end{aligned} .$$

We now prove that when $k_2 = 0$ in equations (2.5), then in equations (2.4) it is $k_1 = 1$ and $n_1 > 2$. For $k_2 = 0$, from equations (2.5) we obtain

$\chi = 2\psi - 1$

and from equations (2.4) we get

$$k_1 = \frac{2\psi + 1 - (2\psi - 1)}{2} = 1.$$

$$n_1 = \chi - 1 = 2\psi - 2 \geq 2$$

We now prove that at least one of the k_2 and k_2 is positive. Let

$$k_1 < 0 \wedge k_2 < 0.$$

Then from equations (2.4) and (2.5) we have that

$$2\psi + 1 - \chi < 0 \wedge \chi + 1 - 2\psi < 0. \tag{2.6}$$

Taking into account that $\chi > 1$ is odd, that is $\chi = 2\rho + 1, \rho \in \mathbb{N}$, we obtain from inequalities (2.6)

$$2\psi + 1 - (2\rho - 1) < 0 \wedge (2\rho + 1) + 1 - 2\psi < 0$$

$$2\psi - 2\rho < 0 \wedge 2\rho - 2\psi + 2 > 0$$

$$\psi < \rho \wedge \psi > \rho + 1$$

which is absurd. Thus, at least one of k_1 and k_2 is positive.

For equations (2.4) we take

$$\mu(k_1, n_1) = \frac{(n_1 + 1)(2k_1 + n_1)}{2}$$

$$\mu(k_1, n_1) = \frac{(\chi - 1 + 1)\left(2\dfrac{2\psi + 1 - \chi}{2} + \chi - 1\right)}{2} = \frac{\chi \cdot 2 \cdot \psi}{2} = \chi\psi = N.$$

For equations (2.5) we obtain

$$\mu(k_2, n_2) = \frac{(n_2 + 1)(2k_2 + n_2)}{2}$$

$$\mu(k_2, n_2) = \frac{(2\psi - 1 + 1)\left(2\dfrac{\chi + 1 - 2\psi}{2} + 2\psi - 1\right)}{2} = \frac{2\psi\chi}{2} = \chi\psi = N.$$

Thus, there are always exist $k \in \mathbb{N}^*$ and $n \in A = \{2, 3, 4, ...\}$ such as

$N = \chi\psi = \mu(k, n)$ for every N which is not a prime number and is not a power of 2. □

Example 2.1. For the natural number $N = 40$ we have

$$N = 40 = 5 \cdot 8$$
$$\chi = 5$$
$$\psi = 8$$

and from equations (2.4) we get

$$k = k_1 = \frac{16+1-5}{2} = 6$$
$$n = n_1 = 5-1 = 4$$

thus, we obtain

$$40 = \mu(6,4) \ .$$

Example 2.2. For the natural number $N = 51$,

$$N = 51 = 3 \cdot 17 = 17 \cdot 3$$

there are two cases. First case:

$$N = 51 = 3 \cdot 17$$
$$\chi = 3$$
$$\psi = 17$$

and from equations (2.4) we obtain

$$k = k_1 = \frac{34+1-3}{2} = 16$$
$$n = n_1 = 3-1 = 2$$

thus,

$$51 = \mu(16,2) .$$

Second case:

$$N = 51 = 17 \cdot 3$$
$$\chi = 17$$
$$\psi = 3$$

and from equations (2.5) we obtain

$$k = k_2 = \frac{17+1-6}{2} = 6$$
$$n = n_2 = 6-1 = 5$$

thus,

$51 = \mu(6,5)$.

The second example expresses a general property of the sequence $\mu(k,n)$. The more composite an odd number that is not prime (or an even number that is not a power of 2) is, the more are the $\mu(k,n)$ combinations that generate it.

Example 2.3.

$135 = 15 \cdot 9 = 27 \cdot 5 = 9 \cdot 15 = 45 \cdot 3 = 5 \cdot 27 = 3 \cdot 45$

$135 = \mu(2,14) = \mu(9,9) = \mu(11,8) = \mu(20,5) = \mu(25,4) = \mu(44,2)$

a. $135 = 9 \cdot 15 = \mu(2,14) = \mu(11,8)$

$135 = 2+3+4+......+15+16 = 11+12+13.....+18+19$.

b. $135 = 5 \cdot 27 = \mu(9,9) = \mu(25,4)$

$135 = 9+10+11+.....+17+18 = 25+26+27+28+29$.

c. $135 = 3 \cdot 45 = \mu(20,5) = \mu(44,2)$

$135 = 20+21+22+23+24+25 = 44+45+46$.

In the transitive property of multiplication, when writing a composite odd number or an even number that is not a power of 2 as a product of two natural numbers, we use the same natural numbers $\chi, \psi \in \mathbb{N}$:

$\Phi = \chi \cdot \psi = \psi \cdot \chi$.

On the contrary, the natural number Φ can be written in the form $\Phi = \mu(k,n)$ using different natural numbers $k \in \mathbb{N}^*$ and $n \in A = \{2,3,4,...\}$, through equations (2.4), (2.5). This difference between the product and the sum can also become evident in example 2.3:

$135 = 3 \cdot 45 = 45 \cdot 3$

$135 = 44+45+46 = 20+21+22+23+24+25$.

From Theorem 2.1 the following corollary is derived:

Corollary 2.1. 1. *Every natural number which is not a power of* 2 *and is not a prime can be written as the sum of three or more consecutive natural numbers.*

2. *Every power of* 2 *and every prime number cannot be written as the sum of three or more consecutive natural numbers.*

Proof. Corollary 2.1 is a direct consequence of Theorem 2.1. □

3 The concept of rearrangement

In this paragraph, we present the concept of rearrangement of the composite odd numbers and even numbers that are not power of 2. Moreover, we prove some of the consequences of the rearrangement in the Diophantine analysis. The concept of rearrangement is given from the following definition:

Definition 3.1. *We say that the sequence* $\mu(k,n), k\in\mathbb{N}^*, n\in A=\{2,3,4,...\}$ *is rearranged if there exist natural numbers* $k_1\in\mathbb{N}^*, n_1\in A$, $(k_1,n_1)\neq(k,n)$ *such as*

$$\mu(k,n)=\mu(k_1,n_1). \tag{3.1}$$

From equation (2.1) written in the form of

$$\mu(k,n)=k+(k+1)+(k+2)+.....+(k+n)$$

two different types of rearrangement are derived: The "compression", during which n decreases with a simultaneous increase of k. The «decompression», during which n increases with a simultaneous decrease of k. The following theorem provides the criterion for the rearrangement of the sequence $\mu(k,n)$.

Theorem 3.1. 1. *The sequence* $\mu(k_1,n_1),(k_1,n_1)\in\mathbb{N}^*\times A$ *can be compressed*

$$\mu(k_1,n_1)=\mu(k_1+\varphi,n_1-\omega) \tag{3.2}$$

if and only if there exist $\varphi,\omega\in\mathbb{N}^*,\omega\leq n_1-2$ *which satisfies the equation*

$$\begin{aligned}&\omega^2-(2k_1+2n_1+1+2\varphi)\omega+2(n_1+1)\varphi=0\\&\varphi,\omega\in\mathbb{N}^*\\&\omega\leq n_1-2\end{aligned}\quad. \tag{3.3}$$

2. *The sequence* $\mu(k_2,n_2),(k_2,n_2)\in\mathbb{N}^*\times A$ *can be decompressed*

$$\mu(k_2,n_2)=\mu(k_2-\varphi,n_2+\omega) \tag{3.4}$$

if and only if there exist $\varphi,\omega\in\mathbb{N}^*,\varphi\leq k_2-1$ *which satisfies the equation*

$$\begin{aligned}&\omega^2+(2k_2+2n_2+1-2\varphi)\omega-2(n_2+1)\varphi=0\\&\varphi,\omega\in\mathbb{N}^*\\&\varphi\leq k_2-1\end{aligned}\quad. \tag{3.5}$$

3. *The odd number* $\Pi \neq 1$ *is prime if and only if the sequence*

$$\begin{aligned} &\mu(k,n) = \Pi \cdot 2^l \\ &l,k \in \mathbb{N}^*, n \in A \end{aligned} \tag{3.6}$$

cannot be rearranged.

4. *The odd* Π is *prime if and only if the sequence*

$$\mu\left(\frac{\Pi+1}{2}, \Pi-1\right) = \Pi^2 \tag{3.7}$$

cannot be rearranged.

Proof. 1, 2. We prove part 1 of the corollary and similarly number 2 can also be proven. From equation (4.1) we conclude that the sequence $\mu(k_1, n_1)$ can be compressed if and only if there exist $\varphi, \omega \in \mathbb{N}^*$ such as

$$\mu(k_1, n_1) = \mu(k_1+\varphi, n_1-\omega).$$

In this equation the natural number $n_1-\omega$ belongs to the set $A = \{2,3,4,...\}$ and thus $n_1 - \omega \geq 2 \Leftrightarrow \omega \leq n_1 - 2$.

Next, from equations (2.1) we obtain

$$\mu(k_1, n_1) = \mu(k_1+\varphi, n_1-\omega)$$

$$\frac{(n_1+1)(2k_1+n_1)}{2} = \frac{(n_1-\omega+1)\left[2(k_1+\varphi)+n_1-\omega\right]}{2}$$

and after the calculations we get equation (3.3).

3. The sequence (3.6) is derived from equations (2.4) or (2.5) for $\chi = \Pi$ and $\psi = 2^l$. Thus, in the product $\chi\psi$ the only odd number is Π. If the sequence $\mu(k,n)$ in equation (3.6) cannot be rearranged then the odd number Π has no divisors. Thus, Π is prime. Obviously, the inverse also holds.

4. First, we prove equations (3.7). From equation (2.1) we obtain:

$$\mu\left(\frac{\Pi+1}{2}, \Pi-1\right) = \frac{(\Pi-1+1)\left(2\frac{\Pi+1}{2}+\Pi-1\right)}{2} = \Pi^2.$$

In case that the odd number Π is prime in equations (2.4), (2.5) the natural numbers χ, ψ are unique $\chi = \Pi \wedge \psi = \Pi$, and from equation (2.5) we get

$k = \frac{\Pi+1}{2} \wedge n = \Pi - 1$.

Thus, the sequence

$$\mu(k,n) = \mu\left(\frac{\Pi+1}{2}, \Pi-1\right)$$

cannot be rearranged. Conversely, if the sequence

$$\mu\left(\frac{\Pi+1}{2}, \Pi-1\right) = \Pi^2 = \Pi \cdot \Pi$$

cannot be rearranged the odd number Π cannot be composite and thus Π is prime.□

We now prove the following corollary:

Corollary 3.1. 1. *The odd number* Φ ,

$$\begin{aligned} &\Phi = \Pi^2 = \mu\left(\frac{\Pi+1}{2}, \Pi-1\right) \\ &\Pi = odd \\ &\Pi \neq 1 \end{aligned} \tag{3.8}$$

is decompressed and compressed if and only if the odd number Π *is composite.*

2. *The even number* α_1 ,

$$\begin{aligned} &\alpha_1 = 2^l \Pi = \mu\left(2^l - \frac{\Pi-1}{2}, \Pi-1\right) \\ &\Pi = odd \\ &3 \leq \Pi \leq 2^l - 1 \\ &l \in \mathbb{N}, l \geq 2 \end{aligned} \tag{3.9}$$

cannot be decompressed, while it compresses if and only if the odd number Π *is composite.*

3. *The even number* α_2 ,

$$\begin{aligned} &\alpha_2 = 2^l \Pi = \mu\left(\frac{\Pi+1}{2} - 2^l, 2^{l+1} - 1\right) \\ &\Pi = odd \\ &\Pi \geq 2^{l+1} + 1 \\ &l \in \mathbb{N}^* \end{aligned} \tag{3.10}$$

cannot be compressed, while it decompresses if and only if the odd number Π *is composite.*

4. *Every even number that is not a power of can be written either in the form of equation* (3.9) *or in the form of equation* (3.10).

Proof. 1. It is derived directly through number (4) of Theorem 3.1. A second proof can be derived through equations (2.4), (2.5) since every composite odd Π can be written in the form of $\Pi = \chi\psi$, $\chi,\psi \in \mathbb{N}$, χ,ψ odds.

2, 3. Let the even number α ,

$$\begin{aligned} &\alpha = 2^l \Pi \\ &\Pi = odd\,. \\ &l \in \mathbb{N}^* \end{aligned} \tag{3.11}$$

From equation (2.4) we obtain

$$\begin{aligned} &k = \frac{2 \cdot 2^l + 1 - \Pi}{2} = 2^l - \frac{\Pi - 1}{2} \\ &n = \Pi - 1 \end{aligned} \tag{3.12}$$

and since $k,n \in \mathbb{N}, k \geq 1 \wedge n \geq 2$ we get

$$\begin{aligned} &\frac{2 \cdot 2^l + 1 - \Pi}{2} \geq 1 \\ &\Pi - 1 \geq 2 \end{aligned}$$

and equivalently

$$3 \leq \Pi \leq 2^{l+1} - 1.$$

In the second of equations (3.12) the natural number n obtains the maximum possible value of $n = \Pi - 1$, and thus the natural number k takes the minimum possible value in the first of equations (3.12). Thus, the even number

$$\alpha_1 = \mu\left(2^l - \frac{\Pi - 1}{2}, \Pi - 1\right)$$

cannot decompress. If the odd number Π is composite then it can be written in the form of $\Pi = \chi\psi$, $\chi,\psi \in \mathbb{N}^*$, χ,ψ odds, $\chi,\psi < \Pi$, $\alpha_1 = 2^l \chi\psi$. Therefore, the natural number $\alpha_1 = 2^l \chi\psi$ decompresses since from equations (3.11) it can be written in the form of $\alpha_1 = \mu(k,n)$ with $n = \chi - 1 < \Pi - 1$. Similarly, the proof of 3 is derived from equations (2.5).

4. From the above proof process it follows that every even number that is not a power of 2 can be written either in the form of equation (3.9) or in the form of equation (3.10).□

By substituting $\Pi = P = prime$ in equations of Theorem 3.1 and of corollary 3.1 four sets of equations are derived, each including infinite impossible diophantine equations.

Example 3.1. The odd number $P = 999961$ is prime. Thus, combining (1) of Theorem 3.1 with (1) of corollary 3.1 we conclude that there is no pair $(\omega, \varphi) \in \mathbb{N}^2$ with $\omega \leq 999958$ which satisfies the diophantine equation

$$\omega^2 - (2999883 + 2\varphi)\omega + 1999922\varphi = 0 .$$

We now prove the following corollary:

Corollary 3.2. *The square of every prime number can be uniquely written as the sum of consecutive natural numbers.*

Proof. For $\Pi = P = prime$ in equation (3.5) we obtain

$$P^2 = \mu\left(\frac{P+1}{2}, P-1\right). \tag{3.13}$$

According with 4 of Theorem 3.1 the odd P^2 cannot be rearranged. Thus, the odd can be uniquely written as the sum of consecutive natural numbers, as given from equation (3.13). □

Example 3.2. The odd $P = 17$ is prime. From equation (3.13) for $P = 17$ we obtain

$$289 = \mu(9,16)$$

and from equation (2.1) we get

$$289 = 9+10+11+12+13+14+15+16+17+18+19+20+21+22+23+24+25$$

which is the only way in which the odd number 289 can be written as a sum of consecutive natural numbers.

4 Natural numbers as linear combination of consecutive powers of 2

According to the fundamental theorem of arithmetic, every natural number can be uniquely written as a product of powers of prime numbers. The previously presented study reveals a correspondence between odd prime numbers and the powers of 2. Thus, the question arises whether there exists a theorem for the powers of 2 corresponding to the fundamental theorem of arithmetic. The answer is given by the following theorem:

Theorem 4.1. *Every natural number, with the exception of* 0 *and* 1, *can be uniquely written as a linear combination of consecutive powers of* 2, *with the coefficients of the linear combination being* -1 *or* +1.

Proof. Let the odd number Π as given from equation

$$\Pi = \Pi(\nu, \beta_i) = 2^{\nu+1} + 2^{\nu} \pm 2^{\nu-1} \pm 2^{\nu-2} \pm ....\pm 2^1 \pm 2^0 = 2^{\nu+1} + 2^{\nu} + \sum_{i=0}^{\nu-1} \beta_i 2^i$$
$$\beta_i = \pm 1, i = 0,1,2,........,\nu-1 \quad . \tag{4.1}$$
$$\nu \in \mathbb{N}$$

From equation (4.1) for $\nu = 0$ we obtain

$$\Pi = 2^1 + 2^0 = 2 + 1 = 3.$$

We now examine the case where $\nu \in \mathbb{N}^*$. The lowest value that the odd number Π of equation (4.1) can obtain is

$$\Pi_{\min} = \Pi(\nu) = 2^{\nu+1} + 2^{\nu} - 2^{\nu-1} - 2^{\nu-1} - ....2^1 - 1$$

$$\Pi_{\min} = \Pi(\nu) = 2^{\nu+1} + 1. \tag{4.2}$$

The largest value that the odd number Π of equation (4.1) can obtain is

$$\Pi_{\max} = \Pi(\nu) = 2^{\nu+1} + 2^{\nu} + 2^{\nu-1} + ...2^1 + 1$$

$$\Pi_{\max} = \Pi(\nu) = 2^{\nu+2} - 1. \tag{4.3}$$

Thus, for the odd numbers $\Pi = \Pi(\nu, \beta_i)$ of equation (4.1) the following inequality holds

$$\Pi_{\min} = 2^{\nu+1} + 1 \le \Pi(\nu, \beta_i) \le 2^{\nu+2} - 1 = \Pi_{\max} . \tag{4.4}$$

The number $N(\Pi(\nu, \beta_i))$ of odd numbers in the closed interval $\left[2^{\nu+1} + 1, 2^{\nu+2} - 1\right]$ is

$$N(\Pi(\nu, \beta_i)) = \frac{\Pi_{\max} - \Pi_{\min}}{2} + 1 = \frac{(2^{\nu+2} - 1) - (2^{\nu+1} + 1)}{2} + 1$$

$$N(\Pi(\nu, \beta_i)) = 2^{\nu} . \tag{4.5}$$

The integers $\beta_i, i = 0,1,2,........,\nu-1$ in equation (4.1) can take only two values, $\beta_i = -1 \vee \beta_i = +1$, thus equation (4.1) gives exactly $2^{\nu} = N(\Pi(\nu, \beta_i))$ odd numbers. Therefore, for every $\nu \in \mathbb{N}^*$ equation (4.1) gives all odd numbers in the interval $\left[2^{\nu+1} + 1, 2^{\nu+2} - 1\right]$.

We now prove the theorem for the even numbers. Every even number α which is a power of 2 can be uniquely written in the form of $\alpha = 2^{\nu}, \nu \in \mathbb{N}^*$. We now consider the case where the even number α is not a power of 2. In that case, according to corollary 3.1 the even number α is written in the form of

$$\alpha = 2^l \Pi, \Pi = \text{odd}, \Pi \neq 1, l \in \mathbb{N}^* . \qquad (4.6)$$

We now prove that the even number α can be uniquely written in the form of equation (4.6). If we assume that the even number α can be written in the form of

$$\begin{aligned} &\alpha = 2^l \Pi = 2^{l'} \Pi' \\ &l \neq l' (l > l') \\ &\Pi \neq \Pi' \\ &l, l' \in \mathbb{N}^* \\ &\Pi, \Pi' = odd \end{aligned} \qquad (4.7)$$

the we obtain

$$\begin{aligned} &2^l \Pi = 2^{l'} \Pi' \\ &2^{l-l'} \Pi = \Pi' \end{aligned}$$

which is impossible, since the first part of this equation is even and the second odd. Thus, it is $l = l'$ and we take that $\Pi = \Pi'$ from equation (4.7). Therefore, every even number α that is not a power of 2 can be uniquely written in the form of equation (4.6). The odd number Π of equation (4.6) can be uniquely written in the form of equation (4.1), thus from equation (4.6) it is derived that every even number α that is not a power of 2 can be uniquely written in the form of equation

$$\begin{aligned} &\alpha = \alpha(l, \nu, \beta_i) = 2^l \left(2^{\nu+1} + 2^{\nu} + \sum_{i=0}^{\nu-1} \beta_i 2^i \right) \\ &l \in \mathbb{N}^*, \nu \in \mathbb{N} \\ &\beta_i = \pm 1, i = 0, 1, 2, \ldots\ldots\ldots, \nu - 1 \end{aligned} \qquad (4.8)$$

and equivalently

$$\begin{aligned} &\alpha = \alpha(l, \nu, \beta_i) = 2^{l+\nu+1} + 2^{l+\nu} + \sum_{i=0}^{\nu-1} \beta_i 2^{l+i} \\ &l \in \mathbb{N}^*, \nu \in \mathbb{N} \qquad . \\ &\beta_i = \pm 1, i = 0, 1, 2, \ldots\ldots\ldots, \nu - 1 \end{aligned} \qquad (4.9)$$

For 1 we take

$$1 = 2^0$$

$$1 = 2^1 - 2^0$$

thus, it can be written in two ways in the form of equation (4.1). Both the odds of equation (4.1) and the evens of the equation (4.8) are positive. Thus, 0 cannot be written either in the form of equation (4.1) or in the form of equation (4.8). □

In order to write an odd number $\Pi \neq 1,3$ in the form of equation (4.1) we initially define the $\nu \in \mathbb{N}^*$ from inequality (4.4). Then, we calculate the sum

$$2^{\nu+1} + 2^{\nu}.$$

If it holds that $2^{\nu+1} + 2^{\nu} < \Pi$ we add the $2^{\nu-1}$, whereas if it holds that $2^{\nu+1} + 2^{\nu} > \Pi$ then we subtract it. By repeating the process exactly ν times we write the odd number Π in the form of equation (4.1). The number of ν steps needed in order to write the odd number Π in the form of equation (4.1) is extremely low compared to the magnitude of the odd number Π, as derived from inequality (4.4).

Example 4.1. For the odd number $\Pi = 23$ we obtain from inequality (4.4)

$$2^{\nu+1} + 1 < 23 < 2^{\nu+2} - 1$$

$$2^{\nu+1} + 2 < 24 < 2^{\nu+2}$$

$$2^{\nu} < 12 < 2^{\nu+1}$$

thus $\nu = 3$. Then, we have

$2^{\nu+1} + 2^{\nu} = 2^4 + 2^3 = 24 > 23$ (thus 2^2 is subtracted)

$2^4 + 2^3 - 2^2 = 20 < 23$ (thus 2^1 is added)

$2^4 + 2^3 - 2^2 + 2^1 = 22 < 23$ (thus $2^0 = 1$ is added)

$$2^4 + 2^3 - 2^2 + 2^1 + 1 = 23.$$

Fermat numbers F_s can be written directly in the form of equation (4.1), since they are of the form $\Pi_{\min}$,

$$\begin{gathered} F_s = 2^{2^s} + 1 = \Pi_{\min}\left(2^s - 1\right) = 2^{2^s} + 2^{2^s-1} - 2^{2^s-2} - 2^{2^s-3} - \dots\dots\dots - 2^1 - 1 \\ s \in \mathbb{N} \end{gathered} \tag{4.10}$$

Mersenne numbers M_p can be written directly in the form of equation (4.1), since they are of the form $\Pi_{\max}$,

$$\begin{gathered} M_p = 2^p - 1 = \Pi_{\max}\left(p - 2\right) = 2^{p-1} + 2^{p-2} + 2^{p-3} + \dots\dots\dots + 2^1 + 1 \\ p = prime \end{gathered} \tag{4.11}$$

In order to write an even number α that is not a power of 2 in the form of equation (4.1), initially it is consecutively divided by 2 and it takes of the form of equation (4.6). Then, we write the odd number Π in the form of equation (4.1).

Example 4.2. By consecutively dividing the even number $\alpha = 368$ by 2 we obtain

$$\alpha = 368 = 2^4 \cdot 23.$$

Then, we write the odd number $\Pi = 23$ in the form of equation (4.1),

$$23 = 2^4 + 2^3 - 2^2 + 2^1 + 1,$$

and we get

$$368 = 2^4\left(2^4 + 2^3 - 2^2 + 2^1 + 1\right)$$

$$368 = 2^8 + 2^7 - 2^6 + 2^5 + 2^4.$$

This equation gives the unique way in which the even number $\alpha = 368$ can be written in the form of equation (4.9).

From inequality (4.4) we obtain

$$2^{\nu+1} + 1 \le \Pi \le 2^{\nu+2} - 1$$

$$2^{\nu+1} < 2^{\nu+1} + 1 \le \Pi \le 2^{\nu+2} - 1 < 2^{\nu+2}$$

$$2^{\nu+1} < \Pi < 2^{\nu+2}$$

$$(\nu+1)\log 2 < \log \Pi < (\nu+2)\log 2$$

from which we get

$$\frac{\log \Pi}{\log 2} - 1 < \nu + 1 < \frac{\log \Pi}{\log 2}$$

and finally

$$\nu + 1 = \left[\frac{\log \Pi}{\log 2}\right] \tag{4.12}$$

where $\left[\frac{\log \Pi}{\log 2}\right]$ the integer part of $\frac{\log \Pi}{\log 2} \in \mathbb{R}$.

We now give the following definition:

Definition 4.1. *We define as the conjugate of the odd*

$$\begin{aligned} &\Pi = \Pi(\nu, \beta_i) = 2^{\nu+1} + 2^{\nu} + \sum_{i=0}^{\nu-1} \beta_i 2^i \\ &\beta_i = \pm 1, i = 0,1,2,........,\nu - 1 \\ &\nu \in \mathbb{N}^* \end{aligned} \tag{4.13}$$

the odd Π^*,

$$\Pi^* = \Pi^*\left(\nu, \gamma_j\right) = 2^{\nu+1} + 2^{\nu} + \sum_{j=0}^{\nu-1} \gamma_j 2^j$$
$$\gamma_i = \pm 1,\ j = 0,1,2,........,\nu - 1 \qquad (4.14)$$
$$\nu \in \mathbb{N}^*$$

for which it holds

$$\gamma_k = -\beta_k \forall k = 0,1,2,........,\nu - 1\,. \qquad (4.15)$$

For conjugate odds, the following corollary holds:

Corollary 4.1. *For the conjugate odds* $\Pi = \Pi\left(\nu, \beta_i\right)$ *and* $\Pi^* = \Pi^*\left(\nu, \gamma_i\right)$ *the following hold:*

1. $\left(\Pi^*\right)^* = \Pi$. (4.16)

2. $\Pi + \Pi^* = 3 \cdot 2^{\nu+1}$. (4.17)

3. Π *is divisible by* 3 *if and only if* Π^* *is divisible by* 3.

Proof. 1. The 1 of the corollary is an immediate consequence of definition 4.1.

2. From equations (4.13), (4.14) and (4.15) we get

$$\Pi + \Pi^* = \left(2^{\nu+1} + 2^{\nu}\right) + \left(2^{\nu+1} + 2^{\nu}\right)$$

and equivalently

$$\Pi + \Pi^* = 3 \cdot 2^{\nu+1}\,.$$

3. If the odd Π is divisible by 3 then it is written in the form $\Pi = 3x, x = odd$ and from equation (4.17) we get $3x + \Pi^* = 3 \cdot 2^{\nu+1}$ and equivalently $\Pi^* = 3\left(2^{\nu+1} - x\right)$. Similarly we can prove the inverse.□

5 The T symmetry and a method for defining large prime numbers

We now give the following definition:

Definition 5.1. *Define as "symmetry" every specific algorithm which determines the signs of* $\beta_i = \pm 1, i = 0,1,2,........,\nu - 1$ *in equation* (4.1):

$$\Pi = \Pi\left(\nu, \beta_i\right) = 2^{\nu+1} + 2^{\nu} \pm 2^{\nu-1} \pm 2^{\nu-2} \pm .... \pm 2^1 \pm 2^0 = 2^{\nu+1} + 2^{\nu} + \sum_{i=0}^{\nu-1} \beta_i 2^i$$
$$\beta_i = \pm 1, i = 0,1,2,........,\nu - 1 \,.$$
$$\nu \in \mathbb{N}$$

Next, we develop a specific symmetry, the *T symmetry*.

If the natural number ν , in the equation (4.1), is not a prime and is not a power of 2, the equation (2.1) gives

$$\nu=\mu(k,n)=\frac{(n+1)(2k+n)}{2}=k+(k+1)+(k+2)+........+(k+n) . \qquad (5.1)$$
$$k\in\mathbb{N}^*, n\in A=\{2,3,4,...\}$$

We define the odd number $T_1=\Pi(\nu=\mu(k,n))=T_1(k,n)$ as follows: In the right side of equation (4.1), from left to right, we take k signs -1, and then (k+1) signs +1, (k+2) signs -1, (k+3) signs +1 etc., according to the right side of equation (5.1). After making some calculations we have

$$T_1=T_1(k,n)=2^{\frac{(n+1)(2k+n)}{2}+1}+\left(\sum_{j=0}^{n}(-1)^{j}\times 2^{\frac{(n+1)(2k+n)}{2}+1-\sum_{i=0}^{j}(k+i)}\right)-(-1)^{n} \qquad (5.2)$$
$$k\in\mathbb{N}^*, n\in A$$

and

$$T_1^*=T_1^*(k,n)=3\times 2^{\mu(k,n)+1}-T_1(k,n)$$

and equivalently

$$T_1^*=T_1^*(k,n)=2^{\frac{(n+1)(2k+n)}{2}+2}-\left(\sum_{j=1}^{n}(-1)^{j}\times 2^{\frac{(n+1)(2k+n)}{2}+1-\sum_{i=0}^{j}(k+i)}\right)+(-1)^{n} . \qquad (5.3)$$
$$k\in\mathbb{N}^*, n\in A$$

We write the equation (5.1) in the form

$$\nu=\mu(k,n)=\frac{(n+1)(2k+n)}{2}=(k+n)+(k+n-1)+(k+n-2)+........+k . \qquad (5.4)$$
$$k\in\mathbb{N}^*, n\in A$$

We define the odd number $T_2=\Pi(\nu=\mu(k,n))=T_2(k,n)$ by the same way as we defined $T_1=\Pi(\nu=\mu(k,n))=T_1(k,n)$ but the signs in equation (4.1) are now determined according to the right side of equation (5.4), (k+n) signs -1, (k+n-1) signs +1, (k+n-2) signs -1, (k+n-3) signs +1 etc. After making some calculations we have

$$T_2 = T_2(k,n) = 2^{\frac{(n+1)(2k+n)}{2}+1} + \left(\sum_{j=0}^{n}(-1)^{j} \times 2^{\frac{(n+1)(2k+n)}{2}+1-\sum_{i=0}^{j}(k+n-i)}\right) - (-1)^n \qquad (5.5)$$

$k \in \mathbb{N}^*, n \in A$

and

$$T_2^* = T_2^*(k,n) = 3 \times 2^{\mu(k,n)+1} - T_2(k,n)$$

and equivalently

$$T_2^* = T_2^*(k,n) = 2^{\frac{(n+1)(2k+n)}{2}+2} - \left(\sum_{j=0}^{n}(-1)^{j} \times 2^{\frac{(n+1)(2k+n)}{2}+1-\sum_{i=0}^{j}(k+n-i)}\right) + (-1)^n. \qquad (5.6)$$

$k \in \mathbb{N}^*, n \in A$

Equations (5.2), (5.3), (5.5) and (5.6) define the T symmetry.

A method for the determination of large prime numbers emerges from the study we presented. This method is completely different from previous methods [1-5]. For the T symmetry holds:

"*There are pairs* $(k,n) \in \mathbb{N}^* \times A$,

$$n \neq 3 + 4l, l \in \mathbb{N} \qquad (5.7)$$

for which one or more of $T_1(k,n)$, $T_1^*(k,n)$, $T_2(k,n)$, $T_2^*(k,n)$ *are prime numbers.*"

We will present three examples:

1. The number

$T_2(11,5) = 2^{82} + 2^{66} - 2^{51} + 2^{37} - 2^{24} + 2^{12} - 2^1 + 1$ =4 835777 063183 149145 526271 is a prime.

The number

$T_1^*(11,5) = 2^{83} - 2^{71} + 2^{59} - 2^{46} + 2^{32} - 2^{17} + 2^1 - 1$ =9 669045 950065 986429 124609 is a prime.

2. The number

$T_1(23,4) = 2^{126} + 2^{103} - 2^{79} + 2^{54} - 2^{28} + 2^1 - 1$ = 85 070601 871438 813228 787070 915221 389313 is a prime.

3. The number $T_1^*(80,2) = 2^{245} - 2^{164} + 2^{83} - 2^1 + 1$ = 56 539106 072908 298546 665496 639747 195212 032793 441072 154605 920000 979840 794623 (74 digits) is a prime.

The number of digits of the primes calculated by the method is of order

$$\left(\mu(k,n)+1\right)\log 2=\left(\frac{(n+1)(2k+n)}{2}+1\right)\log 2. \qquad (5.8)$$

The smallest prime number given by the method is $T_1(2,2)=2^{10}+2^8-2^5+2^1-1$ =1249. Also, it doesn't give prime numbers Fermat and Mersenne. The method may be further investigated for the form of the pairs $(k,n)\in\mathbb{N}^*\times A$ in equations (5.2), (5.3), (5.5) and (5.6).

We now cite three remarkable properties of the T symmetry. When the numbers of the T symmetry are not primes, with high probability, one or more of them are the product of a set of small primes with a large prime (with ratio of the number of digits at least 3:1 in the decimal system). We give an example for n=4 and k=1, 2, 3,......., 23.

Example 5.1.

1. $T_1^*(1,4)$ = 5 × 21107.

2. $T_1(2,4)$ = 3 × 853291

   $T_2(2,4)$ = 3 × 709651

   $T_2^*(2,4)$ = 3 × 1 387501.

3. $T_1^*(3,4)$ =126 337279 (9 digits) *is a prime*

   $T_2^*(3,4)$ =133 701391 (9 digits) *is a prime.*

4. $T_2^*(4,4)$ = 3 × 13 × 109 913929.

5. $T_2(5,4)$ = 68853 174209 (11 digits) *is a prime*

   $T_2^*(5,4)$ = 19 × 7226 592421.

6. $T_2(6,4)$ =3 × 7 × 104817 455293.

7. $T_2(7,4)$ =37 × 1 902785 687213

   $T_2^*(7,4)$ =11 × 12 791196 555101.

8. $T_1(8,4)$ =3 × 47 × 16 032473 358917.

9. $T_1^*(11,4)$ =1301 × 113403 483925 962179.

10. $T_1^*(12,4)$ =3 × 13 × 121 071540 832866 439273

$T_2(12,4)$ =3 × 7 × 112 439012 815828 430653

$T_2^*(12,4)$ =3 × 89 × 17 686630 918247 456093.

11. $T_1^*(13,4)$ =5 × 30221 300928 544913 175347

$T_2^*(13,4)$ =2239 × 67 492251 451483 773121.

12. $T_1^*(14,4)$ =3 × 19 × 107 × 792 844025 087630 419877

13. $T_1^*(15,4)$ =23 × 6 727832 337541 722681 821273.

14. $T_2(16,4)$ =3 × 825 294146 583166 134057 740971.

15. $T_2(17,4)$ =3541 × 22 374527 052572 768094 438269.

16. $T_1^*(18,4)$ =3 × 73 × 2903 × 7 975677 388569 543733 588379.

17. $T_2^*(19,4)$ =11 × 641 × 23012 234740 860744 903766 035421.

18. $T_1^*(20,4)$ =3 × 6 643069 × 260 536928 672371 642740 686521.

19. $T_2(21,4)$ =7 × 79 × 150 229208 340754 381651 561471 195673 (33 digits).

20. $T_1(22,4)$ =3 × 29 × 1259 × 24 270828 201501 431550 885053 400181 (32 digits)

$T_1^*(22,4)$ =3 × 1933 × 916 866933 835909 456002 715952 336617 (33 digits).

21. $T_1(23,4)$ = 85 070601 871438 813228 787070 915221 389313 (38 digits) *is a prime*

$T_1^*(23,4)$ =18269 × 9313 108178 842029 359502 101081 537291 (34 digits)

$T_2^*(23,4)$ =19 × 89 × 100615 720181 338817 896100 110722 568301 (36 digits).

For

$$n = 3 + 4l, l \in \mathbb{N} \tag{5.9}$$

the numbers of the T symmetry have 3 as a factor. In these cases, we factorize the numbers of the T symmetry in order to identify the ones which are the product of a set of small primes with a large prime (with ratio of the number of digits at least 3:1 in the decimal system). We give an example for l=0 and k=1, 2, 3,…….., 33.

Example 5.2.

1. $T_2^*(1,3)$ =3 × 1327.

2. $T_2^*(2,3)$ =3 × 21523.

3. $T_2^*(4,3)$ =3 × 5 570891.

4. $T_1(5,3)$ =3 × 46 115669.

5. $T_2^*(7,3)$ =3 × 5 × 4579 065839.

6. $T_2^*(9,3)$ =3^2 × 1 954448 845369.

7. $T_1(10,3)$ =3 × 73 × 643264 201901.

8. $T_1(11,3)$ = 3 × 5 × 23 × 6 530142 193943.

9. $T_1(12,3)$ = 3^2 × 4004 176893 145543.

10. $T_2(13,3)$ =3 × 7 × 31 × 107 × 151 × 54806 826689.

11. $T_1(14,3)$ = 3 × 11 × 279513 180897 836063.

12. $T_1^*(16,3)$ =3^3 × 7 × 19 × 73 × 18014 329790 791679.

13. $T_1(17,3)$ =3 × 12593 073364 077934 630229.

14. $T_1(18,3)$ =3^2 × 9239 × 7 269488 227993 959889.

15. $T_1(19,3)$ =3 × 5 × 73 × 331 × 26 683841 696377 422587

    $T_1^*(19,3)$ =3 × 7^2 × 127 × 269 × 337 × 11429 204013 400937

    $T_2^*(19,3)$ =3 × 5 × 557 × 2315 117990 184578 945803.

16. $T_2(20,3)$ =3 × 23 × 89 × 683 × 9041 × 4080 688125 380017.

17. $T_1^*(21,3)$ =3 × 47 × 178481 × 196 765246 663328 879957

$T_2^*(21,3)$ =3² × 83 × 23473 × 282 403703 315945 507251.

18. $T_2^*(22,5)$ =3 × 5273 × 5 008417 809828 231066 746851.

19. $T_2^*(23,3)$ =3 × 5² × 137 × 211 × 379 × 1542 751819 716389 148523.

20. $T_2(24,3)$ =3 × 2731 × 7487 × 8191 × 20 183749 276015 547071.

21. $T_1(25,3)$ =3 × 13 × 269 × 636697 × 24291 809038 323309 475541.

22. $T_1(26,3)$ =3 × 47 × 829 × 22210 374525 858205 252016 927831

$T_2(26,3)$ =3 × 5 × 13 × 29 × 43 × 83 × 113 × 127 × 151 × 59 359638 928368 977041.

23. $T_1^*(27,3)$ =3 × 233 × 1103 × 2089 × 48091 × 1072 567317 671651 381903

$T_2^*(27,3)$ =3² × 5 × 23 × 139 × 257 × 6 560737 × 342 482665 485076 269161.

24. $T_1^*(28,3)$ =3² × 7 × 11 × 31 × 151 × 331 × 1237 940038 132458 773513 502719.

25. $T_2^*(30,3)$ =3² × 5801 × 288383 × 22600 831355 114079 948328 119407.

26. $T_1^*(31,3)$ =3 × 7 × 23 × 83 × 89 × 109 × 599479 × 23 353056 263230 084539 231539.

27. $T_1(32,3)$ =3 × 37 × 392 397684 468660 613729 344084 167488 872817 (39 digits).

28. $T_1(33,3)$ =3² × 23 × 857 × 3 928422 863348 787826 215906 015441 564473 (37 digits)

$T_2^*(33,3)$ =3⁴ × 20 286419 × 848 220926 630659 241732 391340 317419 (33 digits).

Fermat and Mersenne, for odds $N \neq 3$ of the form $N = 2^n + 1 = 2^n + 1^n, n \in \mathbb{N}^*$ and $N = 2^n - 1 = 2^n - 1^n, n \in \mathbb{N}^*$, respectively, chose the values of $n \in \mathbb{N}$ for which the odd N , firstly, does not have 3 as a factor ($n = 2^s, s \in \mathbb{N}$ and $n = prime$, respectively). This has as a consequence that the Fermat and Mersenne numbers are not divisible by 3, that is, they are not divisible by ⅓ of the odd numbers (that are smaller than N). This non-divisibility by 3, is a property of the numbers of the T symmetry for n=5. Consequently, the odds $T_1(k,5)$, $T_1^*(k,5)$, $T_2(k,5)$, $T_2^*(k,5)$, $k \in \mathbb{N}^*$ are not divisible by ⅓ of the odd numbers (that are smaller than $T_1(k,5)$, $T_1^*(k,5)$, $T_2(k,5)$, $T_2^*(k,5)$, $k \in \mathbb{N}^*$). Because of this, the method is particularly efficient for n=5. We give an example for n=5 and for small values of k, k=1, 2, 3,........, 18.

Example 5.3.

1. $T_2(2,5)$ =270 500807 (9 digits) *is a prime.*

2. $T_2(3,5)$ =17246 461711 (11digits) *is a prime*

   $T_1^*(3,5)$ =32342 343169 (11 digits) *is a prime.*

3. $T_1^*(4,5)$ =2 132417 969153 (13 digits) *is a prime.*

4. $T_1^*(8,5)$ =36 821571 153497 669633 (20 digits) *is a prime.*

5. $T_2(9,5)$ =1180 663669 517502 645247 (22 digits) *is a prime.*

6. $T_1(10,5)$ =75631 614682 207162 007551 (23 digits) *is a prime.*

7. $T_2(11,5)$ =4 835777 063183 149145 526271 (25 digits) *is a prime*

   $T_1^*(11,5)$ =9 669045 950065 986429 124609 (25 digits) *is a prime.*

8. $T_1^*(12,5)$ =618 894471 001773 327207 104513 (27 digits) *is a prime.*

9. $T_2(16,5)$ =5192 299334 412545 020553 193752 494079 (34 digits) *is a prime.*

10. $T_1^*(18,5)$ =42 535214 735633 635683 576920 453379 260417 (38 digits) *is a prime.*

From the above study it emerges that the method is applied in two ways:

a. *We factorize the numbers of the T symmetry and identify the ones that are products of a set of prime numbers with a comparatively larger prime number.*

b. *We identify the prime numbers of the T symmetry, via a primality test, when the relation* (5.7) *holds.*

We suggest, in both cases, that a specific $n \in A = \{2,3,4,...\}$ should be chosen, and then the values k=1, 2, 3,… can be given in equations (5.2), (5.3), (5.5) and (5.6).

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