The internal structure of natural numbers and the sets of odd numbers that are not primes and even numbers that are not powers of two

Emmanuil Manousos

APM Institute for the Advancement of Physics and Mathematics, 13 Pouliou str., 11 523

Athens, Greece

Abstract

Natural numbers have a strictly defined internal structure that is being revealed in the present article. This structure is inherent of the natural numbers and is not derived through the introduction of any axioms for the set of natural numbers. In the present article, we prove the fundamental theorems that determine this structure. As a consequence of this structure, a mathematical expression for the set of odd numbers that are not primes is derived. Given the set of odd numbers, we can identify the set of prime numbers. Additionally, a new method for expressing odd composite numbers as the product of powers of prime numbers is derived.

1. Introduction

It holds that every product of natural numbers can also be written as a sum. The inverse (i.e. each sum of natural numbers can be written as a product) does not hold when 1 is excluded from the product. This is due to prime numbers p which can be written as a product only in the form of $p = 1 \cdot p$. For this reason, the investigation of natural numbers should be done through their sum and not through their product. Such an investigation is presented in the present article.

We prove that each natural number can be written as a sum of three or more **consecutive** natural numbers except of the powers of 2 and the prime numbers. Each power of 2 and each prime number cannot be written as a sum of three or more consecutive natural numbers. Primes play the same role for odd numbers as the powers of 2 for even numbers, and vice versa.

The present study reveals a strictly defined internal structure of natural numbers. This structure is inherent of the natural numbers and is not derived through the introduction of any axioms for the set of natural numbers $\mathbb{N}$. As a consequence of this structure, a mathematical expression for the set of odd numbers which are not primes is derived.

For practical reasons, we denote $\mathbb{N}$ the set of natural numbers excluding 0, $\mathbb{N} = \{1, 2, 3, ...\}$.

2. THE SEQUENCE $\mu(k,n)$

We consider the sequence of natural numbers

$$\begin{aligned}&\mu(k,n)=k+(k+1)+(k+2)+\ldots+(k+n)=\frac{(n+1)(2k+n)}{2}\\&k\in\mathbb{N}=\{1,2,3,\ldots\}\\&A=\{2,3,4,\ldots\}\end{aligned} \quad . \tag{2.1}$$

For the sequence $\mu(k,n)$ the following theorem holds:

Theorem 2.1. (First theorem for the sequence $\mu(k,n)$)

" For the sequence $\mu(k,n)$ the following hold:

1. $\mu(k,n)\in\mathbb{N}=\{1,2,3,\ldots\}$
2. No element of the sequence is a prime number.
3. No element of the sequence is a power of 2 .
4. The range of the sequence is all natural numbers that are not primes and are not powers of 2 .

Proof.

1. $\mu(k,n)\in\mathbb{N}$ as a sum of natural numbers.

2. $n\in A=\{2,3,4,\ldots\}$ and therefore it holds that

$$\begin{aligned}&n\geq 2\\&n+1\geq 3\end{aligned} .$$

Also we have that

$$2k+n\geq 4$$
$$\frac{2k+n}{2}\geq\frac{3}{2}>1$$

since $k\in\mathbb{N}$ and $n\in A=\{2,3,4,\ldots\}$. Thus, the product

$$\frac{(n+1)(2k+n)}{2}=\mu(k,n)$$

is always a product of two natural numbers different than 1, thus the natural number $\mu(k,n)$ cannot be prime.

3. Let that the natural number $\mu(k,n)=\dfrac{(n+1)(2k+n)}{2}$ is a power of 2 . Then, it exists $\lambda\in\mathbb{N}$ such as

$$\frac{(n+1)(2k+n)}{2} = 2^{\lambda}$$
$$(n+1)(2k+n) = 2^{\lambda+1} \quad . \tag{2.2}$$

Equation (2.2) can hold if and only if there exist $\lambda_1, \lambda_2 \in \mathbb{N}$ such as
$n+1 = 2^{\lambda_1} \wedge 2k+n = 2^{\lambda_2}$

and equivalently

$$\left.\begin{array}{l} n = 2^{\lambda_1} - 1 \\ n = 2^{\lambda_2} - 2k \end{array}\right\} . \tag{2.3}$$

We eliminate n from equations (2.3) and we obtain
$2^{\lambda_1} - 1 = 2^{\lambda_2} - 2k$
and equivalently
$2k - 1 = 2^{\lambda_2} - 2^{\lambda_1}$
which is impossible since the first part of the equation is an odd number and the second part is an even number. Thus, the range of the sequence $\mu(k,n)$ does not include the powers of 2.

4. We now prove that the range of the sequence $\mu(k,n)$ includes all natural numbers that are not primes and are not powers of 2. Let a random natural number N which is not a prime nor a power of 2. Then, N can be written in the form
$N = \chi\psi$
where at least one of the χ, ψ is an odd number ≥ 3. Let χ be an odd number ≥ 3. We will prove that there are always exist $k \in \mathbb{N}$ and $n \in A = \{2,3,4,\ldots\}$ such as
$N = \chi \cdot \psi = \mu(k,n)$.

We consider the following two pairs of k and n:

$$\begin{array}{l} \chi \leq 2\psi + 1, \chi, \psi \in \mathbb{N} \\ k = k_1 = \dfrac{2\psi + 1 - \chi}{2} \\ n = n_1 = \chi - 1 \end{array} \tag{2.4}$$

$$\chi \geq 2\psi - 1, \chi, \psi \in \mathbb{N}$$
$$k = k_2 = \frac{\chi + 1 - 2\psi}{2} \qquad (2.5)$$
$$n = n_2 = 2\psi - 1$$

For every $\chi, \psi \in \mathbb{N}$ it holds either the inequality $\chi \leq 2\psi + 1$ or the inequality $\chi \geq 2\psi + 1$. Thus, for each pair of naturals (χ, ψ), where χ is odd, at least one of the pairs (k_1, n_1), (k_2, n_2) of equations (2.4), (2.5) is defined. We now prove that "when the natural number k_1 of equation (2.4) is $k_1 = 0$ then the natural number k_2 of equation (2.5) is $k_2 = 1$ and additionally it holds that $n_2 > 2$.". For $k_1 = 0$ from equations (2.4) we take

$$\chi = 2\psi + 1$$

and from equations (2.5) we have that

$$k_2 = \frac{(2\psi + 1) + 1 - 2\psi}{2} = 1$$
$$n_2 = 2\psi - 1$$

and because $\psi \geq 2$ we obtain

$$k_2 = 1$$
$$n_2 = 2\psi - 1 \geq 3 > 2.$$

We now prove that when $k_2 = 0$ in equations (2.5), then in equations (2.4) it is $k_1 = 1$ and $n_1 > 2$. For $k_2 = 0$, from equations (2.5) we obtain

$$\chi = 2\psi - 1$$

and from equations (2.4) we get

$$k_1 = \frac{2\psi + 1 - (2\psi - 1)}{2} = 1.$$
$$n_1 = \chi - 1 = 2\psi - 2 \geq 2$$

We now prove that at least one of the k_1 and k_2 is positive. Let

$$k_1 < 0 \wedge k_2 < 0.$$

Then from equations (2.4) and (2.5) we have that

$$2\psi + 1 - \chi < 0 \wedge \chi + 1 - 2\psi < 0. \qquad (2.6)$$

Taking into account that $\chi > 1$ is odd, that is $\chi = 2\rho + 1, \rho \in \mathbb{N}$, we obtain from inequalities (2.6)

$$2\psi+1-(2\rho-1)<0\wedge(2\rho+1)+1-2\psi<0$$
$$2\psi-2\rho<0\wedge 2\rho-2\psi+2>0$$
$$\psi<\rho\wedge\psi>\rho+1$$

which is absurd. Thus, at least one of k_1 and k_2 is positive.

For equations (2.4) we take

$$\mu(k_1,n_1)=\frac{(n_1+1)(2k_1+n_1)}{2}$$
$$=\frac{(\chi-1+1)\left(2\frac{2\psi+1-\chi}{2}+\chi-1\right)}{2}=\frac{\chi(2)\psi}{2}=\chi\psi=N.$$

For equations (2.5) we obtain

$$\mu(k_2,n_2)=\frac{(n_2+1)(2k_2+n_2)}{2}$$
$$=\frac{(2\psi-1+1)\left(2\frac{\chi+1-2\psi}{2}+2\psi-1\right)}{2}=\frac{2\psi\chi}{2}=\chi\psi=N.$$

Thus, there are always exist $k\in\mathbb{N}$ and $n\in A=\{2,3,4,\ldots\}$ such as $N=\chi\psi=\mu(k,n)$ for every N which is not a prime number and is not a power of 2. □

Example 2.1. For the natural number $N=40$ we have

$$N=40=5\cdot 8$$
$$\chi=5$$
$$\psi=8$$

and from equations (2.4) we get

$$k=k_1=\frac{16+1-5}{2}=6$$
$$n=n_1=5-1=4$$

thus, we obtain

$$40=\mu(6,4).$$

Example 2.2. For the natural number $N=51$,

$$N=51=3\cdot 17=17\cdot 3$$

there are two cases. First case:

$$N=51=3\cdot 17$$
$$\chi=3$$
$$\psi=17$$

and from equations (2.4) we obtain

$$k = k_1 = \frac{34+1-3}{2} = 16$$
$$n = n_1 = 3-1 = 2$$
thus,
$$51 = \mu(16,2).$$
Second case:
$$N = 51 = 17\cdot 3$$
$$\chi = 17$$
$$\psi = 3$$
and from equations (2.5) we obtain
$$k = k_2 = \frac{17+1-6}{2} = 6$$
$$n = n_2 = 6-1 = 5$$
thus,
$$51 = \mu(6,5).$$

The second example expresses a general property of the sequence $\mu(k,n)$. The more composite an odd number that is not prime (or an even number that is not a power of 2) is, the more are the $\mu(k,n)$ combinations that generate it.

Example 2.3.
$$135 = 15\cdot 9 = 27\cdot 5 = 9\cdot 15 = 45\cdot 3 = 5\cdot 27 = 3\cdot 45$$
$$135 = \mu(2,14) = \mu(9,9) = \mu(11,8) = \mu(20,5) = \mu(25,4) = \mu(44,2).$$

We now prove the following corollary:

Corollary 2.1. "For the sequence $\mu(k,n), k\in\mathbb{N}, n\in A=\{2,3,4,...\}$ the following hold:

1. $$\left.\begin{matrix}\mu(k,n)=\mu(k_1,n)\\ k_1\in\mathbb{N}\end{matrix}\right\}\Leftrightarrow k_1 = k. \tag{2.7}$$

2. $$\left.\begin{matrix}\mu(k,n)=\mu(k,n_1)\\ n_1\in A=\{2,3,4,...\}\end{matrix}\right\}\Leftrightarrow n_1 = n. \tag{2.8}$$

3. If they exist

$$k = 1,2,3,.....,k_{\max}$$
$$n = 2,3,4,.....,n_{\max}$$

possible natural numbers such as a natural number Φ can be written in the form $\Phi = \mu(k,n)$, then it can be written **at most** with

$$T = T(\Phi) = \min\{k_{\max}, n_{\max}\} \tag{2.9}$$

different ways.''

Proof. 1. From equation (2.1) we get equivalently

$$\begin{aligned}
&\mu(k,n) = \mu(k_1,n) \\
&\frac{(n+1)(2k+n)}{2} = \frac{(n+1)(2k_1+n)}{2} . \\
&2k+n = 2k_1+n \\
&k_1 = k
\end{aligned}$$

The inverse is obvious.

2. From equation (2.1) we get equivalently

$$\mu(k,n) = \mu(k,n_1)$$

$$k+(k+1)+(k+2)+....+(k+n) = k+(k+1)+(k+2)+....+(k+n_1) . \tag{2.10}$$

For $n_1 < n$ from equation (2.10) we get

$$\begin{aligned}
&k+(k+1)+(k+2)+.....+(k+n_1)+(k+n_1+1)+.....+(k+n) \\
&= k+(k+1)+(k+2)+.....+(k+n_1)
\end{aligned}$$

and equivalently we get

$$(k+n_1+1)+(k+n_1+2)+(k+n_1+3).....+(k+n) = 0$$

which is absurd. Similarly, we arrive at absurd for $n_1 > n$. Thus, $n_1 = n$.

3. Based on 1 and 2 of the corollary, in $\mu(k,n) = \Phi$ sequence every k cannot be combined with more than one n. Also, every n cannot be combined with more than one k. Thus, if $k_{\max} < n_{\max}$ the maximum number of possible pairs in $\mu(k,n) = \Phi$ sequences is equal to $k_{\max}$. If $n_{\max} < k_{\max}$ the maximum number of possible pairs is equal to $n_{\max}$. Thus, equation (2.10) holds. □

From Theorem 2.1 the following corollary is derived:

Corollary 2.2. "1. Every natural number which is not a power of 2 and is not a prime can be written as the sum of three or more *consecutive* natural numbers.

2. Every power of 2 and every prime number cannot be written as the sum of three or more *consecutive* natural numbers.”

Proof. Corollary 2.1 is a direct consequence of Theorem 2.1.□

3. The fundamental theorems for sequence $\mu(k,n)$

In this chapter we prove three of the fundamental theorems for sequence $\mu(k,n)$. For the proof of the first of these theorems we first prove the following lemma:

Lemma 3.1 ″For the sets A_2, A_3, A_4, A_5,

$$A_2 = \{x / x = 2 + 4\lambda, \lambda = 0,1,2,...\} \quad (3.1)$$

$$A_3 = \{x / x = 3 + 4\lambda, \lambda = 0,1,2,...\} \quad (3.2)$$

$$A_4 = \{x / x = 4 + 4\lambda, \lambda = 0,1,2,...\} \quad (3.3)$$

$$A_5 = \{x / x = 5 + 4\lambda, \lambda = 0,1,2,...\} \quad (3.4)$$

the following hold:

1. $A_i \cap A_j = \varnothing$ for every $i \neq j, i, j \in \{1,2,3,4\}$. (3.5)

2. $A_2 \cup A_3 \cup A_4 \cup A_5 = A = \{2,3,4,...\}$. ″ (3.6)

Proof. 1. We will prove that it holds

$$A_2 \cap A_4 = \varnothing$$

and the proof is similar for the rest of the pairwise intersections of the sets A_2, A_3, A_4, A_5. Let there exist a common element x between the sets A_2 and A_4, $x \in A_2 \wedge x \in A_4$. Then, there exist $\lambda_1, \lambda_2 \in \{0,1,2,...\}$ such as

$$x = 2 + 4\lambda_1$$
$$x = 4 + 4\lambda_2$$

thus we get

$$2 + 4\lambda_1 = 4 + 4\lambda_2$$
$$4\lambda_1 = 2 + 4\lambda_2$$
$$2\lambda_1 = 1 + 2\lambda_2$$

which is impossible, since the natural number in the first part of the equation is even and in the second part is odd.

2. For every $x \in A_2$ it holds:

$x \in A_2 \Rightarrow x+1 \in A_3 \Rightarrow x+2 \in A_4 \Rightarrow x+3 \in A_5$. Thus, starting from the set A_2 with $x=2$ and $\lambda=0$ and by increasing x continually by 1, the natural number $(x+4\lambda)$ passes successively and repeatedly through all of the sets $A_2, A_3, A_4, A_5, A_2, \ldots$, producing the set $A=\{2,3,4,\ldots\}$. □

We now prove the following theorem:

Theorem 3.1 (Second theorem for the sequence $\mu(k,n)$)

''1. The even numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$,

$$\begin{aligned}
&\alpha_1 = \alpha(\xi,\lambda) = \mu(2\xi+1, 2+4\lambda) = 2(\xi+\lambda+1)(4\lambda+3) \\
&\alpha_2 = \alpha_2(\xi,\lambda) = \mu(2\xi, 3+4\lambda) = 2(4\xi+4\lambda+3)(\lambda+1), \xi \neq 0 \\
&\alpha_3 = \alpha_3(\xi,\lambda) = \mu(2\xi+1, 3+4\lambda) = 2(4\xi+4\lambda+5)(\lambda+1) \\
&\alpha_4 = \alpha_4(\xi,\lambda) = \mu(2\xi, 4+4\lambda) = 2(\lambda+\xi+1)(4\lambda+5), \xi \neq 0 \\
&\xi, \lambda \in \{0,1,2,\ldots\} \\
&\alpha_1(\xi,\lambda) \le \alpha_2(\xi,\lambda) \le \alpha_3(\xi,\lambda) \le \alpha_4(\xi,\lambda) \\
&\alpha_1(\xi,\lambda) = \alpha_2(\xi,\lambda) \Leftrightarrow \xi = 0 \\
&\alpha_2(\xi,\lambda) = \alpha_3(\xi,\lambda) \Leftrightarrow \lambda = 1 \\
&\alpha_3(\xi,\lambda) = \alpha_4(\xi,\lambda) \Leftrightarrow \xi = 0
\end{aligned} \tag{3.7}$$

give, generate, all of the even numbers which are not a power of 2.

2. The odd numbers $\beta_1, \beta_2, \beta_3, \beta_4$,

$$\begin{aligned}
&\beta_1 = \beta_1(\xi,\lambda) = \mu(2\xi, 2+4\lambda) = (2\xi+2\lambda+1)(4\lambda+3), \xi \neq 0 \\
&\beta_2 = \beta_2(\xi,\lambda) = \mu(2\xi+1, 4+4\lambda) = (2\xi+2\lambda+3)(4\lambda+5) \\
&\beta_3 = \beta_3(\xi,\lambda) = \mu(2\xi, 5+4\lambda) = (4\xi+4\lambda+5)(2\lambda+3), \xi \neq 0 \\
&\beta_4 = \beta_4(\xi,\lambda) = \mu(2\xi+1, 5+4\lambda) = (4\xi+4\lambda+7)(2\lambda+3) \\
&\xi, \lambda \in \{0,1,2,\ldots\} \\
&\beta_1(\xi,\lambda) < \beta_2(\xi,\lambda) \le \beta_3(\xi,\lambda) < \beta_4(\xi,\lambda) \\
&\beta_2(\xi,\lambda) = \beta_3(\xi,\lambda) \Leftrightarrow \xi = 0
\end{aligned} \tag{3.8}$$

give, generate, all of the odd numbers which are not primes.''

Proof. We consider the natural numbers N as given from equations

$$\begin{aligned}&N=\mu(2\xi+1,n)\\&\xi\in\{0,1,2,\ldots\}\\&n\in A=\{2,3,4,\ldots\}\end{aligned} \qquad . \qquad \begin{aligned}&N=\mu(2\xi,n)\\&\xi\in\{1,2,3,\ldots\}\\&n\in A=\{2,3,4,\ldots\}\end{aligned} \tag{3.9}$$

The way that we defined the natural numbers N in equations (3.9) allows k,

$$\left.\begin{aligned}&k=2\xi+1\\&\xi\in\{0,1,2,\ldots\}\end{aligned}\right\}\vee\left\{\begin{aligned}&k=2\xi\\&\xi\in\{1,2,3,\ldots\}\end{aligned}\right.$$

to obtain all values $k=1,2,3\ldots$ in $\mu(k,n)$ sequence, and additionally the natural number n takes all values $n=2,3,4,\ldots$. Thus, according to Theorem 2.1, equations (3.9) give all natural numbers which are not primes and are not powers of 2. We now consider the natural numbers α and β as given from equations

$$\begin{aligned}&\alpha=\mu(2\xi+1,n)\\&\xi\in\{0,1,2,\ldots\}\\&n\in A_2\cup A_3\end{aligned} \qquad \begin{aligned}&\alpha=\mu(2\xi,n)\\&\xi\in\{1,2,3,\ldots\}\\&n\in A_3\cup A_4\end{aligned} \tag{3.10}$$

$$\begin{aligned}&\beta=\mu(2\xi+1,n)\\&\xi\in\{0,1,2,\ldots\}\\&n\in A_4\cup A_5\end{aligned} \qquad . \qquad \begin{aligned}&\beta=\mu(2\xi,n)\\&\xi\in\{1,2,3,\ldots\}\\&n\in A_2\cup A_5\end{aligned} \tag{3.11}$$

The natural numbers α of equations (3.10) are even and the natural numbers β of equations (3.11) are odd. Indicatively, we prove that the natural numbers α of the first of equations (3.10) are even. Similarly, we can prove the second of equations (3.10) as well as the equations (3.11).

In the first of equations (3.10) we have that $n \in A_2 \cup A_3$. When $n \in A_2$ then from equations (2.1) and (3.1) we get

$$\alpha = \mu(2\xi+1, 2+4\lambda) = \frac{(2+4\lambda+1)\left[2(2\xi+1)+2+4\lambda\right]}{2}$$
$$= \frac{(4\lambda+3)(4\xi+4\lambda+4)}{2} = 2(4\lambda+3)(\xi+\lambda+1)$$

which is an even number. When $n \in A_3$ then from equations (2.1) and (3.2) we obtain

$$\alpha = \mu(2\xi+1, 3+4\lambda) = \frac{(3+4\lambda+1)\left[2(2\xi+1)+3+4\lambda\right]}{2}$$
$$= \frac{(4\lambda+4)(4\xi+4\lambda+5)}{2} = 2(\lambda+1)(4\xi+4\lambda+5)$$

which is an even. Following the same proof procedure we get

$$\begin{aligned}
&\alpha = \alpha_1(\xi,\lambda) = \mu(2\xi+1, 2+4\lambda) = 2(\xi+\lambda+1)(4\lambda+3)\\
&\alpha = \alpha_3(\xi,\lambda) = \mu(2\xi+1, 3+4\lambda) = 2(4\xi+4\lambda+5)(\lambda+1)\\
&\alpha = \alpha_2(\xi,\lambda) = \mu(2\xi, 3+4\lambda) = 2(4\xi+4\lambda+3)(\lambda+1), \xi \neq 0\\
&\alpha = \alpha_4(\xi,\lambda) = \mu(2\xi, 4+4\lambda) = 2(\lambda+\xi+1)(4\lambda+5), \xi \neq 0\\
&\xi, \lambda \in \{0,1,2,\ldots\}
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
&\beta = \beta_2(\xi,\lambda) = \mu(2\xi+1, 4+4\lambda) = (2\xi+2\lambda+3)(4\lambda+5)\\
&\beta = \beta_4(\xi,\lambda) = \mu(2\xi+1, 5+4\lambda) = (4\xi+4\lambda+7)(2\lambda+3)\\
&\beta = \beta_1(\xi,\lambda) = \mu(2\xi, 2+4\lambda) = (2\xi+2\lambda+1)(4\lambda+3), \xi \neq 0\ .\\
&\beta = \beta_3(\xi,\lambda) = \mu(2\xi, 5+4\lambda) = (4\xi+4\lambda+5)(2\lambda+3), \xi \neq 0\\
&\xi, \lambda \in \{0,1,2,\ldots\}
\end{aligned} \tag{3.13}$$

According to Theorem 2.1 and lemma 3.1 equations (3.12) give, generate, all even numbers that are not powers of 2. Equations (3.13) give, generate all odd numbers that are not primes.

For the natural numbers $\alpha_i, \beta_i, i \in \{1,2,3,4\}$ of equations (3.12) and (3.13) the following hold:

$$\alpha_1(\xi,\lambda) \le \alpha_2(\xi,\lambda) \le \alpha_3(\xi,\lambda) \le \alpha_4(\xi,\lambda)$$
$$\alpha_1(\xi,\lambda) = \alpha_2(\xi,\lambda) \Leftrightarrow \xi = 0$$
$$\alpha_2(\xi,\lambda) = \alpha_3(\xi,\lambda) \Leftrightarrow \lambda = 1$$
$$\alpha_3(\xi,\lambda) = \alpha_4(\xi,\lambda) \Leftrightarrow \xi = 0$$
$$\beta_1(\xi,\lambda) < \beta_2(\xi,\lambda) \le \beta_3(\xi,\lambda) < \beta_4(\xi,\lambda)$$
$$\beta_2(\xi,\lambda) = \beta_3(\xi,\lambda) \Leftrightarrow \xi = 0$$

Indicatively, we prove inequality $\beta_2(\xi,\lambda) \le \beta_3(\xi,\lambda)$:

$$\beta_2(\xi,\lambda) \le \beta_3(\xi,\lambda) \Leftrightarrow$$
$$(2\xi + 2\lambda + 3)(4\lambda + 5) \le (4\xi + 4\lambda + 5)(2\lambda + 3) \Leftrightarrow$$
$$8\xi\lambda + 10\xi + 8\lambda^2 + 10\lambda + 12\lambda + 15 \le 8\xi\lambda + 12\xi + 8\lambda^2 + 12\lambda + 10\lambda + 15 \Leftrightarrow$$
$$0 \le 2\xi$$

which holds. Equality, $\beta_2(\xi,\lambda) = \beta_3(\xi,\lambda)$, holds for $\xi = 0$ ($\xi = 0$ cannot be substituted in $\beta_3(\xi,\lambda)$). Thus, we rewrite equations (3.12), (3.13) by substituting in ascending order the even numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and the odd numbers $\beta_1, \beta_2, \beta_3, \beta_4$ and we obtain equations (3.7) and (3.8).□

We now prove the following theorems:

Theorem 3.2 (Third theorem for the sequence $\mu(k,n)$)

'For the sequence $\mu(k,n)$, $k \in \mathbb{N}, n \in A = \{2,3,4,...\}$ the following equations hold

$$\begin{aligned} &\mu(k,n) = \mu(k,\omega-1) + \mu(k+\omega, n-\omega) \\ &3 \le \omega \le n-2 \\ &\omega \in \mathbb{N} \end{aligned} \tag{3.14}$$

$$\begin{aligned} &\mu(k,n) = \mu(k-\theta, n) + \theta(n+1) \\ &\theta \le k-1 \\ &\theta \in \mathbb{Z} \end{aligned} \tag{3.15}$$

where $\mathbb{Z}$ the set of integers.''

Proof. 1. From equation (2.1) we obtain

$$\mu(k,n) = k + (k+1) + (k+2) + ...(k+n)$$

and since

$$\omega - 1 \ge 2 \wedge n - \omega \ge 2$$

we get

$$3 \le \omega \le n-2$$
$$\omega \in \mathbb{N}$$

thus,

$$\mu(k,n) = \left[k+(k+1)+(k+2)+\ldots+(k+\omega-1)\right]+\left[k+\omega+(k+\omega+1)+\ldots+\left((k+\omega)+(n-\omega)\right)\right]$$

and with equation (2.1) we obtain

$$\mu(k,n) = \mu(k,\omega-1)+\mu(k+\omega,n-\omega).$$

and since

$$k-\theta \ge 1,$$

we get

$$\theta \le k-1, \theta \in \mathbb{Z} \ ,$$

and finally we obtain from equation (2.1)

$$\begin{aligned}\mu(k-\theta,n)+\theta(n+1) &= \frac{(n+1)(2k-2\theta+n)}{2}+\frac{2\theta(n+1)}{2} \\ &= \frac{(n+1)(2k-2\theta+n+2\theta)}{2} = \frac{(n+1)(2k+n)}{2}\end{aligned} . \square$$

Theorem 3.4. (Fourth theorem for the sequence $\mu(k,n)$)

''The sequence $\Phi = \mu(k,n)$, that is every even which is not a power of 2 and every odd which is not a prime, can be written in the form of equations

$$\begin{array}{ll} \Phi = \mu\left(\frac{3n-2k+6}{4}, k+\frac{n}{2}-1\right) & \Phi = \mu\left(\frac{2k-3n-2}{4}, 2n+1\right) \\ \left(\frac{3n-2k+6}{4}, k+\frac{n}{2}-1\right) \in \mathbb{N}\times A & \left(\frac{2k-3n-2}{4}, 2n+1\right) \in \mathbb{N}\times A \\ k \le \frac{3n}{2}+3 & k \ge \frac{3n}{2}+1 \end{array} \tag{3.16}$$

$$\Phi = \mu\left(\frac{8k+3n+1}{4}, \frac{n-1}{2}\right)$$
$$\left(\frac{8k+3n+1}{4}, \frac{n-1}{2}\right) \in \mathbb{N} \times A$$
.''

Proof. From equation (2.1) we obtain

$$\Phi = \left(k + \frac{n}{2}\right)(n+1) = \left(\frac{n}{2} + \frac{1}{2}\right)(2k+n)$$

and next we use equations (2.4) and (2.5).□

We now give the following definition:

Definition. ''We say that the sequence $\mu(k,n), k \in \mathbb{N}, n \in A = \{2,3,4,...\}$ is rearranged if there exist natural numbers $k_1 \in \mathbb{N}, n_1 \in A$, $(k_1,n_1) \neq (k,n)$ such as

$$\mu(k,n) = \mu(k_1,n_1).'' \qquad (3.17)$$

From equation (2.1) written in the form of

$$\mu(k,n) = k + (k+1) + (k+2) + ... + (k+n)$$

two different types of rearrangement are derived: The "compression", during which n decreases with a simultaneous increase of k. The «decompression», during which n increases with a simultaneous decrease of k. Such a rearrangement is given by the equation (3.16) of theorem 3.4.

Example 3.1.

For the pairs of example 2.3 we obtain

a. $135 = \mu(2,14) = \mu(11,8)$ through the first of equations (3.16), ($135 = 15 \cdot 9 = 9 \cdot 15$).

b. $135 = \mu(9,9) = \mu(25,4)$ through the third of equations (3.16), ($135 = 5 \cdot 27 = 27 \cdot 5$).

c. $135 = \mu(20,5) = \mu(44,2)$ through the third of equations (3.16), ($135 = 3 \cdot 45 = 45 \cdot 3$).

We now prove the following corollary:

Corollary 3.1. ''For the natural numbers $\alpha_i, \beta_i, i \in \{1,2,3,4\}$ the following equations hold:

$$\left.\begin{array}{l}
\beta_1(\xi,\lambda)=\beta_4(\xi,\lambda-1)+2\xi \\
\xi,\lambda\in\{1,2,3,\ldots\} \\
\\
\beta_3(\xi,\lambda)=\beta_2(\xi,\lambda)+2\xi \\
\xi\in\{1,2,3,\ldots\} \\
\lambda\in\{0,1,2,\ldots\} \\
\\
\alpha_2(\xi,\lambda)=\alpha_1(\xi,\lambda)+2\xi \\
\xi\in\{1,2,3,\ldots\} \\
\lambda\in\{0,1,2,\ldots\} \\
\\
\alpha_4(\xi,\lambda)=\alpha_3(\xi,\lambda)+2\xi \\
\xi\in\{1,2,3,\ldots\} \\
\lambda\in\{0,1,2,\ldots\} \\
\\
\alpha_1(\xi,\lambda)=\beta_1(\xi,\lambda)+4\lambda+3 \\
\xi\in\{1,2,3,\ldots\} \\
\lambda\in\{0,1,2,\ldots\} \\
\\
\beta_2(\xi,\lambda)=\alpha_4(\xi,\lambda)+4\lambda+5 \\
\xi\in\{1,2,3,\ldots\} \\
\lambda\in\{0,1,2,\ldots\}
\end{array}\right\} \tag{3.18}$$

Proof. We prove the first two of equations (3.18) and in a similar way the rest of equations can be proved. From equations (3.8) and (3.7) we get, respectively, the following equations

$$\begin{array}{l}
\beta_1(\xi,\lambda)=\mu(2\xi,2+4\lambda),\xi\neq 0 \\
\beta_2(\xi,\lambda)=\mu(2\xi+1,4+4\lambda) \\
\beta_3(\xi,\lambda)=\mu(2\xi,5+4\lambda),\xi\neq 0 \\
\beta_4(\xi,\lambda)=\mu(2\xi+1,5+4\lambda) \\
\xi,\lambda\in\{0,1,2,\ldots\}
\end{array} \tag{3.19}$$

$$
\begin{array}{l}
\alpha_1 = \alpha_1(\xi, \lambda) = \mu(2\xi + 1, 2 + 4\lambda) \\
\alpha_2 = \alpha_2(\xi, \lambda) = \mu(2\xi, 3 + 4\lambda), \xi \neq 0 \\
\alpha_3 = \alpha_3(\xi, \lambda) = \mu(2\xi + 1, 3 + 4\lambda) \\
\alpha_4 = \alpha_4(\xi, \lambda) = \mu(2\xi, 4 + 4\lambda), \xi \neq 0 \\
\xi, \lambda \in \{0, 1, 2, ...\}
\end{array} \quad . \tag{3.20}
$$

We now observe that in equation (3.14) when $\mu(k,n) = \beta_j$ and $\mu(k+\omega, n-\omega) = \beta_i$, $i, j \in \{1, 2, 3, ...\}$ then the natural number $\mu(k, \omega - 1)$ is necessarily even. Thus, for $\mu(k,n) = \beta_j$ and $\mu(k+\omega, n-\omega) = \beta_i$, $i, j \in \{1, 2, 3, ...\}$ we have that

$$\mu(k, \omega - 1) = 2\xi, \xi \in \mathbb{N} \tag{3.21}$$

in equation (3.14). From the sum

$$k + (k+1) + (k+2) + ... + (k+n)$$

in equation (2.1) we conclude that, since the natural number $\mu(k, \omega-1)$ is equal to $2\xi, \xi\{1, 2, 3, ...\}$, in equation (3.14) the natural number $k + \omega$ is equal to $2\xi + 1$, that is

$$k + \omega = 2\xi + 1, \xi \in \{1, 2, 3, ...\} \text{ .} \tag{3.22}$$

We solve the system of equations (3.21) and (3.22):

$$
\begin{array}{l}
\mu(k, \omega - 1) = 2\xi \wedge k + \omega = 2\xi + 1, \xi \in \{1, 2, 3, ...\} \\
\dfrac{\omega(2k + \omega - 1)}{2} = 2\xi \wedge k = 2\xi + 1 - \omega \geq 0, \xi \in \{1, 2, 3, ...\} \\
\omega(2k + \omega - 1) = 4\xi \wedge k = 2\xi + 1 - \omega, \omega \leq 2\xi + 1, \xi \in \{1, 2, 3, ...\} \\
\omega\left[2(2\xi + 1 - \omega) + \omega - 1\right] = 4\xi \wedge k = 2\xi + 1 - \omega, \omega \leq 2\xi + 1, \xi \in \{1, 2, 3, ...\} \\
\omega(-\omega + 4\xi + 1) - 4\xi = 0 \wedge k = 2\xi + 1 - \omega, \omega \leq 2\xi + 1, \xi \in \{1, 2, 3, ...\} \\
-\omega^2 + (4\xi + 1)\omega - 4\xi = 0 \wedge k = 2\xi + 1 - \omega, \omega \leq 2\xi + 1, \xi \in \{1, 2, 3, ...\}
\end{array}
$$

$$\omega = 1 \wedge k = 2\xi, \xi \in \{1, 2, 3, ...\}$$

and from equation (3.14) we obtain

$$
\begin{array}{l}
\mu(2\xi, n) = 2\xi + \mu(2\xi + 1, n - 1) \\
\xi \in \{1, 2, 3, ...\} \\
n \in A = \{2, 3, 4, ...\}
\end{array} \quad . \tag{3.23}
$$

From equations (3.19) we conclude that the natural number $\mu(2\xi,n)$ in equation (3.23) can only be $\beta_1(\xi,\lambda)$ or $\beta_3(\xi,\lambda)$. Thus, there are two cases:

For $\mu(2\xi,n)=\beta_1$ from the first of equations (3.19) we get $n=2+4\lambda$ and from equation (3.23) we obtain

$$\mu(2\xi,2+4\lambda)=2\xi+\mu(2\xi+1,1+4\lambda)=2\xi+\mu(2\xi+1,5+4(\lambda-1)),\lambda-1\geq 0$$

and with the first and fourth of equations (3.19) we obtain

$$\beta_1(\xi,\lambda)=2\xi+\beta_4(\xi,\lambda-1),\lambda\geq 1$$

which is the first of equations (3.18). For $\mu(2\xi,n)=\beta_3$ from the third of equations (3.19) we get $n=5+4\lambda$ and from equations (3.23) we obtain

$$\mu(2\xi,5+4\lambda)=2\xi+\mu(2\xi+1,4+4\lambda)$$

and with the second and third of equations (3.19) we obtain

$$\beta_3(\xi,\lambda)=2\xi+\beta_2(\xi,\lambda)$$

which is the second of equations (3.18). Equations (3.18) give all of the possible cases that are derived from the combination of equations (3.23) and (3.19), (3.20).□

The first four of the equations (3.18) are independent of the natural number λ, while the fifth and six are independent of the natural number ξ. A similar corollary is also derived from Theorem 3.1:

Corollary 3.2. "For natural numbers $\alpha_i,\beta_i,i\in(1,2,3,4\}$ the following equations hold

$$\begin{aligned}
&\alpha_1(\xi,\lambda)=\alpha_1(\xi-1,\lambda)+6+8\lambda\\
&\alpha_2(\xi,\lambda)=\alpha_2(\xi-1,\lambda)+8+8\lambda\\
&\alpha_3(\xi,\lambda)=\alpha_3(\xi-\iota,\lambda)+8+8\lambda\\
&\alpha_4(\xi,\lambda)=\alpha_4(\xi-1,\lambda)+10+8\lambda\\
&\xi\in\{1,2,3,...\}\\
&\lambda\in\{0,1,2,...\}
\end{aligned}\tag{3.24}$$

$$\begin{aligned}&\beta_1(\xi,\lambda)=\beta_1(\xi-1,\lambda)+6+8\lambda\\&\beta_2(\xi,\lambda)=\beta_2(\xi-1,\lambda)+10+8\lambda\\&\beta_3(\xi,\lambda)=\beta_3(\xi-1,\lambda)+12+8\lambda\\&\beta_4(\xi,\lambda)=\beta_4(\xi-1,\lambda)+12+8\lambda\\&\xi\in\{1,2,3,\ldots\}\\&\lambda\in\{0,1,2,\ldots\}\end{aligned}." \qquad (3.25)$$

Proof. We conduct the calculations in equations (3.24) and (3.25) taking into account equations (3.7) and (3.8).□

Equations (3.24) and (3.25) are independent of the natural number ξ.

According to equation (3.17) the sequence $\mu(k,n), k\in\mathbb{N}, n\in A=\{2,3,4,\ldots\}$ is rearranged if there are exist natural numbers $k_1\in\mathbb{N}, n_1\in A$, $(k_1,n_1)\neq(k,n)$ such as the following equation holds

$$\mu(k,n)=\mu(k_1,n_1). \qquad (3.26)$$

The following corollary provides the criterion for the rearrangement of the sequence $\mu(k,n)$.

Corollary 3.3. "1. The sequence $\mu(k,n)$, $(k,n)\in\mathbb{N}\times A$ can be compressed if and only if there exist $\varphi,\omega\in\mathbb{N}, \omega\le n-2$ which verify the equation

$$\begin{aligned}&\omega^2-(2k+2n+2\varphi+1)\omega+2(n+1)\varphi=0\\&\varphi,\omega\in\mathbb{N}\\&\omega\le n-2\end{aligned}. \qquad (3.27)$$

2. The sequence $\mu(k,n)$, $(k,n)\in\mathbb{N}\times A$ can be decompressed if and only if there exist $\rho,\sigma\in\mathbb{N}, \rho\le k$ which verify the equation

$$\begin{aligned}&\sigma^2+(2k+2n-2\rho+1)\sigma-2(n+1)\rho=0\\&\rho,\sigma\in\mathbb{N}\\&\rho\le k\end{aligned}. \qquad (3.28)$$

3. The odd number $\Pi\neq1$ is prime if and only if the sequence

$$\begin{aligned}&\mu(k,n)=\Pi\cdot2^\nu\\&\nu,k\in\mathbb{N}, n\in A\end{aligned} \qquad (3.29)$$

cannot be rearranged.

4. The odd Π is prime if and only if the sequence

$$\mu\left(\frac{\Pi+1}{2},\Pi-1\right)=\Pi^2 \tag{3.30}$$

cannot be rearranged.''

Proof. 1,2. We prove part 1 of the corollary and similarly number 2 can also be proven. From equation (3.26) we conclude that the sequence $\mu(k,n)$ can be compressed if and only if there exist $\varphi,\omega\in\mathbb{N}$ such as

$$\mu(k,n)=\mu(k+\varphi,n-\omega).$$

In this equation the natural number $n-\omega$ belongs to the set $A=\{2,3,4,...\}$ and thus $n-\omega\geq 2\Leftrightarrow\omega\leq n-2$. Next, from equations (2.1) we obtain

$$\mu(k,n)=\mu(k+\varphi,n-\omega)$$

$$\frac{(n+1)(2k+n)}{2}=\frac{(n-\omega+1)\left[2(k+\varphi)+n-\omega\right]}{2}$$

and after the calculations we get equation (3.27).

3. The sequence (3.29) is derived from equations (2.4) or (2.5) for $\chi=\Pi$ and $\psi=2^{\nu}$. Thus, in the product $\chi\psi$ the only odd number is Π. If the sequence $\mu(k,n)$ in equation (3.29) cannot be rearranged then the odd number Π has no divisors. Thus, Π is prime. Obviously, the inverse also holds.

4. First, we prove equations (3.30). From equation (2.1) we obtain:

$$\mu\left(\frac{\Pi+1}{2},\Pi-1\right)=\frac{(\Pi-1+1)\left(2\frac{\Pi+1}{2}+\Pi-1\right)}{2}=\Pi^2.$$

In case that the odd number Π is prime in equations (2.4), (2.5) the natural numbers χ,ψ are unique $\chi=\Pi\wedge\psi=\Pi$, and from equation (2.15) we get $k=\frac{\Pi+1}{2}\wedge n=\Pi-1$. Thus, the sequence $\mu(k,n)=\mu\left(\frac{\Pi+1}{2},\Pi-1\right)$ cannot be rearranged. Conversely, if the sequence $\mu\left(\frac{\Pi+1}{2},\Pi-1\right)=\Pi^2=\Pi\cdot\Pi$ cannot be rearranged the odd number Π cannot be composite and thus Π is prime. □

Observing equations (3.27) and (3.28) we conclude that they exchange roles in the transformation

$$(\varphi,\omega)\leftrightarrow(-\rho,-\sigma). \tag{3.31}$$

Transformation (3.31) is the pivotal characteristics of the rearrangement. Compression and decompression are two inverse processes between two "conditions" of the same natural number $\mu(k,n)$.

Equations (3.7) give the even numbers which are not powers of 2 using four mathematical expression $\alpha_1,\alpha_2,\alpha_3,\alpha_4$. We will now prove that all even numbers that are not powers of 2 can be expressed through two mathematical expressions. We now prove the following corollary:

Corollary 3.4. ''1. Each even number which is not a power of 2 can be written in a unique way either in the form of

$$\delta_1 = \Pi\cdot 2^{\nu} = \mu\left(\frac{2^{\nu+1}+1-\Pi}{2},\Pi-1\right) = \mu\left(2^{\nu}-\frac{\Pi-1}{2},\Pi-1\right) \tag{3.32}$$

$$\Pi < 2^{\nu+1}+1$$

or in the form of

$$\delta_2 = \Pi\cdot 2^{\nu} = \mu\left(\frac{\Pi+1-2^{\nu+1}}{2},2^{\nu+1}-1\right) = \mu\left(\frac{\Pi+1}{2}-2^{\nu},2^{\nu+1}-1\right) \tag{3.33}$$

$$\Pi+1 > 2^{\nu+1}$$

where $\nu\in\mathbb{N}$ and $\Pi\neq 1$ is odd.

2. Even numbers in the form of δ_1 have the minimum value of k and the maximum value of n among all possible rearrangements of the natural number $\delta_1 = \mu(k,n)$. Even numbers in the form of δ_2 have the maximum value of k and the minimum value of n among all possible rearrangements of the natural number $\delta_2 = \mu(k,n)$.

3. If the even number δ_1 cannot be compressed then the odd number Π is prime. If the even number δ_2 cannot be decompressed then Π is prime.''

Proof . 1. By consecutively dividing with 2 an even number δ which is not a power of 2 we bring it in the form of

$$\delta = \Pi\cdot 2^{\nu} \tag{3.34}$$

where $\nu\in\mathbb{N}$ and $\Pi\neq 1$ is odd.

If we assume that the same even number δ is written in the form of $\delta = \Pi_1\cdot 2^{\nu_1} = \Pi_2\cdot 2^{\nu_2}, \nu_1,\nu_2\in\mathbb{N}, \nu_2>\nu_1$ we take $\Pi_1 = \Pi_2\cdot 2^{\nu_2-\nu_1}$ which is impossible, since the first part of the equations is odd and the second is even. Thus, every even number which is not a power of 2 can be written in a unique way in the form of equations (3.34).

From equation (3.34) it is derived that

$$\chi = \Pi \text{ and } \psi = 2^{\nu} \qquad 3.35)$$

in equations (2.4) and (2.5). For

$$\chi < 2\psi + 1 \Leftrightarrow \Pi < 2 \cdot 2^{\nu} + 1$$

we obtain from equation (2.4) equation (3.32). For

$$\chi > 2\psi - 1 \Leftrightarrow \Pi > 2 \cdot 2^{\nu} - 1$$

we get from equation (2.5) equation (3.33).

2. If the odd number Π is prime, from corollary 3.3 it follows that the even numbers δ_1 and δ_2 cannot be rearranged. If the odd number Π is composite then it can be written as the product of prime numbers p_i,

$$\begin{gathered} \Pi = p_1 p_2 p_3 \ldots\ldots p_s \\ s \in \mathbb{N} \end{gathered}. \qquad (3.36)$$

In that case, from equations (2.4) και (2.5) we obtain

$$\begin{gathered} \chi = p_1 p_2 p_3 \ldots p_l \wedge \psi = p_{l+1} p_{l+2} p_{l+3} \ldots p_s \\ 1 \le l < s, l \in \mathbb{N} \end{gathered} \qquad (3.37)$$

and by rearranging the primes $p_i, i = 1,2,3,\ldots\ldots,s$ in the product of equations (3.36) different χ and ψ in equation (3.37) are derived. The maximum number of χ derived from equation (3.36) is $\chi = \Pi$. Thus, in equation (3.32), $n = \Pi - 1$ has the maximum value of n, where for the specific natural number in the form of $\mu(k,n)$ is equivalent to the minimum value of k, as derived from equation (2.1) written in the form of $\mu(k,n) = k + (k+1) + (k+2) + \ldots\ldots + (k+n)$. Similarly, in equation (3.33), n has the minimum possible value for the natural number δ_2, $n = 2^{\nu+1} - 1$ and consequently k has the maximum possible value.

3. Part 3 of corollary 3.4 is derived from Theorem 3.3, taking into account that the even number δ_1 cannot be decompressed and the even number δ_2 cannot be compressed.□

Corollary 3.4 gives every even number δ which is not a power of 2 either in its more “extended” form (equation (3.32)) or in its more “condensed” form (equation (3.33)). We now prove the following corollary:

Corollary 3.5 “Every odd number $\Pi \neq 1$ can be written through at least one of the following mathematical expressions:

$$\Pi = \delta_1 \pm 1 \tag{3.38}$$

$$\Pi = \delta_2 \pm 1 .\text{''} \tag{3.39}$$

Proof. If the odd number Π follows a power of 2 then it precedes an even number δ and thus it can be written in the form of $\Pi = \delta_1 - 1 \vee \Pi = \delta_2 - 1$. If the odd number Π precedes a power of 2, then it follows an even number δ and thus it can be written in the form of $\Pi = \delta_1 + 1 \vee \Pi = \delta_2 + 1$. In all other cases, the odd number Π is between two even numbers in the form of δ and thus it can be written with at least one of the forms of equations (3.38), (3.39). □

The study presented in chapters 2 and 3 reveals the internal structure of the natural numbers. The volume of information derived from this structure is extremely large. In the present article, we will present only two of the most important applications, which are a direct consequence of this structure: A method of calculating the set of prime numbers and a method of expressing a composite odd number as a product of powers of prime numbers.

4. TWO METHODS OF CALCULATING THE SET OF PRIME NUMBERS.

An initial method of calculating the set of prime numbers is derived from Theorem 2.1 and corollary 2.1. Every natural number Φ in the form of $\Phi = \mu(k,n)$, that is every natural number which is not a power of 2 and is not a prime can be written in the form of equation (2.1):

$$\begin{gathered} \Phi = \frac{(n+1)(2k+n)}{2} \\ k \in \mathbb{N}, n \in A = \{2,3,4,...\} \end{gathered} \tag{4.1}$$

and solving for k we obtain equation

$$k = \frac{\Phi}{n+1} - \frac{n}{2} . \tag{4.2}$$

In equation (4.2) the natural number k belongs to the set $\mathbb{N}$, $k \in \mathbb{N}$. Thus,

$$k \geq 1$$

and equivalently we get

$$\frac{\Phi}{n+1} - \frac{n}{2} \geq 1$$

and equivalently

$$n^2 + 3n + 2 - 2\Phi \leq 0 . \tag{4.3}$$

From inequality (4.3) and taking into account that $n \in A = \{2,3,4,...\} \Leftrightarrow n \geq 2$ we obtain inequality

$$2 \leq n \leq \frac{\sqrt{8\Phi+1}-3}{2}. \tag{4.4}$$

From inequality (4.4) we obtain for the maximum value $n_{\max}$ of n,

$$n_{\max} = \left[\frac{\sqrt{8\Phi+1}-3}{2}\right] \leq \frac{\sqrt{8\Phi+1}-3}{2} \tag{4.5}$$

where $[x]$ the integer part of $x \in \mathbb{R}$.

From equation (4.1) solving for n we get

$$n^2 + (2k+1)n + 2k - 2\Phi = 0$$

and equivalently (since $n \in \mathbb{N}, n \geq 2$) we obtain

$$n = \frac{\sqrt{(2k+1)^2 + 8\Phi} - (2k+1)}{2}. \tag{4.6}$$

Taking into account that $n \geq 2$ from equation (4.6) we obtain

$$\frac{\sqrt{(2k-1)^2 + 8\Phi} - (2k+1)}{2} \geq 2$$

and after the calculations we get

$$k \leq \frac{\Phi - 3}{3}. \tag{4.7}$$

From inequality (4.7) we get for the maximum value $k_{\max}$ of k

$$k_{\max} = \left[\frac{\Phi-3}{3}\right]. \tag{4.8}$$

Easily, it can be proved that for every $\Phi \in \mathbb{N}$ it holds that

$$\frac{\sqrt{8\Phi+1}-3}{2} < \frac{\Phi-3}{3}$$

and thus it is

$$\left[\frac{\sqrt{8\Phi+1}-3}{2}\right]<\left[\frac{\Phi-3}{3}\right]$$

and thus from equations (4.5) and (4.8) we get

$$n_{\max}<k_{\max}. \tag{4.9}$$

From inequality (4.9) and corollary 2.1 we arrive at the conclusion that the number of pairs (k,n) in the sequence $\mu(k,n)=\Phi$ is at most $n_{\max}$.

If a natural number Φ is not a power of 2 and is not a prime, then there exist at least one pair

$$(k,n), k\in\mathbb{N}, n\in\left\{2,3,4,.....,\left[\frac{\sqrt{8\Phi+1}-3}{2}\right]\leq\frac{\sqrt{8\Phi+1}-3}{2}\right\}$$

which satisfies equation (4.2). By conducting all of the trials, which are $n_{\max}$, we obtain all of the pairs (k,n) of the sequence $\mu(k,n)=\Phi$.

In no natural number n exists,

$$n\in\left\{2,3,4,.....,\left[\frac{\sqrt{8\Phi+1}-3}{2}\right]\leq\frac{\sqrt{8\Phi+1}-3}{2}\right\}$$

for which the number $k=\dfrac{\Phi}{n+1}-\dfrac{n}{2}$ belongs to the set $\mathbb{N}$, then the natural number Φ cannot be of the form $\Phi=\mu(k,n)$. Thus, in that case, the natural number Φ is either a power of 2 or prime.

According to equation (4.5) in order to examine if an odd number Π is prime at most $n_{\max}=\left[\dfrac{\sqrt{8\Pi+1}-3}{2}\right]\sim\sqrt{2\Pi}$ trials in equation (4.2) are required; which is the same number as the natural numbers n that we have to test in equation (4.2) in order to examine if the derived number k belongs to the set of $\mathbb{N}$.

If the odd number Π is prime, all of the trials will have to be conducted. It is easily proven that the number of these trials is of the same or greater order of magnitude with the calculations (i.e. divisions) required using the standard method [1-8] when the odd number Π is prime. Thus, the aforementioned method, just like the standard one, has the same limitations in its application: The large number of calculations that have to be performed in order to define large primes.

A question posed is whether the internal structure of natural numbers allows us to define the set of prime numbers, overcoming the aforementioned limitations. Theorem 3.1 provides an answer to this question: Equations (3.8) generate all of the odd numbers which are not primes. For $\xi \to +\infty$ and $\lambda \to +\infty$ in equations (3.8) the set of odd numbers C is derived, $C \subset \mathbb{N}$; this set contains all of the odd numbers which are not primes (above 9). Thus we can define the set of prime numbers P_C through the empty positions of the odd numbers of the set C. Giving such a command to a computer, and not commands for calculating primes, we can define the set of prime numbers P_C. After an initial calculation, sets C and P_C will continuously expand over time.

Theorem 3.1 also applies to the statistical definition of individual large primes. Giving appropriate values to the natural numbers ξ, λ in equations (3.8) we can define and then remove a set of odd composite numbers from every subset D, $D \subset \mathbb{N}$, of the set $\mathbb{N}$. In this way we increase the density of the possible prime numbers in the set D. Moreover, all of the odd numbers which are included in every open interval in the form of $(\beta_i, \beta_j), i, j \in \{1,2,3,4\}, \beta_i < \beta_j$ are primes when the odd numbers β_i, β_j are *consecutive*.

5. A METHOD OF EXPRESSING AN ODD COMPOSITE NUMBER AS A PRODUCT OF POWERS OF PRIME NUMBERS

Equations (3.8) provide a method of expressing an odd composite number as a product of powers of prime numbers. In order to apply this method, which is presented next, it is necessary that we know the sets $\Omega_{1C}, \Omega_{2C}, \Omega_{3C} \subseteq P_C$ of the prime numbers p in the form of

$$\begin{aligned} &\Omega_{1C} = \{ p / p = 2\lambda + 3, \lambda = 0,1,2,..., p = prime \} = P_C \\ &\Omega_{2C} = \{ p / p = 4\lambda + 3, \lambda = 0,1,2,..., p = prime \} \subset P_C \\ &\Omega_{3C} = \{ p / p = 4\lambda + 5, \lambda = 0,1,2,..., p = prime \} \subset P_C \end{aligned} \quad . \qquad (5.1)$$

Let the composite odd number Π, $\Pi \geq 9$. The odd number Π has at least one of the mathematical expressions of the equations (3.8):

$$\begin{aligned} &\Pi = \beta_1 = (2\xi + 2\lambda + 1)(4\lambda + 3), (\xi, \lambda) \neq (0,0) \\ &\Pi = \beta_2 = (2\xi + 2\lambda + 3)(4\lambda + 5) \\ &\Pi = \beta_3 = (4\xi + 4\lambda + 5)(2\lambda + 3) \\ &\Pi = \beta_4 = (4\xi + 4\lambda + 7)(2\lambda + 3) \\ &\xi, \lambda \in \{0,1,2,...\} \end{aligned} \quad . \qquad (5.2)$$

The composite odd number Π is always written as a product of powers of prime numbers. In combination with equations (2.4), (2.5) we conclude that there always exist $\lambda \in \{0,1,2,...\}$ such as the odd numbers $(4\lambda+3)$ or $(4\lambda+5)$ or $(2\lambda+3)$ in equations (5.2) are prime numbers. These factors are only dependent on $\lambda \in \{0,1,2,...\}$ and can be defined for the odd number Π, since we know the sets $\Omega_{1C}, \Omega_{2C}, \Omega_{3C} \subseteq P_C$. Repeating the procedure for the odd numbers

$$\frac{\Pi}{4\lambda+3} \vee \frac{\Pi}{4\lambda+5} \vee \frac{\Pi}{2\lambda+3}$$

we finally obtain the odd number Π as a product of powers of prime numbers.

In the third and fourth of equations (5.2) the following inequalities hold

$$\begin{aligned} (2\lambda+3) &< (4\xi+4\lambda+5) \\ (2\lambda+3) &< (4\xi+4\lambda+7) \end{aligned} \tag{5.3}$$

thus inequality

$$(2\lambda+3) < \sqrt{\Pi} \tag{5.4}$$

also holds.

The respective inequalities

$$\begin{aligned} (4\lambda+3) &< (2\xi+2\lambda+1) \\ (4\lambda+5) &< (2\xi+2\lambda+3) \end{aligned} \tag{5.5}$$

$$(4\lambda+3) < (4\lambda+5) < \sqrt{\Pi} \tag{5.6}$$

for the first and the second of equations (5.2) hold in case that

$$\lambda < \xi + 1. \tag{5.7}$$

The aforementioned inequalities decrease the possible values of $\xi, \lambda \in \{0,1,2,...\}$ for the odd number Π. From equations (5.2) a further investigation for the factorization of the odd number Π is derived.

The previously described method is completely different from the currently used methods [9-11] for the factorization of an odd composite number. The calculation of the set C and, through it, of the sets P_C and $\Omega_{1C}, \Omega_{2C}, \Omega_{3C} \subseteq P_C$ is necessary for applications of the Number Theory in multiple fields.

6. Equation $\mu(\chi, \psi)$

We consider equation $\mu(\chi,\psi)$,

$$\mu:(\chi,\psi)\in\mathbb{C}\times\mathbb{C}\to\mu(\chi,\psi)=\frac{(\psi+1)(2\chi+\psi)}{2}\in\mathbb{C} \tag{6.1}$$

where $\mathbb{C}$ is the set of complex numbers.

We have proven that equations (3.19) give all odd numbers which are not primes and equations (3.20) all even numbers which are not powers of 2. In these equations we have predefined the domain of the sequence $\mu(k,n)$, since

$$\begin{aligned}&(k,n)\in\mathbb{N}\times A\\&A=\{2,3,4,\ldots\}\end{aligned}.$$

However, the general expression of the domain of the $\mu(k,n)$ is given by equation

$$\mathbb{Z}\times\mathbb{Z}\subset\mathbb{C}\times\mathbb{C} \tag{6.2}$$

where $\mathbb{Z}$ is the set of integers.

With an appropriate domain, the sequence $\mu(k,n)$ is also defined for negative values of k and n. For example, it is easily proven, through conducting the calculations of equations (6.1), that the following equations hold

$$\begin{aligned}&\mu(k+n+1,-n-2)=\mu(k+n+1,-2k-n-1)\\&=\mu(-k-n,n)=\mu(-k-n,2k+n-1)\\&=-\mu(k,n)\\&(k,n)\in\mathbb{N}\times A\\&A=\{2,3,4,\ldots\}\end{aligned}. \tag{6.3}$$

Taking as the domain of function $\mu(\chi,\psi)$ the set

$$\mathbb{Q}\times\mathbb{Z}\subset\mathbb{C}\times\mathbb{C}$$

where $\mathbb{Q}$ the set of rational numbers it is easily proven, after the performing the calculations through equation (6.1),), that the following equations hold

$$\begin{aligned}&\mu\left(\frac{\nu+2}{2},\nu\right)=(\nu+1)^2\\&\mu\left(\frac{\nu}{2},\nu-2\right)=(\nu-1)^2.\\&\nu\in\mathbb{Z}\end{aligned} \tag{6.4}$$

The domain's choice depends on the problem that we wish to solve using equations $\mu(\chi,\psi)$. For domains in the form of $\mathbb{C}\times\mathbb{C}$ or $\mathbb{R}\times\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, we can apply the methods of Calculus, in order to derive conclusions for the sequence $\mu(k,n)$. By writing equation (4.2) in the forms

$$\left.\begin{array}{l} k(x)=\dfrac{\Phi}{x+1}-\dfrac{x}{2}\geq 1 \\ x\in\mathbb{R}, x\geq 2 \\ \Phi\in\mathbb{N} \end{array}\right\} \qquad \left.\begin{array}{l} x=\dfrac{\Phi}{n(x)+1}-\dfrac{n(x)}{2}\geq 1 \\ x\in\mathbb{R}, n(x)\geq 2 \\ \Phi\in\mathbb{N} \end{array}\right\} \tag{6.5}$$

we can easily prove the inequalities of Chapter 4 using derivatives. In a similar manner, we can prove number 2 of Corollary 3.4.

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