The internal structure of natural numbers and one method for the definition of large prime numbers

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Abstract

It holds that every product of natural numbers can also be written as a sum. The inverse does not hold when 1 is excluded from the product. For this reason, the investigation of natural numbers should be done through their sum and not through their product. Such an investigation is presented in the present article. We prove that primes play the same role for odd numbers as the powers of 2 for even numbers, and vice versa. The following theorem is proven: ''Every natural number, except for 0 and 1, can be uniquely written as a linear combination of consecutive powers of 2 with the coefficients of the linear combination being -1 or +1.'' This theorem reveals a set of symmetries in the internal order of natural numbers which cannot be derived when studying natural numbers on the basis of the product. From such a symmetry a method for identifying large prime numbers is derived.

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1. INTRODUCTION

It holds that every product of natural numbers can also be written as a sum. The inverse (i.e. each sum of natural numbers can be written as a product) does not hold when 1 is excluded from the product. This is due to prime numbers p which can be written as a product only in the form of $p=1\cdot p$. For this reason, the investigation of natural numbers should be done through their sum and not through their product. Such an investigation is presented in the present article.

We prove that each natural number can be written as a sum of three or more consecutive natural numbers except of the powers of 2 and the prime numbers. Each power of 2 and each prime number cannot be written as a sum of three or more consecutive natural numbers. Primes play the same role for odd numbers as the powers of 2 for even numbers, and vice versa.

We prove a theorem which is analogous to the fundamental theorem of arithmetic, when we study the positive integers with respect to addition: ''Every natural number, with the exception of 0 and 1, can be written in a unique way as a linear combination of consecutive powers of 2, with the coefficients of the linear combination being -1 or +1.'' This theorem reveals a set of symmetries in the internal order of natural numbers which cannot be derived when studying natural numbers on the basis of the product. From such a symmetry a method for identifying large prime numbers is derived.

2. THE SEQUENCE $\mu(k,n)$

We consider the sequence of natural numbers

$$\begin{aligned}&\mu(k,n)=k+(k+1)+(k+2)+\ldots+(k+n)=\frac{(n+1)(2k+n)}{2}\\&k\in\mathbb{N}^*=\{1,2,3,\ldots\}\\&n\in A=\{2,3,4,\ldots\}\end{aligned} \qquad (2.1)$$

For the sequence $\mu(k,n)$ the following theorem holds:

Theorem 2.1.

'' For the sequence $\mu(k,n)$ the following hold:

1. $\mu(k,n)\in\mathbb{N}^*$.
2. No element of the sequence is a prime number.
3. No element of the sequence is a power of 2.
4. The range of the sequence is all natural numbers that are not primes and are not powers of 2.

Proof.

1. $\mu(k,n)\in\mathbb{N}^*$ as a sum of natural numbers.

2. $n\in A=\{2,3,4,\ldots\}$ and therefore it holds that

$$\begin{aligned}&n\geq 2\\&n+1\geq 3\end{aligned}.$$

Also we have that

$$\begin{aligned}&2k+n\geq 4\\&\frac{2k+n}{2}\geq\frac{3}{2}>1\end{aligned}$$

since $k\in\mathbb{N}^*$ and $n\in A=\{2,3,4,\ldots\}$. Thus, the product

$$\frac{(n+1)(2k+n)}{2}=\mu(k,n)$$

is always a product of two natural numbers different than 1, thus the natural number $\mu(k,n)$ cannot be prime.

3. Let that the natural number $\mu(k,n)=\frac{(n+1)(2k+n)}{2}$ is a power of 2. Then, it exists $\lambda\in\mathbb{N}$ such as

$$\frac{(n+1)(2k+n)}{2}=2^{\lambda}$$

$$(n+1)(2k+n)=2^{\lambda+1}\quad. \tag{2.2}$$

Equation (2.2) can hold if and only if there exist $\lambda_1,\lambda_2\in\mathbb{N}$ such as

$$n+1=2^{\lambda_1}\wedge 2k+n=2^{\lambda_2}$$

and equivalently

$$\left.\begin{array}{l} n=2^{\lambda_1}-1 \\ n=2^{\lambda_2}-2k \end{array}\right\}. \tag{2.3}$$

We eliminate n from equations (2.3) and we obtain

$$2^{\lambda_1}-1=2^{\lambda_2}-2k$$

and equivalently

$$2k-1=2^{\lambda_2}-2^{\lambda_1}$$

which is impossible since the first part of the equation is an odd number and the second part is an even number. Thus, the range of the sequence $\mu(k,n)$ does not include the powers of 2.

4. We now prove that the range of the sequence $\mu(k,n)$ includes all natural numbers that are not primes and are not powers of 2. Let a random natural number N which is not a prime nor a power of 2. Then, N can be written in the form

$$N=\chi\psi$$

where at least one of the χ,ψ is an odd number ≥ 3. Let χ be an odd number ≥ 3. We will prove that there are always exist $k\in\mathbb{N}$ and $n\in A=\{2,3,4,\ldots\}$ such as

$$N=\chi\cdot\psi=\mu(k,n).$$

We consider the following two pairs of k and n:

$$\begin{array}{l} \chi\leq 2\psi-1,\ \chi,\psi\in\mathbb{N} \\ k=k_1=\dfrac{2\psi+1-\chi}{2} \\ n=n_1=\chi-1 \end{array} \tag{2.4}$$

$$\begin{array}{l} \chi\geq 2\psi+1,\ \chi,\psi\in\mathbb{N} \\ k=k_2=\dfrac{\chi+1-2\psi}{2}\quad. \\ n=n_2=2\psi-1 \end{array} \tag{2.5}$$

For every $\chi,\psi\in\mathbb{N}$ it holds either the inequality $\chi\leq 2\psi-1$ or the inequality $\chi\geq 2\psi+1$. Thus, for each pair of naturals (χ,ψ), where χ is odd, at least one of the pairs (k_1,n_1), (k_2,n_2) of equations (2.4), (2.5) is defined. We now prove that "when the natural number k_1 of equation (2.4) is $k_1=0$ then the natural number k_2 of equation (2.5) is $k_2=1$ and additionally it holds that $n_2>2$.". For $k_1=0$ from equations (2.4) we take

$$\chi=2\psi+1$$

and from equations (2.5) we have that

$$k_2=\frac{(2\psi+1)+1-2\psi}{2}=1$$
$$n_2=2\psi-1$$

and because $\psi\geq 2$ we obtain

$$k_2=1$$
$$n_2=2\psi-1\geq 3>2\,.$$

We now prove that when $k_2=0$ in equations (2.5), then in equations (2.4) it is $k_1=1$ and $n_1>2$. For $k_2=0$, from equations (2.5) we obtain

$$\chi=2\psi-1$$

and from equations (2.4) we get

$$k_1=\frac{2\psi+1-(2\psi-1)}{2}=1\,.$$
$$n_1=\chi-1=2\psi-2\geq 2$$

We now prove that at least one of the k_1 and k_2 is positive. Let

$$k_1<0\wedge k_2<0\,.$$

Then from equations (2.4) and (2.5) we have that

$$2\psi+1-\chi<0\wedge\chi+1-2\psi<0\,. \qquad (2.6)$$

Taking into account that $\chi>1$ is odd, that is $\chi=2\rho+1,\rho\in\mathbb{N}$, we obtain from inequalities (2.6)

$$2\psi+1-(2\rho-1)<0\wedge(2\rho+1)+1-2\psi<0$$
$$2\psi-2\rho<0\wedge 2\rho-2\psi+2>0$$
$$\psi<\rho\wedge\psi>\rho+1$$

which is absurd. Thus, at least one of k_1 and k_2 is positive.

For equations (2.4) we take

$$\mu(k_1,n_1)=\frac{(n_1+1)(2k_1+n_1)}{2}$$
$$=\frac{(\chi-1+1)\left(2\frac{2\psi+1-\chi}{2}+\chi-1\right)}{2}=\frac{\chi(2)\psi}{2}=\chi\psi=N.$$

For equations (2.5) we obtain

$$\mu(k_2,n_2)=\frac{(n_2+1)(2k_2+n_2)}{2}$$
$$=\frac{(2\psi-1+1)\left(2\frac{\chi+1-2\psi}{2}+2\psi-1\right)}{2}=\frac{2\psi\chi}{2}=\chi\psi=N.$$

Thus, there are always exist $k\in\mathbb{N}^*$ and $n\in A=\{2,3,4,...\}$ such as

$N=\chi\psi=\mu(k,n)$ for every N which is not a prime number and is not a power of 2. □

Example 2.1. For the natural number $N=40$ we have

$N=40=5\cdot 8$

$\chi=5$

$\psi=8$

and from equations (2.4) we get

$$k=k_1=\frac{16+1-5}{2}=6$$

$n=n_1=5-1=4$

thus, we obtain

$40=\mu(6,4)$.

Example 2.2. For the natural number $N=51$,

$N=51=3\cdot 17=17\cdot 3$

there are two cases. First case:

$N=51=3\cdot 17$

$\chi=3$

$\psi=17$

and from equations (2.4) we obtain

$$k=k_1=\frac{34+1-3}{2}=16$$

$n=n_1=3-1=2$

thus,

$51=\mu(16,2)$.

Second case:

$N=51=17\cdot 3$
$\chi=17$
$\psi=3$
and from equations (2.5) we obtain

$$k=k_2=\frac{17+1-6}{2}=6$$

$$n=n_2=6-1=5$$

thus,

$$51=\mu(6,5).$$

The second example expresses a general property of the sequence $\mu(k,n)$. The more composite an odd number that is not prime (or an even number that is not a power of 2) is, the more are the $\mu(k,n)$ combinations that generate it.

Example 2.3.

$$135=15\cdot 9=27\cdot 5=9\cdot 15=45\cdot 3=5\cdot 27=3\cdot 45$$
$$135=\mu(2,14)=\mu(9,9)=\mu(11,8)=\mu(20,5)=\mu(25,4)=\mu(44,2)$$

a. $135=9\cdot 15=\mu(2,14)=\mu(11,8)$

$$135=2+3+4+......+15+16=11+12+13.....+18+19.$$

b. $135=5\cdot 27=\mu(9,9)=\mu(25,4)$

$$135=9+10+11+.....+17+18=25+26+27+28+29.$$

c. $135=3\cdot 45=\mu(20,5)=\mu(44,2)$

$$135=20+21+22+23+24+25=44+45+46.$$

In the transitive property of multiplication, when writing a composite odd number or an even number that is not a power of 2 as a product of two natural numbers, we use the same natural numbers $\chi,\psi\in\mathbb{N}$:

$$\Phi=\chi\cdot\psi=\psi\cdot\chi.$$

On the contrary, the natural number Φ can be written in the form $\Phi=\mu(k,n)$ using different natural numbers $k\in\mathbb{N}^*$ and $n\in A=\{2,3,4,...\}$, through equations (2.4), (2.5). This difference between the product and the sum can also become evident in example 2.3:

$$135=3\cdot 45=45\cdot 3$$
$$135=44+45+46=20+21+22+23+24+25.$$

From Theorem 2.1 the following corollary is derived:

Corollary 2.1. “1. Every natural number which is not a power of 2 and is not a prime can be written as the sum of three or more *consecutive* natural numbers.

2. Every power of 2 and every prime number cannot be written as the sum of three or more *consecutive* natural numbers.”

Proof. Corollary 2.1 is a direct consequence of Theorem 2.1.□

3. THE CONCEPT OF REARRANGEMENT

In this paragraph, we present the concept of rearrangement of the composite odd numbers and even numbers that are not power of 2. Moreover, we prove some of the consequences of the rearrangement in the Diophantine analysis. The concept of rearrangement is given from the following definition:

Definition. ‘’We say that the sequence $\mu(k,n), k \in \mathbb{N}^*, n \in A = \{2,3,4,...\}$ is rearranged if there exist natural numbers $k_1 \in \mathbb{N}^*, n_1 \in A$, $(k_1,n_1) \neq (k,n)$ such as

$$\mu(k,n) = \mu(k_1,n_1) . \text{''} \tag{3.1}$$

From equation (2.1) written in the form of

$$\mu(k,n) = k + (k+1) + (k+2) + ... + (k+n)$$

two different types of rearrangement are derived: The “compression”, during which n decreases with a simultaneous increase of k. The «decompression», during which n increases with a simultaneous decrease of k. The following theorem provides the criterion for the rearrangement of the sequence $\mu(k,n)$.

Theorem 3.1. ‘’ ’1. The sequence $\mu(k_1,n_1), (k_1,n_1) \in \mathbb{N}^* \times A$ can be compressed

$$\mu(k_1,n_1) = \mu(k_1+\varphi, n_1-\omega) \tag{3.2}$$

if and only if there exist $\varphi, \omega \in \mathbb{N}^*, \omega \leq n_1 - 2$ which satisfies the equation

$$\begin{aligned} &\omega^2 - (2k_1 + 2n_1 + 1 + 2\varphi)\omega + 2(n_1+1)\varphi = 0 \\ &\varphi, \omega \in \mathbb{N}^* \\ &\omega \leq n_1 - 2 \end{aligned} . \tag{3.3}$$

2. The sequence $\mu(k_2,n_2), (k_2,n_2) \in \mathbb{N}^* \times A$ can be decompressed

$$\mu(k_2,n_2) = \mu(k_2-\varphi, n_2+\omega) \tag{3.4}$$

if and only if there exist $\varphi, \omega \in \mathbb{N}^*, \varphi \le k_2 - 1$ which satisfies the equation

$$\begin{aligned}&\omega^2 + (2k_2 + 2n_2 + 1 - 2\varphi)\omega - 2(n_2 + 1)\varphi = 0\\&\varphi, \omega \in \mathbb{N}^* \qquad .\\&\varphi \le k_2 - 1\end{aligned} \tag{3.5}$$

3. The odd number $\Pi \ne 1$ is prime if and only if the sequence

$$\begin{aligned}&\mu(k,n) = \Pi \cdot 2^l\\&l, k \in \mathbb{N}^*, n \in A\end{aligned} \tag{3.6}$$

cannot be rearranged.

4. The odd Π is prime if and only if the sequence

$$\mu\left(\frac{\Pi+1}{2}, \Pi - 1\right) = \Pi^2 \tag{3.7}$$

cannot be rearranged.''

Proof. 1,2. We prove part 1 of the corollary and similarly number 2 can also be proven. From equation (4.1) we conclude that the sequence $\mu(k_1, n_1)$ can be compressed if and only if there exist $\varphi, \omega \in \mathbb{N}^*$ such as

$$\mu(k_1, n_1) = \mu(k_1 + \varphi, n_1 - \omega).$$

In this equation the natural number $n_1 - \omega$ belongs to the set $A = \{2, 3, 4, ...\}$ and thus $n_1 - \omega \ge 2 \Leftrightarrow \omega \le n_1 - 2$. Next, from equations (2.1) we obtain

$$\mu(k_1, n_1) = \mu(k_1 + \varphi, n_1 - \omega)$$

$$\frac{(n_1+1)(2k_1+n_1)}{2} = \frac{(n_1 - \omega + 1)\left[2(k_1+\varphi) + n_1 - \omega\right]}{2}$$

and after the calculations we get equation (3.3).

3. The sequence (3.6) is derived from equations (2.4) or (2.5) for $\chi = \Pi$ and $\psi = 2^l$. Thus, in the product $\chi\psi$ the only odd number is Π. If the sequence $\mu(k,n)$ in equation (3.6) cannot be rearranged then the odd number Π has no divisors. Thus, Π is prime. Obviously, the inverse also holds.

4. First, we prove equations (3.7). From equation (2.1) we obtain:

$$\mu\left(\frac{\Pi+1}{2},\Pi-1\right)=\frac{(\Pi-1+1)\left(2\frac{\Pi+1}{2}+\Pi-1\right)}{2}=\Pi^2.$$

In case that the odd number Π is prime in equations (2.4), (2.5) the natural numbers χ,ψ are unique $\chi=\Pi\wedge\psi=\Pi$, and from equation (2.5) we get $k=\frac{\Pi+1}{2}\wedge n=\Pi-1$. Thus, the sequence $\mu(k,n)=\mu\left(\frac{\Pi+1}{2},\Pi-1\right)$ cannot be rearranged. Conversely, if the sequence $\mu\left(\frac{\Pi+1}{2},\Pi-1\right)=\Pi^2=\Pi\cdot\Pi$ cannot be rearranged the odd number Π cannot be composite and thus Π is prime.□

We now prove the following corollary:

Corollary 3.1. ''1. The odd number Φ,

$$\begin{aligned}&\Phi=\Pi^2=\mu\left(\frac{\Pi+1}{2},\Pi-1\right)\\&\Pi=odd\\&\Pi\neq1\end{aligned}\tag{3.8}$$

is decompressed and compressed if and only if the odd number Π is composite.

2. The even number α_1,

$$\begin{aligned}&\alpha_1=2^l\Pi=\mu\left(2^l-\frac{\Pi-1}{2},\Pi-1\right)\\&\Pi=odd\\&3\leq\Pi\leq2^l-1\\&l\in\mathbb{N},l\geq2\end{aligned}\tag{3.9}$$

cannot be decompressed, while it compresses if and only if the odd number Π is composite.

3. The even number α_2,

$$\begin{aligned}&\alpha_2=2^l\Pi=\mu\left(\frac{\Pi+1}{2}-2^l,2^{l+1}-1\right)\\&\Pi=odd\\&\Pi\geq2^{l+1}+1\\&l\in\mathbb{N}^*\end{aligned}\tag{3.10}$$

cannot be compressed, while it decompresses if and only if the odd number Π is composite.

4. Every even number that is not a power of can be written either in the form of equation (3.9) or in the form of equation (3.10).’’

Proof.

1. It is derived directly through number (4) of Theorem 3.1. A second proof can be derived through equations (2.4), (2,5) since every composite odd Π can be written in the form of $\Pi = \chi\psi$, $\chi,\psi \in \mathbb{N}$, χ,ψ odds.

2,3.

Let the even number α ,

$$\begin{aligned} &\alpha = 2^l \Pi \\ &\Pi = odd\,. \\ &l \in \mathbb{N}^* \end{aligned} \tag{3.11}$$

From equation (2.4) we obtain

$$\begin{aligned} &k = \frac{2 \cdot 2^l + 1 - \Pi}{2} = 2^l - \frac{\Pi - 1}{2} \\ &n = \Pi - 1 \end{aligned} \tag{3.12}$$

and since $k,n \in \mathbb{N}, k \geq 1 \wedge n \geq 2$ we get

$$\begin{aligned} &\frac{2 \cdot 2^l + 1 - \Pi}{2} \geq 1 \\ &\Pi - 1 \geq 2 \end{aligned}$$

and equivalently

$$3 \leq \Pi \leq 2^{l+1} - 1.$$

In the second of equations (3.12) the natural number n obtains the maximum possible value of $n = \Pi - 1$, and thus the natural number k takes the minimum possible value in the first of equations (3.12). Thus, the even number

$$\alpha_1 = \mu\left(2^l - \frac{\Pi - 1}{2}, \Pi - 1\right)$$

cannot decompress. If the odd number Π is composite then it can be written in the form of $\Pi = \chi\psi$, $\chi,\psi \in \mathbb{N}^*$, χ,ψ odds, $\chi,\psi < \Pi$, $\alpha_1 = 2^l \chi\psi$. Therefore, the natural number $\alpha_1 = 2^l \chi\psi$ decompresses since from equations (3.11) it can be written in the form of $\alpha_1 = \mu(k,n)$ with $n = \chi - 1 < \Pi - 1$. Similarly, the proof of 3 is derived from equations (2.5).

4. From the above proof process it follows that every even number that is not a power of 2 can be written either in the form of equation (3.9) or in the form of equation (3.10). □

By substituting $\Pi = P = prime$ in equations of Theorem 3.1 and of corollary 3.1 four sets of equations are derived, each including infinite *impossible* diophantine equations.

Example 3.1. The odd number $P = 999961$ is prime. Thus, combining (1) of Theorem 3.1 with (1) of corollary 3.1 we conclude that there is no pair $(\omega, \varphi) \in \mathbb{N}^2$ with $\omega \leq 999958$ which satisfies the diophantine equation

$$\omega^2 - (2999883 + 2\varphi)\omega + 1999922\varphi = 0.$$

We now prove the following corollary:

Corollary 3.2 ''The square of every prime number can be uniquely written as the sum of consecutive natural numbers.''

Proof. For $\Pi = P = prime$ in equation (3.5) we obtain

$$P^2 = \mu\left(\frac{P+1}{2}, P-1\right). \tag{3.13}$$

According with 4 of Theorem 3.1 the odd P^2 cannot be rearranged. Thus, the odd can be uniquely written as the sum of consecutive natural numbers, as given from equation (3.13). □

Example 3.2. The odd $P = 17$ is prime. From equation (3.13) for $P = 17$ we obtain

$$289 = \mu(9,16)$$

and from equation (2.1) we get

$$289 = 9+10+11+12+13+14+15+16+17+18+19+20+21+22+23+24+25$$

which is the only way in which the odd number 289 can be written as a sum of consecutive natural numbers.

4. NATURAL NUMBERS AS LINEAR COMBINATION OF CONSECUTIVE POWERS OF 2

According to the fundamental theorem of arithmetic, every natural number can be uniquely written as a product of powers of prime numbers. The previously presented study reveals a correspondence between odd prime numbers and the powers of 2. Thus, the question arises whether there exists a theorem for the powers of 2 corresponding to the fundamental theorem of arithmetic. The answer is given by the following theorem:

Theorem 4.1. 'Every natural number, with the exception of 0 and 1, can be uniquely written as a linear combination of consecutive powers of 2, with the coefficients of the linear combination being -1 or +1.''

Proof. Let the odd number Π as given from equation

$$\Pi = \Pi\left(\nu,\beta_i\right) = 2^{\nu+1} + 2^{\nu} \pm 2^{\nu-1} \pm 2^{\nu-2} \pm .... \pm 2^1 \pm 2^0 = 2^{\nu+1} + 2^{\nu} + \sum_{i=0}^{\nu-1} \beta_i 2^i$$

$$\beta_i = \pm 1, i = 0,1,2,........,\nu-1 \qquad . \qquad (4.1)$$

$$\nu \in \mathbb{N}$$

From equation (4.1) for $\nu = 0$ we obtain

$$\Pi = 2^1 + 2^0 = 2 + 1 = 3.$$

We now examine the case where $\nu \in \mathbb{N}^*$. The lowest value that the odd number Π of equation (4.1) can obtain is

$$\Pi_{\min} = \Pi\left(\nu\right) = 2^{\nu+1} + 2^{\nu} - 2^{\nu-1} - 2^{\nu-1} - .... 2^1 - 1$$

$$\Pi_{\min} = \Pi\left(\nu\right) = 2^{\nu+1} + 1. \qquad (4.2)$$

The largest value that the odd number Π of equation (4.1) can obtain is

$$\Pi_{\max} = \Pi\left(\nu\right) = 2^{\nu+1} + 2^{\nu} + 2^{\nu-1} + ... 2^1 + 1$$

$$\Pi_{\max} = \Pi\left(\nu\right) = 2^{\nu+2} - 1. \qquad (4.3)$$

Thus, for the odd numbers $\Pi = \Pi\left(\nu,\beta_i\right)$ of equation (4.1) the following inequality holds

$$\Pi_{\min} = 2^{\nu+1} + 1 \le \Pi\left(\nu,\beta_i\right) \le 2^{\nu+2} - 1 = \Pi_{\max} . \qquad (4.4)$$

The number $N\left(\Pi\left(\nu,\beta_i\right)\right)$ of odd numbers in the closed interval $\left[2^{\nu+1}+1, 2^{\nu+2}-1\right]$ is

$$N\left(\Pi\left(\nu,\beta_i\right)\right) = \frac{\Pi_{\max} - \Pi_{\min}}{2} + 1 = \frac{\left(2^{\nu+2}-1\right) - \left(2^{\nu+1}+1\right)}{2} + 1$$

$$N\left(\Pi\left(\nu,\beta_i\right)\right) = 2^{\nu} . \qquad (4.5)$$

The integers $\beta_i, i = 0,1,2,........,\nu-1$ in equation (4.1) can take only two values, $\beta_i = -1 \vee \beta_i = +1$, thus equation (4.1) gives exactly $2^{\nu} = N\left(\Pi\left(\nu,\beta_i\right)\right)$ odd numbers. Therefore, for every $\nu \in \mathbb{N}^*$ equation (4.1) gives all odd numbers in the interval $\left[2^{\nu+1}+1, 2^{\nu+2}-1\right]$.

We now prove the theorem for the even numbers. Every even number α which is a power of 2 can be uniquely written in the form of $\alpha = 2^{\nu}, \nu \in \mathbb{N}^*$. We now consider the case where the even number α is not a power of 2. In that case, according to corollary 3.1 the even number α is written in the form of

$$\alpha = 2^l \Pi, \Pi = \mathrm{odd}, \Pi \ne 1, l \in \mathbb{N}^* . \qquad (4.6)$$

We now prove that the even number α can be uniquely written in the form of equation (4.6). If we assume that the even number α can be written in the form of

$$\begin{aligned}
&\alpha = 2^{l}\Pi = 2^{l'}\Pi' \\
&l \neq l' (l > l') \\
&\Pi \neq \Pi' \qquad\qquad (4.7) \\
&l, l' \in \mathbb{N}^* \\
&\Pi, \Pi' = odd
\end{aligned}$$

the we obtain

$$2^{l}\Pi = 2^{l'}\Pi'$$
$$2^{l-l'}\Pi = \Pi'$$

which is impossible, since the first part of this equation is even and the second odd. Thus, it is $l = l'$ and we take that $\Pi = \Pi'$ from equation (4.7). Therefore, every even number α that is not a power of 2 can be uniquely written in the form of equation (4.6). The odd number Π of equation (4.6) can be uniquely written in the form of equation (4.1), thus from equation (4.6) it is derived that every even number α that is not a power of 2 can be uniquely written in the form of equation

$$\begin{aligned}
&\alpha = \alpha(l, \nu, \beta_i) = 2^{l}\left(2^{\nu+1} + 2^{\nu} + \sum_{i=0}^{\nu-1}\beta_i 2^{i}\right) \\
&l \in \mathbb{N}^*, \nu \in \mathbb{N} \qquad\qquad (4.8) \\
&\beta_i = \pm 1, i = 0, 1, 2, ..., \nu - 1
\end{aligned}$$

and equivalently

$$\begin{aligned}
&\alpha = \alpha(l, \nu, \beta_i) = 2^{l+\nu+1} + 2^{l+\nu} + \sum_{i=0}^{\nu-1}\beta_i 2^{l+i} \\
&l \in \mathbb{N}^*, \nu \in \mathbb{N} \qquad . \qquad\qquad (4.9) \\
&\beta_i = \pm 1, i = 0, 1, 2, ..., \nu - 1
\end{aligned}$$

For 1 we take

$$1 = 2^{0}$$
$$1 = 2^{1} - 2^{0}$$

thus, it can be written in two ways in the form of equation (4.1). Both the odds of equation (4.1) and the evens of the equation (4.8) are positive. Thus, 0 cannot be written either in the form of equation (4.1) or in the form of equation (4.8). □

In order to write an odd number $\Pi \neq 1, 3$ in the form of equation (4.1) we initially define the $\nu \in \mathbb{N}^*$ from inequality (4.4). Then, we calculate the sum

$2^{\nu+1}+2^{\nu}$.

If it holds that $2^{\nu+1}+2^{\nu}<\Pi$ we add the $2^{\nu-1}$, whereas if it holds that $2^{\nu+1}+2^{\nu}>\Pi$ then we subtract it. By repeating the process exactly ν times we write the odd number Π in the form of equation (4.1). The number of ν steps needed in order to write the odd number Π in the form of equation (4.1) is extremely low compared to the magnitude of the odd number Π, as derived from inequality (4.4).

Example 4.1. For the odd number $\Pi=23$ we obtain from inequality (4.4)

$$2^{\nu+1}+1<23<2^{\nu+2}-1$$
$$2^{\nu+1}+2<24<2^{\nu+2}$$
$$2^{\nu}<12<2^{\nu+1}$$

thus $\nu=3$. Then, we have

$2^{\nu+1}+2^{\nu}=2^4+2^3=24>23$ (thus 2^2 is subtracted)

$2^4+2^3-2^2=20<23$ (thus 2^1 is added)

$2^4+2^3-2^2+2^1=22<23$ (thus $2^0=1$ is added)

$2^4+2^3-2^2+2^1+1=23.$

Fermat numbers F_s can be written directly in the form of equation (4.1), since they are of the form $\Pi_{\min}$,

$$\begin{aligned}&F_s=2^{2^s}+1=\Pi_{\min}\left(2^s-1\right)=2^{2^s}+2^{2^s-1}-2^{2^s-2}-2^{2^s-3}-\dots\dots-2^1-1\\&s\in\mathbb{N}^*\end{aligned}. \quad (4.10)$$

Mersenne numbers M_p can be written directly in the form of equation (4.1), since they are of the form $\Pi_{\max}$,

$$\begin{aligned}&M_p=2^p-1=\Pi_{\max}\left(p-2\right)=2^{p-1}+2^{p-2}+2^{p-3}+\dots\dots+2^1+1\\&p=prime\end{aligned}. \quad (4.11)$$

In order to write an even number α that is not a power of 2 in the form of equation (4.1), initially it is consecutively divided by 2 and it takes of the form of equation (4.6). Then, we write the odd number Π in the form of equation (4.1).

Example 4.2. By consecutively dividing the even number $\alpha=368$ by 2 we obtain $\alpha=368=2^4\cdot 23$. Then, we write the odd number $\Pi=23$ in the form of equation (4.1), $23=2^4+2^3-2^2+2^1+1$, and we get

$$\begin{aligned}&368=2^4\left(2^4+2^3-2^2+2^1+1\right)\\&368=2^8+2^7-2^6+2^5+2^4\end{aligned}.$$

This equation gives the unique way in which the even number $\alpha = 368$ can be written in the form of equation (4.9).

From inequality (4.4) we obtain

$$2^{\nu+1} + 1 \le \Pi \le 2^{\nu+2} - 1$$
$$2^{\nu+1} < 2^{\nu+1} + 1 \le \Pi \le 2^{\nu+2} - 1 < 2^{\nu+2}$$
$$2^{\nu+1} < \Pi < 2^{\nu+2}$$
$$(\nu+1)\log 2 < \log \Pi < (\nu+2)\log 2$$

from which we get

$$\frac{\log \Pi}{\log 2} - 1 < \nu + 1 < \frac{\log \Pi}{\log 2}$$

and finally

$$\nu + 1 = \left[\frac{\log \Pi}{\log 2}\right] \tag{4.12}$$

'where $\left[\frac{\log \Pi}{\log 2}\right]$ the integer part of $\frac{\log \Pi}{\log 2} \in \mathbb{R}$.

We now give the following definition:

Definition 4.1. We define as the conjugate of the odd

$$\begin{gathered} \Pi = \Pi(\nu, \beta_i) = 2^{\nu+1} + 2^{\nu} + \sum_{i=0}^{i=\nu-1} \beta_i 2^i \\ \beta_i = \pm 1, i = 0,1,2,........,\nu - 1 \\ \nu \in \mathbb{N}^* \end{gathered} \tag{4.13}$$

the odd Π^*,

$$\begin{gathered} \Pi^* = \Pi^*(\nu, \gamma_j) = 2^{\nu+1} + 2^{\nu} + \sum_{j=0}^{j=\nu-1} \gamma_j 2^j \\ \gamma_i = \pm 1, \; j = 0,1,2,........,\nu - 1 \\ \nu \in \mathbb{N}^* \end{gathered} \tag{4.14}$$

for which it holds

$$\gamma_k = -\beta_k \forall k = 0,1,2,........,\nu - 1 . \tag{4.15}$$

For conjugate odds, the following corollary holds:

Corollary 4.1. '' For the conjugate odds $\Pi = \Pi(\nu, \beta_i)$ and $\Pi^* = \Pi^*(\nu, \gamma_i)$ the following hold:

1. $\left(\Pi^*\right)^* = \Pi$. (4.16)

2. $\Pi + \Pi^* = 3 \cdot 2^{\nu+1}$. (4.17)

3. Π is divisible by 3 if and only if Π^* is divisible by 3.''

Proof. 1.The 1 of the corollary is an immediate consequence of definition 4.1.

2. From equations (4.13), (4.14) and (4.15) we get

$$\Pi + \Pi^* = \left(2^{\nu+1} + 2^{\nu}\right) + \left(2^{\nu+1} + 2^{\nu}\right)$$

and, equivalently

$$\Pi + \Pi^* = 3 \cdot 2^{\nu+1}.$$

3. If the odd Π is divisible by 3 then it is written in the form $\Pi = 3x, x = odd$ and from equation (4.17) we get $3x + \Pi^* = 3 \cdot 2^{\nu+1}$ and equivalently $\Pi^* = 3\left(2^{\nu+1} - x\right)$. Similarly we can prove the inverse. □

5. THE HARMONIC ODD NUMBERS AND A METHOD FOR DEFINING LARGE PRIME NUMBERS

The harmonic symmetry: We define as harmonic the odd numbers of equation (4.1) for which the signs of $\beta_i = \pm 1, i = 0,1,2,3........,\nu - 1$ alternate:

$$\begin{aligned} &\Pi_1 = 2^{\nu+1} + 2^{\nu} - 2^{\nu-1} + 2^{\nu-2} - .... - 2^1 + 1 \\ &\Pi_2 = 2^{\nu+1} + 2^{\nu} + 2^{\nu-1} - 2^{\nu-2} + ... + 2^1 - 1 \ . \\ &\nu = 2\lambda, \lambda \in \mathbb{N}^* \end{aligned} \quad (5.1)$$

$$\begin{aligned} &\Pi_1 = 2^{\nu+1} + 2^{\nu} - 2^{\nu-1} + 2^{\nu-2} - .... - 2^1 + 1 \\ &\Pi_2 = 2^{\nu+1} + 2^{\nu} + 2^{\nu-1} - 2^{\nu-2} + ... + 2^1 - 1 . \\ &\nu = 2\lambda + 1, \lambda \in \mathbb{N}^* \end{aligned} \quad (5.2)$$

From equations (5.1), (5.2) and definition 4.1 we obtain

$$\Pi_2 = \Pi_1^* = 3 \times 2^{\nu+1} - \Pi_1 \quad (5.3)$$

for the pair of harmonic odd numbers.

A method for the determination of large prime numbers emerges from the study we presented. This method is completely different from previous methods [1-11].When we consider the prime factorization of the odd integers

$$\begin{aligned} &\Phi_1 = 2 + \Pi_1 = 2 + 2^{\nu+1} + 2^{\nu} - 2^{\nu-1} + 2^{\nu-2} - .... - 2^1 + 1 \\ &\nu = 2\lambda, \lambda \in \mathbb{N}^* \end{aligned} \quad (5.4)$$

$$\Phi_2 = -2 + \Pi_2 = -2 + 2^{\nu+1} + 2^{\nu} + 2^{\nu-1} - 2^{\nu-2} + \dots\dots + 2^1 - 1$$
$$\nu = 2\lambda, \lambda \in \mathbb{N}^* \qquad (5.5)$$

$$\Phi_2 = \Phi_1^* = 3 \times 2^{\nu+1} - \Phi_1 \qquad (5.6)$$

we have the following statement:

The factors of either Φ_1 or $\Phi_2 = \Phi_1^*$ consist of a set of small prime factors and one large factor. Hence from the factorization of Φ_1 and $\Phi_2 = \Phi_1^*$ of equations (5.4), (5.5) we get a large prime number. Following are 11 examples where we have chosen arbitrary even ν , $600 \le \nu \le 1000$, in equations (5.4), (5.5).

1. $\nu = 604$

$\Phi_1 =$ 3 × 5 × 1423 × 2677 × 1 039667 × 1 465469 × 2 033624 136455 907062 140355 606581 460617 960329 378244 909713 340374 546035 722007 834481 807880 893223 943637 129816 307143 853799 454110 280390 103176 523414 883666 509589 711687 765791

2. $\nu = 626$

$\Phi_1 =$ 13 × 186653 × 306 032599 340492 581270 323029 570138 222136 733600 420877 092183 139417 574185 782109 955578 496315 765962 131603 014089 519221 871827 181120 845674 859725 387186 219442 305406 755275 821605 426602 403741 599957

3. $\nu = 644$

$\Phi_1 =$ 5 × 79 × 12 671297 × 38892 671359 559494 324882 180204 888273 078001 950134 412751 881230 225550 378061 442396 379471 711953 850899 474720 409489 565536 036909 109253 945965 590266 361910 559333 142120 493266 182138 997818 136400 630503

4. $\nu = 688$

$\Phi_1 =$ 3^3 × 5 × 137 × 2357 × 84239 × 14 276659 × 111598 463167 × 164995 567141 × 3547 493034 864246 374604 223939 439254 117526 195183 644765 258504 745395 441461 348003 624541 265182 053620 319595 210678 493117 621150 188802 864705 030169 622562 000148 389984 593085 080457

5. $\nu = 732$

$\Phi_1 =$ 5^5 × 19 × 4357 × 10093 × 2 901193 × 373 058471 × 21 318693 003272 810610 223875 009176 985967 454565 131359 547239 807330 702392 207730 665072 351378 572215 387223 133953 567092 456647 869354 874941 347502 746701 928543 247909 407783 975122 056127 018272 539991 430427 637981

6. $\nu = 818$

$\Phi_2 =$ 5 826599 918309 521729 414628 892756 111346 582385 085483 938095 996388 692690 239258 901551 139189 714409 550909 093308 382061 608683 211278 156913 402724 889465 422572 029940 710372 011896 628413 739441 379150 647273 555252 328499 350633 086191 299627 798178 454098 229036 211569 513811 is prime

7. $\nu = 838$?

8. $\nu = 842$

$\Phi_1 =$ 13 × 811 × 7789 × 15271 × 66809 × 933 419184 297225 688884 848133 741618 091582 561157 362135 750330 558036 085494 747230 138415 970602 017694 350758 458917 589235 971861 548843 635060 827053 633582 882443 092203 262135 552296 661334 709021 156021 405492 515100 671199 284761 072521 866782 927154 434480 887521

9. $\nu = 914$?

10. $\nu = 986$?

11. $\nu = 998$

$\Phi_2 =$ 23 × 277 × 4211 × 1 385899 × 240154 091459 652243 015812 929515 159070 212159 918817 425875 611004 712759 052716 135663 441910 181493 025014 669780 274245 881010 561780 858639 784499 969926 885693 756207 174479 909272 942309 784548 553831 369221 141895 942976 579419 394048 307219 568666 715750 728448 387606 183250 921312 430705 694057 415487 884739 523892 723969

Equations (5.4), (5.5) and (5.6) are a special case of equations

$$\begin{aligned}
&\Phi_1\left(\nu,\xi\right) = \Phi_1\left(\nu, 2^{2\xi+1}\right) = 2^{2\xi+1} + \Pi_1\left(\nu\right) = 2^{2\xi+1} + 2^{\nu+1} + 2^{\nu} - 2^{\nu-1} + 2^{\nu-2} - \dots\dots\dots - 2^1 + 1 \\
&\nu = 2\lambda, \lambda \in \mathbb{N}^* \qquad (5.7) \\
&\xi = 0,1,2,\dots\dots\dots,\frac{\nu-2}{2} = \lambda - 1
\end{aligned}$$

$$\begin{aligned}
&\Phi_2\left(\nu,\xi\right) = \Phi_2\left(\nu, 2^{2\xi+1}\right) = -2^{2\xi+1} + \Pi_2\left(\nu\right) = -2^{2\xi+1} + 2^{\nu+1} + 2^{\nu} + 2^{\nu-1} - 2^{\nu-2} + \dots\dots\dots + 2^1 - 1 \\
&\nu = 2\lambda, \lambda \in \mathbb{N}^* \qquad (5.8) \\
&\xi = 0,1,2,\dots\dots\dots,\frac{\nu-2}{2} = \lambda - 1
\end{aligned}$$

$$\Phi_2\left(\nu,\xi\right) = 3\times 2^{\nu+1} - \Phi_1\left(\nu,\xi\right) = \Phi_1^*\left(\nu,\xi\right) \qquad (5.9)$$

for $\xi = 0$. The general equations (5.7), (5.8) and (5.9) give all possible variations of the method. For example, for $\nu = 838$, a value of ν that did not give a large prime number in the previous examples, from equation (5.7) for $\xi = 1$ we get

$\Phi_1(838,1) =$ 3 × 251 × 124 958179 125661 642577 × 51 945201 394308 356447 274374 943957 749268 889249 128703 205379 933327 597692 534177 000888 147927 160249 734500 867000 722765 431922 957290 626876 299700 840201 468643 187688 745195 339241 792572 155819 582073 320776 475981 870379 650986 830637 696975 455178 897139

For $\nu = 66$ and $\xi = 1,2,3,\ldots\ldots,\dfrac{\nu-2}{2} = 32$ from equation (5.7) we get

$\Pi_1 =$ 196765270119568550571

$\Pi_2 =$ 245956587649460688213=3^2 × 27 328509 738828 965357

$\Phi_2(\nu,2) = \Phi_1^*(\nu,2) =$ 73 × 3 369268 323965 214907

$\Phi_1(\nu,2^3) =$ 1 645337 × 119 589646 449067

$\Phi_2(\nu,2^5) =$ 4140 643009 × 59400 577909

$\Phi_2(\nu,2^7) = \Phi_1^*(\nu,2^7) =$ 5 × 13907 × 3537 162402 379531

$\Phi_2(\nu,2^9) = \Phi_1^*(\nu,2^9) =$ 13 × 601 × 25184 342777 367023

$\Phi_2(\nu,2^{11}) = \Phi_1^*(\nu,2^{11}) =$ 5 × 49 191317 529892 137233

$\Phi_1(\nu,2^{13}) =$ 196 765270 119568 558763 is prime

$\Phi_2(\nu,2^{15}) = \Phi_1^*(\nu,2^{15}) =$ 5 × 3259 × 36269 × 416167 836559

$\Phi_2(\nu,2^{17}) = \Phi_1^*(\nu,2^{17}) =$ 157 × 12 248491 × 127901 671243

$\Phi_1(\nu,2^{19}) =$ 13 × 19 × 4643 × 32083 × 5347 833013

$\Phi_1(\nu,2^{21}) =$ 271 × 5903 × 123 000357 013771

$\Phi_1(\nu,2^{23}) =$ 7 × 257879 × 2 350441 × 46 375123

$\Phi_1(\nu,2^{25}) =$ 634853 × 309 938316 617551

$\Phi_2(\nu,2^{27}) = \Phi_1^*(\nu,2^{27}) =$ 5 × 49 191317 529865 294097

$\Phi_1(\nu,2^{29}) =$ 7^2 × 107 × 37529 137921 057681

$\Phi_1(\nu, 2^{31}) =$ 13 × 17 749367 × 852750 974689

$\Phi_1(\nu, 2^{33}) =$ 233 × 1231 × 5227 × 131244 759703

$\Phi_2(\nu, 2^{35}) = \Phi_1^*(\nu, 2^{35}) =$ 5 × 14683 × 3350 222537 834243

$\Phi_2(\nu, 2^{37}) = \Phi_1^*(\nu, 2^{37}) =$ 73 × 3 369268 322082 489517

$\Phi_2(\nu, 2^{39}) = \Phi_1^*(\nu, 2^{39}) =$ 7 × 29 × 2293 × 528 394593 740437

$\Phi_1(\nu, 2^{41}) =$ 7 × 37 × 759711 476133 559097

$\Phi_2(\nu, 2^{43}) = \Phi_1^*(\nu, 2^{43}) =$ 5 × 41 × 93 472763 × 12835 698347

$\Phi_1(\nu, 2^{45}) =$ 541 × 13963 × 26 047888 286741

$\Phi_1(\nu, 2^{47}) =$ 7 × 10061 × 2793 891701 436337

$\Phi_2(\nu, 2^{49}) = \Phi_1^*(\nu, 2^{49}) =$ 245 956024 699507 266901 is prime

$\Phi_2(\nu, 2^{51}) = \Phi_1^*(\nu, 2^{51}) =$ 5 × 229 × 8423 × 25 502467 080379

$\Phi_1(\nu, 2^{53}) =$ 7 × 28 110611 045546 184509

$\Phi_2(\nu, 2^{53}) = \Phi_1^*(\nu, 2^{53}) =$ 245 947580 450205 947221 is prime

$\Phi_1(\nu, 2^{55}) =$ 13 × 19 × 796766 392374 848237

$\Phi_1(\nu, 2^{57}) =$ 601 × 327636 248432 020643

$\Phi_1(\nu, 2^{59}) =$ 7 × 1447 × 19482 844394 498171

$\Phi_2(\nu, 2^{61}) = \Phi_1^*(\nu, 2^{61}) =$ 16183 × 75731 × 198808 531457

$\Phi_2(\nu, 2^{63}) = \Phi_1^*(\nu, 2^{63}) =$ 5 × 23 × 41 × 50208 529292 175167

$\Phi_1(\nu, 2^{65}) =$ 7 × 46507 × 717 737600 997047

Theorem 4.1 highlights additional symmetries of the internal structure of the natural numbers. We will not expand upon these symmetries in the current article.

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