An Ideal MHD as a System of Coupled Quaternioni Riccati Equations for MHD Power Generator, and Outline of Their Possible Solutions

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Abstract— **In recent years, there are several proposals of using MHD theory for clean power generators on top of coal plant. But the theory involved appears too complicated, so in this paper we will use a simpler approach using ideal MHD equations which then they can be reduced to a system of coupled quaternionic Riccati equations. Further numerical and experimental investigations are advisable.**

Keywords*— *ideal MHD, coupled Riccati equations, quaternionic algebra, PDE, Maxwell electromagnetic theory.

I. INTRODUCTION

Modern society requires a variety of goods and services which require energy as the diversity of range of services increases so is the demand for energy. Electrical energy because of its versatility takes major share. Coal has to be transported to thermal stations located away from coalfields by railways and power has to be transmitted over large distances from pithead stations. These problems can be eliminated or reduced by converting coal into SNG (synthetic natural gas) at pithead and transporting the gas by pipe-grid to all thermal stations. The efficiency of power station can be increased by adopting combined cycle. Eco-friendly MHD (Thermal Cell) generator is now suggested for development as a topping addition for combined cycle to further improve the efficiency.[1]

Magneto-hydrodynamic (MHD) power generation has been studied as a novel commercial power plant due to its inherent advantage of high-efficiency with high-working temperatures.

MHD equations are a set of equations to extend Maxwell equations with fluid dynamics. Analytical solution of MHD problem is difficult but remains possible to obtain, and such a solution will be discussed in section 2. In section 3, we discuss how ideal MHD reduces to a system of coupled quaternionic Riccati equations. In the last section, we will discuss an outline of possible solutions of such Quaternionic Linear Differential Equations.

II. MHD EQUATIONS AND THEIR ANALYTICAL SOLUTION

This section presents an analytical solution obtained by Ligere et al. [3] of the MHD problem on a fully developed flow of a conducting fluid in a duct with the rectangular cross-section, located in a uniform external magnetic field, and under a slip boundary condition on side walls of the duct. The flow is driven by a constant pressure gradient. The case of perfectly conducting Hartmann walls and insulating side walls is considered. The solution is derived by using integral transforms.

The dimensionless MHD equations, describing the problem, have the form:[3]

$$\frac{\partial^2 U}{\partial z^2}+\frac{\partial^2 U}{\partial y^2}+1+Ha\frac{\partial b_x}{\partial z}=0, \quad (1)$$

$$\frac{\partial^2 b_x}{\partial z^2}+\frac{\partial^2 b_x}{\partial y^2}+1+Ha\frac{\partial U}{\partial z}=0, \quad (2)$$

Where:

$Ha = B_0 h\sqrt{\sigma/\rho\nu}$ is the Hartmann number, which characterizes the ratio of electromagnetic force to viscous force; σ, ρ, ν are the conductivity, the density and the viscosity of the fluid, respectively.

After some steps and integral transforms, the solution is given by:[3]

$$\bar{u}(\lambda_n, y) = i\left(A\cosh(k_1 y) - B\cosh(k_2 y)\right) - \frac{2(-1)^{n+1}Ha}{\lambda_n(\lambda_n^2 + Ha^2)},$$

where the coefficients A and B are determined from the boundary conditions.

III. IDEAL MHD REDUCES TO A SYSTEM OF COUPLED RICCATI EQUATIONS

Quaternions are 4-vectors whose multiplication rules are governed by a simple non-commutative division algebra. The concept was originally invented by Hamilton to generalize complex numbers to R^4.[4]

Gibbon et al. have shown that the equations of ideal incompressible MHD have a quaternionic structure in Elsasser variables but in a non-degenerate form. The equations of ideal incompressible MHD couple the inviscid fluid to a magnetic field B:[4]

$$\frac{Du}{Dt} = B.\nabla B - \nabla p, \quad (4)$$

$$\frac{DB}{Dt} = B\cdot\nabla u, \quad (5)$$

Together with div u=0 and div B=0. It can be shown that the above system of differential equations can be reduced to a system of coupled quaternionic Riccati equations. The 4-vectors $\zeta^{\pm}$ satisfy [4]:

$$\frac{D^{\mp}\zeta^{\pm}}{Dt}+\zeta^{\pm}\otimes\zeta^{\mp}+\zeta_p^{(m)}=0,$$

(6)

subject to the Poisson relation.[4]

It is worth remarking here that the above system of coupled quaternionic Riccati equations are very difficult to solve analytically, but considering our paper elsewhere suggesting that a system of coupled Riccati equations can be solved numerically using computer algebra packages such as Mathematica [6] Therefore, we are optimistic that it is also possible to find numerical solution of equation (6), although it will be much more complicated than what we obtained in [6]. In the next section, we will discuss an outline of possible solutions of such Linear Quaternionic Differential Equations.

IV. Outline of Solution of LQDE

In this section we will give an outline of possible route to solve equations (6). First we split (6) into two coupled QDE equations:

$$\frac{D^{+}\zeta +}{Dt}+\zeta^{+}\otimes\zeta^{+}++\zeta_p^{(m)}=0,$$

(7)

$$\frac{D^{-}\zeta^{-}}{Dt}+\zeta^{-}\otimes\zeta^{-}+\zeta_p^{(m)}=0,$$

(8)

Where those two equations are coupled and should be solved simultaneously.

First, we shall mention here that there are only two papers that we can find so far which discuss how to solve LQDE, and we do not find yet algorithm to put these methods into a working numerical solutions (to be computed with computer algebra package such as Mathematica.)

a. The LQDE can be put into a general form as follows:[9, p. 6]

$$p(X):=A+BX+XC+DX^2E$$

(9)

Then we can use the following theorem to find its Jacobian matrix:

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Theorem 4.3. *Let p be given as in (4.2). Then the Jacobi matrix, $\mathbf{J}$, of p is*

$$\mathbf{J}(x)=i_3(b,c)+i_3(d,e)+i_3(x,1)+i_3(1,x), \quad (4.7)$$

where i_3 is defined in (1.5).

Proof. The Jacobi matrix is the matrix which defines the linear mapping given by $p(x+h)$ with respect to h. Now

$$\begin{aligned} p(x+h) &= a+b(x+h)c+d(x+h)e+(x+h)(x+h) \\ &= a+bxc+bhc+dxe+dhe+x^2+xh+hx+h^2. \end{aligned}$$

The part which is linear in h is

$$L(h):=bhc+dhe+xh1+1hx,$$

where 1 stands for the quaternion $(1,0,0,0)$. An application of (1.6) to L yields

$$\mathrm{col}(L(h))=(\mathrm{i}_3(b,c)+\mathrm{i}_3(d,e)+\mathrm{i}_3(x,1)+\mathrm{i}_3(1,x))\,\mathrm{col}(h),$$

whiche proves (4.7). □

Figure 1. Theorem for finding the Jacobian matrix J of p [9]

b. In ref. [8], the authors prove that the algebraic structure of the solutions to the QDEs is actually a right free module, not a linear vector space. On the non-commutativity of the quaternion algebra, many concepts and properties for the ordinary differential equations (ODEs) can not be used. They should be redefined accordingly. In the following, we will show that there are four profound differences between QDEs and ODEs:[8]

   a. On the non-commutativity of the quaternion algebra, the algebraic structure of the
   solutions to QDEs is not a linear vector space. It is actually a right free module. Consequently, to determine if two solutions of the QDEs are linearly independent or dependent, we have to define the concept of left dependence (independence) and right dependence (independence). While for classical ODEs, left dependence and right dependence are the same.
   b. Wronskian of ODEs is defined by Caley determinant. However, since Caley determinant depends on the expansion of i-th row and j-th column of quaternion, different expansions can lead to different results. Wronskian of ODEs can not be extended to QDEs. It is necessary to define the Wronskian of QDEs by a novel method.
   c. Liouville formula for QDEs and ODEs are different.
   d. For QDEs, it is necessary to treat the eigenvalue problems with left and right, separately. This is a large-difference between QDEs and ODEs.

Procedures and outline of algorithm to find fundamental matrix of QDE can be found in [8].

V. CONCLUDING REMARKS

In recent years, there are several proposals of using MHD theory for clean power generators on top of coal plant. But the theory involved appears too complicated, so in this paper we will use a simpler approach using ideal MHD equations which then they can be reduced to a system of coupled quaternionic Riccati equations.

It is worth remarking here that the above system of coupled quaternionic Riccati equations are very difficult to solve analytically, but considering our paper elsewhere suggesting that a system of coupled Riccati equations can be solved numerically using computer algebra packages such as Mathematica, then we are optimistic that it is also possible to find numerical solution of equation (6). In the last section, we outlined a possible route to solve the QRiccati as LQDE system, albeit not completely. It would be necessary to implement this procedure to become a full algorithm to be executed in computer algebra package such as Mathematica, but this is beyond the scope of our paper. Further computational and experimental investigations are recommended.

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