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Neutrosophic Crisp Mathematical Morphology

Eman.M.El-Nakeeb,[1,a] Hewayda ElGhawalby,[1,b] A.A.Salama,[2,c] S.A.El-Hafeez[2,d]

[1,2]Port Said University, Faculty of Engineering, Physics and Engineering Mathematics Department, Egypt

a) emanmarzouk1991@gmail.com

b) *hewayda2011@eng.psu.edu.eg*

[3,4]Port Said University , Faculty of Science, Department of Mathematics and Computer Science, Egypt

c) *drsalama44@gmail.com*

d) samyabdelhafeez@yahoo.com

Abstract In this paper, we aim to apply the concepts of the neutrosophic crisp sets and its operations to the classical mathematical morphological operations, introducing what we call "Neutrosophic Crisp Mathematical Morphology". Several operators are to be developed, including the neutrosophic crisp dilation, the neutrosophic crisp erosion, the neutrosophic crisp opening and the neutrosophic crisp closing.
Moreover, we extend the definition of some morphological filters using the neutrosophic crisp sets concept. For instance, we introduce the neutrosophic crisp boundary extraction, the neutrosophic crisp Top-hat and the neutrosophic crisp Bottom-hat filters.
The idea behind the new introduced operators and filters is to act on the image in the neutrosophic crisp domain instead of the spatial domain.

Keywords: Neutrosophic Crisp Set, Neutrosophic Sets, Mathematical Morphology, Filter Mathematical Morphology.

1 Introduction

In late 1960's, a relatively separate part of image analysis was developed; eventually known as "The Mathematical Morphology". Mostly, it deals with the mathematical theory of describing shapes using sets in order to extract meaningful information's from images, the concept of neutrosophy was first presented by Smarandache [14]; as the study of original, nature and scape of neutralities, as well as their interactions with different ideational spectra. The mathematical treatment for the neutrosophic phenomena, which already exists in our real world, was introduced in several studies; such as in [2].

The authors in [15], introduced the concept of the neutrosophic set to deduce. Neutrosophic mathematical morphological operations as an extension for the fuzzy mathematical morphology.

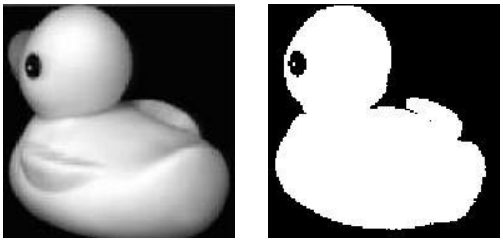
In [9] Salama introduced the concept of neutrosophic crisp sets, to represent any event by a triple crisp structure. In this paper, we aim to use the idea of the neutrosophic crisp sets to develop an alternative extension of the binary morphological operations. The new proposed neutrosophic crisp morphological operations is to be used for image analysis and processing in the neutrosophic domain. To commence, we review the classical operations and some basic filters of mathematical morphology in both §2 and § 3.

A revision of the concepts of neutrosophic crisp sets and its basic operations, is presented in §4 . the remaining sections, (§5, §6 and §7), are devoted for presenting our new concepts for "Neutrosophic crisp mathematical morphology" and its basic operations, as well as some basic neutrosophic crisp morphological filters.

2 Mathematical Morphological Operations:

In this section, we review the definitions of the classical binary morphological operators as given by Heijmans [6]; which are consistent with the original definitions of the Minkowski addition and subtraction [4].

For the purpose of visualizing the effect of these operators, we will use the binary image show in Fig.1(b); which is deduced form the original gray scale image shown in Fig.1(a).



a) b)

Fig.1: a) *the Original grayscale image* b) *the Binary image*

2.1 Binary Dilation: (Minkowski addition)

Based on the concept of Minkowski addition, the dilation is considered to be one of the basic operations in mathematical morphology, the dilations is originally developed for binary images [5]. To commence, we consider any Euclidean space E and a binary image A in E, the Dilation of A by some structuring element B is defined by: $A \oplus B = \bigcup_{b\in B} A_b$, where A_b is the translate of the set A along the vector b, i.e., $A_b = \{a+b \in E / a \in A, b \in B\}$ A_b

The Dilation is commutative, and may also be given by: $A \oplus B = B \oplus A = \cup_{a\in A} B_a$

$A \oplus B = B \oplus B = \bigcup_{a\in A} B_a$

An interpretation of the Dilation of A by B can be understood as, if we put a copy of B at each pixel in A and union all of the copies, then we get $A \oplus B$.

The Dilation can also be obtained by: $A \oplus B = \{b \in E \mid (-B) \cap A \neq \emptyset\}$, where $(-B)$ denotes the reflection of B, that is,

$-B = \{x \in E / -x \in B\}$

Where the reflection satisfies the following property: $-(A \oplus B) = (-A) \oplus (-B)$

$-(A \oplus B) = (-A) \oplus (-B)$

$-(A \oplus B) = (-A) \oplus (-B)$.

a) b)

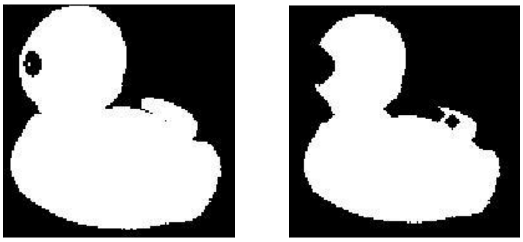
Fig.2: *Applying the dilation operator: a) the Original binary image b) the dilated image.*

2. 2 Binary Erosion: (Minkowski subtraction)

Strongly related to the Minkowski subtraction, the erosion of the binary image A by the structuring element B is defined by: $A \ominus B = \bigcap_{b\in B} A_{-b}$ $A \Theta B = \bigcup_{b\in B} A_{-b}$ Unlike dilation, erosion is not commutative, much like how addition is commutative while subtraction is not [5]. hence $A \Theta B$ is all pixels in A that these copies were translated to. The erosion of A by B is also may be given by the expression:

$A \ominus B = \{p \in E \mid B_p \subseteq A\}$ where B_p is the translation of B by the vector p, i.e., $B_p = \{b + p \in E / b \in B\}, \forall p \in E$

$B_p = \{b + p \in E \mid b \in B\}, \forall p \in E$.



a) b)

Fig.3: *Applying the erosion operator: a) the Original binary image b) the eroted image.*

2. 3 Binary Opening [5]:

The Opening of A by B is obtained by the erosion of A by B, followed by dilation of the resulting image by B: $A \circ B = (A \Theta B) \oplus B$. $A \circ B = (A \ominus B) \oplus B$ The opening is also given by $A \circ B = \bigcup_{B_x \subseteq A} B_x$ $A \circ B = \cup_{B_x \subseteq A} B_x$, which means that, an opening can be consider to be the union of all translated copies of the structuring element that can fit inside the object. Generally, openings can be used to remove small objects and connections between objects.

a) b)

Fig.4: *Applying the opening operator: a) the Original binary image b) the image opening.*

2.4 Binary Closing [5]:

The closing of A by B is obtained by the dilation of A by B, followed by erosion of the resulting structure by B: $A \bullet B = (A \oplus B)\Theta B$ $A \bullet B = (A \oplus B) \ominus B$.

The closing can also be obtained by $A \bullet B = co(coA \circ co(-B))$ $A \bullet B = (A^c \circ (-B))^c$, where coA denotes the complement of A relative to E (that is, $coA = \{a \in E / a \notin A\}$ $A^c = \{a \in E \mid a \notin A\}$).

Whereas opening removes all pixels where the structuring element won't fit inside the image foreground, closing fills in all places where the structuring element will not fit in the image background, that is opening removes small objects, while closing removes small holes.

a) b)

Fig.5: *Applying the closing operator: a) the Original binary image b) the image closing.*

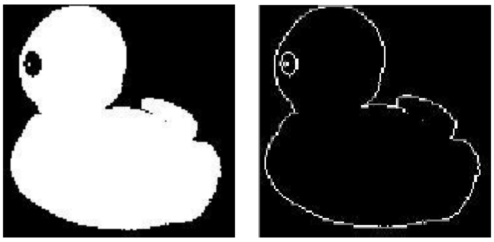
3. Mathematical Morphological Filters [13]:

In image processing and analysis, it is important to extract features of objects, describe shapes, and recognize patterns. Such tasks often refer to geometric concepts, such as size, shape, and orientation. Mathematical Morphology takes these concept from set theory, geometry, and topology to analyse the geometric structures in an image. Most essential image-processing algorithms can be represented in the form of Morphological operations.

In this section we review some basic Morphological filters, such as: the boundary extraction, and the Top-hat and the Bottom-hat filters.

3.1 The Boundary External [13]:

Boundary extraction of a set A requires first the dilating of A by a structuring element B and then taking the set difference between dilation and A. That is, the boundary of a set A is obtained by: . $\partial A = A - (A \Theta B)$



a) b)

Fig.6: *Applying the External Boundary: a) the Original binary image b) the External Boundary.*

3.2 The Hat Filters [13]:

In Mathematical Morphology and digital image processing, top-hat transform is an operation that extracts small elements and details from given images. There exist two types of hat filters: The Top-hat filter is defined as the difference between the input image and its opening by some structuring element; The Bottom-hat filter is defined dually as the difference between the closing and the input image. Top-hat filter are used for various image processing tasks, such as feature extraction, background equalization and image enhancement.

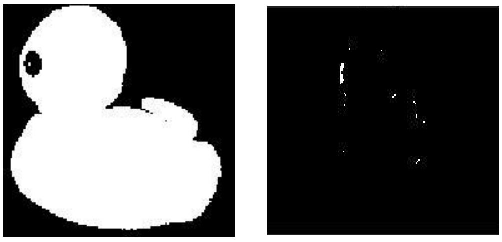
If an opening removes small structures, then the difference of the original image and the opened image should bring them out. This is exactly what the white Top-hat filter does, which is defined as the residue of the original and opening:

Top-hat filter: $T_{hat} = A - (A \circ B\ )$

The counter part of the Top-hat filter is the Bottom-hat filter which is defined as the residue of closing and the original:

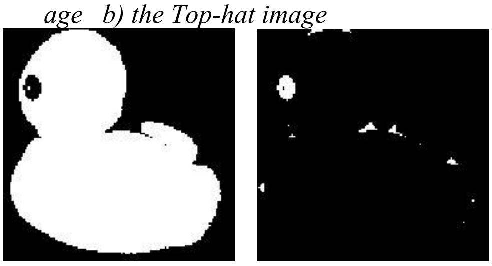
Bottom-hat filter: $B_{hat} = (A \bullet B\ ) - A$

These filters preserve the information removed by the Opening and Closing operations, respectively. They are often cited as white top-hat and black top-hat, respectively.



a) b)

Fig.7: *Applying the Top-hat: a) the Original binary image b) the Top-hat image*



a) b)

Fig.8: *Applying the Bottom-hat filter:*
a) the Original binary image b) Bottom-hat filter image

4. Neutrosophic Crisp Sets Theory [9]:

In this section we review some basic concepts of neutrosophic crisp sets and its operations.

4.1 Neutrosophic Crisp Sets:

4.1.1 Definition [9]

Let X X be a non-empty fixed set, a neutrosophic crisp set A (NCS for short), can be defined as a triple of the form $\langle A^1, A^2, A^3 \rangle$ $\langle A^1,\ A^2,\ A^3 \rangle$, where $A^1, A^2 and\ A^3$ and A^3 are crisp subsets of X. The three components represent a classification of the elements of the space X according to some event A; the subset A^1 contains all the elements of X that are supportive to A, A^3 contains all the elements of X that are against A, and A^2 contains all the elements of X that stand in a distance from being with or against A. A. Consequently, every crisp event A in X can be considered as a NCS having the form: $A = \langle A^1, A^2, A^3 \rangle$. The set of all neutrosophic crisp sets of X will be denoted $\mathcal{NC}(X)$.

Fig.9: *Neutrosophic Crisp Image:tivelycrespe* $\langle A^1, A^2, A^3 \rangle$

4.1.2 Definition [7, 9]:

The null (empty) neutrosophic set φ_N, the absolute (universe) neutrosophic set X_N and
the complement of a neutrosophic crisp set are defined as follows:

1) φ_N may be defined as one of the following two ϕ_N types:

Type 1:, $\varphi_N = \langle \varphi,\ \varphi,\ X \rangle$ $\phi_N = \langle \phi, \phi, X \rangle$

Type 2: $\phi_N = \langle \phi, X, X \rangle$ $\varphi_N = \langle \varphi,\ X,\ X \rangle$

2) 2) X_N may be defined as one of the following two types:

Type 1: $X_N = \langle X, X, \varphi \rangle$,

Type 2: $X_N = \langle X, \varphi, \varphi \rangle$.

3) The complement of a NCS (coA co A for short) may be defined as one of the following two types:

Type 1: $\mathrm{coA} = \langle \mathrm{coA}^1,\ \mathrm{coA}^2,\mathrm{coA}^3 \rangle$,

Type 2: $\mathrm{coA} = \langle \mathrm{A}^3,\ \mathrm{coA}^2,\mathrm{A}^1 \rangle$.

4.2. Neutrosophic Crisp Sets Operations:

In [6, 14], the authors extended the definitions of the crisp sets operations to be defined over Neutrosophic Crisp Sets (in short NCSs). In the following definitions we consider a non-empty set X, and any two Neutrosophic Crisp Sets of X, A and B, where $A = \langle A^1, A^2, A^3 \rangle$ and $B = \langle B^1,\ B^2,\ B^3 \rangle$.

4.2.1 Definition [8, 9]:

For any two sets A, B $\in \mathcal{NC}(X)$, A is said to be a neutrosophic crisp subset of the NCS B, i.e., ($A \subseteq B$), and may be defined as one of the following two types:

$$Type1: A \subseteq B \Leftrightarrow \langle A^1 \subseteq B^1, A^2 \subseteq B^2 and A^3 \supseteq B^3 \rangle$$

$$A \subseteq B \Leftrightarrow A^1 \subseteq B^1, A^2 \subseteq B^2 \text{ and } A^3 \supseteq B^3$$

$$A \subseteq B \Leftrightarrow A^1 \subseteq B^1, A^2 \supseteq B^2 \text{ and } A^3 \supseteq B^3$$

4.2.2 Proposition [7, 9]:

For any neutrosophic crisp set A, the following properties are hold:

a) $\phi_N \subseteq A$ and $\phi_N \subseteq \phi_N$

b) $A \subseteq X_N$ and $X_N \subseteq X_N$

4.2.3 Definition [7, 9]:

The neutrosophic intersection and neutrosophic union of any two neutrosophic crisp sets A, B $\in \mathcal{NC}(X)$, may be defined as follows:

1. The neutrosophic intersection $A \cap B$, may be defined as one of the following two types:

Type1: $A \cap B = \langle A^1 \cap B^1, A^2 \cap\ B^2, A^3 \cup B^3 \rangle$.

Type2: $A \cap B = \langle A^1 \cap B^1, A^2 \cup\ B^2, A^3 \cup B^3 \rangle$.

2. The neutrosophic union $A \cup B$, may be defined as one of the following two types:

Type 1: $A \cup B = \langle A^1 \cup B^1, A^2 \cup\ B^2, A^3 \cap B^3 \rangle$.

Type2: $A \cup B = \langle A^1 \cup B^1, A^2 \cap\ B^2, A^3 \cap B^3 \rangle$.

4.2.4 Proposition [7, 9]:

For any two neutrosophic crisp sets A, $B \in \mathcal{NC}(X)$, then:

$co(A \cap B) = coA \cup coB$

$co(A\ \cap B\ ) = coA\ \cup\ coB$ and

$co(A \cup B) = coA \cap coB$

$co(A\ \cup B\ ) = coA\ \cap\ coB$.

Proof: We can easily prove that the two statements are true for both the complement operators. Defined in definition 4.1.2.

4.2.5 Proposition [9]:

For any arbitrary family $\{A_i : i \in I\}$, of neutrosophic crisp subsets of X, a generalization for the neutrosophic intersection and for the neutrosophic union given in Definition 4.2.3, can be defined as follows:

1) $\bigcap_{i\in I} A_i$ may be defined as one of the following two types:

$$Type1: \bigcap_{i\in I} A_i = \left\langle \bigcap_{i\in I} A_i^1, \bigcap_{i\in I} A_i^2, \bigcup_{i\in I} A_i^3 \right\rangle$$

$$Type2: \bigcap_{i\in I} A_i = \left\langle \bigcap_{i\in I} A_i^1, \bigcup_{i\in I} A_i^2, \bigcup_{i\in I} A_i^3 \right\rangle$$

2) may be defined as one of the $\bigcup_{i\in I} A_i$ $\bigcup_{i\in I} A_i$ following two types:

$$Type1: \bigcup_{i\in I} A_i = \left\langle \bigcup_{i\in I} A_i^1, \bigcup_{i\in I} A_i^2, \bigcap_{i\in I} A_i^3 \right\rangle$$

$$Type2: \bigcup_{i\in I} A_i = \left\langle \bigcup_{i\in I} A_i^1, \bigcap_{i\in I} A_i^2, \bigcap_{i\in I} A_i^3 \right\rangle$$

5. Neutrosophic Crisp Mathematical Morphology:

As a generalization of the classical mathematical morphology, we present in this section the basic operations for the neutrosophic crisp mathematical morphology. To commence, we need to define the translation of a neutrosophic set.

5.1.1 Definition:

Consider the Space X=R^n or Z^n $X = R^n \text{ or } Z^n$ With origin 0 = (0,...,0) given The reflection of the structuring element B mirrored in its Origin is defined as: $-B = \langle -B^1, -B^2, -B^3 \rangle$.

5.1 Definition:

For every the $p \in A$, translation by p is the map $p: X \to X, a \to a+p$ $p: X \to X,\ a \mapsto a+p$ it transforms any Subset A of X into its translate by $p \in Z^2$, $-B = \langle -B^1, -B^2, -B^3 \rangle$, $A_n = \langle A^1{}_n, A^2{}_n, A^3{}_n \rangle A_n = \langle A^1{}_n, A^2{}_n, A^3{}_n \rangle$

Where

$A_p^1 = \{u + p : u \in A^1, p \in B^1\}$

$= \{u + p : u \in A^1,\ p \in B^1\}$

$A^1{}_n(u) = \{u + p : u \in A^1,\ p \in B^1\}$

$A_p^2 = \{u + p : u \in A^2, p \in B^3\}$

$= \{u + p : u \in A^2,\ p \in B^2\}$

$A_p^3 = \{u + p : u \in A^3, p \in B^3\}$

$\{u + p : u \in A^3,\ p \in B^3\}$

5.2 Neutrosophic Crisp Mathematical Morphological Operations:

5.2.1 Neutrosophic Crisp Dilation Operator:

let A, B$\in \mathcal{NC}(X)$, then we define two types of the neutrosophic crisp dilation as follows:

Type1:

$A \tilde{\oplus} B = \langle A^1 \oplus B^1, A^2 \oplus B^2, A^3 \Theta B^3 \rangle$

$(A \oplus B) = \langle A^1 \oplus B^1, A^2 \oplus B^2, A^3 \ominus B^3 \rangle$, where for each u and $v \in Z^2 \in Z^2$.

$A^1 \oplus B^1 = \bigcup_{b\in B^1} A_b^1 \quad A^2 \Theta B^2 = \bigcup_{b\in B^2} A_{-b}^2$

$A^3 \Theta B^3 = \bigcap_{b\in B^3} A_{-b}^3$

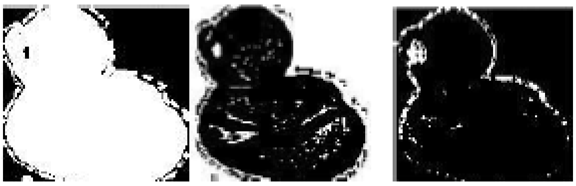


Fig.10(i): *Neutrosophic Crisp Dilation components in type 1* $\langle A^1, A^2, A^3 \rangle$ *respectively*

Type2:

$(A \oplus B) = \langle A^1 \oplus B^1, A^2 \ominus B^2, A^3 \ominus B^3 \rangle$,

where for each u and $v \in Z^2 \in Z^2$.

$A^1 \oplus B^1 = \bigcup_{b\in B^1} A_b^1, \quad A^2 \oplus B^2 = \bigcap_{b\in B^3} A_{-b}^2$

$A^3 \Theta B^3 = \bigcap_{b\in B^3} A_{-b}^3$

Fig.10(ii): *Neutrosophic Crisp Dilation components in type 2* $\langle A^1, A^2, A^3 \rangle$ *respectively*

5.2.2 Neutrosophic Crisp Erosion Operation:

let A and B $\in \mathcal{NC}(X)$; then the neutrosophic dilation is given as two type:

Type 1:

$(A \ominus B) = \langle A^1 \ominus B^1, A^2 \ominus B^2, A^3 \oplus B^3 \rangle$,

where for each u and $v \in Z^2 . \in Z^2$ $A^1 \Theta B^1 = \bigcap_{b\in B^3} A_{-b}^1$

$A^2 \Theta B^2 = \bigcap_{b\in B^3} A_{-b}^2$ and $A^3 \oplus B^3 = \bigcup_{b\in B^1} A_b^3$

Fig.11(i): *Neutrosophic Crisp Erosion components in type 1* $\langle A^1, A^2, A^3 \rangle$ *respectively*

$A^1 \ominus B^1 = \bigcap_{b \in B^1} A^1{}_{-b}$, $A^2 \ominus B^2 = \bigcap_{b \in B^2} A^2{}_{-b}$, $A^3 \oplus B^3 = \bigcup_{b \in B^3} A^3{}_b$

Type2:
$(A \ominus B) = \langle A^1 \ominus B^1, A^2 \oplus B^2, A^3 \oplus B^3 \rangle$,
where for each u and $v \in Z^2 \in Z^2$. $A^1 \Theta B^1 = \bigcap_{b \in B^3} A^1_{-b}$

$A^2 \oplus B^2 = \bigcup_{b \in B^1} A^2_b$ and $A^3 \oplus B^3 = \bigcup_{b \in B^1} A^3_b$

Fig.11(ii): *Neutrosophic Crisp Erosion components in type2* $\langle A^1, A^2, A^3 \rangle$ *respectively*

5.2.3 Neutrosophic Crisp Opening Operation:

let A, B $\in \mathcal{NC}(X)$; then we define two types of the neutrosophic crisp dilation operator as follows:

Type 1: $A \,\tilde{\circ}\, B = \langle A^1 \circ B^1, A^2 \circ B^2, A^3 \bullet B^3 \rangle$

$$A^1 \circ B^1 = (A^1 \Theta B^1) \oplus B^1$$

$$A^2 \circ B^2 = (A^2 \Theta B^2) \oplus B^2$$

$$A^3 \circ B^3 = (A^3 \oplus B^3) \Theta B^3$$

$A^1 \circ B^1 = (A^1 \ominus B^1) \oplus B^1$

$A^2 \circ B^2 = (A^2 \ominus B^2) \oplus B^2$

Fig.12(i): *Neutrosophic Crisp opening components in type1* $\langle A^1, A^2, A^3 \rangle$ *respectively*

$A^3 \bullet B^3 = (A^3 \oplus B^3) \ominus B^3$

Type 2:

$A \,\tilde{\circ}\, B = \langle A^1 \circ B^1, A^2 \bullet B^2, A^3 \bullet B^3 \rangle$

$A^1 \circ B^1 = (A^1 \ominus B^1) \oplus B^1$

$$A^1 \circ B^1 = (A^1 \Theta B^1) \oplus B^1$$

$$A^2 \bullet B^2 = (A^2 \oplus B^2) \Theta B^2$$

$$A^3 \bullet B^3 = (A^3 \oplus B^3) \Theta B^3$$

Fig.12(ii): *Neutrosophic Crisp opening components in type2* $\langle A^1, A^2, A^3 \rangle$ *respectively*

5.2.4 Neutrosophic Crisp Closing Operation:

let A and B $\in \mathcal{NC}(X)$; then the neutrosophic dilation is given as two types:

Type 1: $A \,\tilde{\bullet}\, B = \langle A^1 \bullet B^1, A^2 \bullet B^2, A^3 \circ B^3 \rangle$

$A^1 \bullet B^1 = (A^1 \ominus B^1) \oplus B^1$

$A^3 \circ B^3 = (A^3 \oplus B^3) \ominus B^3$

$$A^3 \circ B^3 = (A^3 \Theta B^3) \oplus B^3$$

Fig.13(i): *Neutrosophic Crisp closing components in type1* $\langle A^1, A^2, A^3 \rangle$ *respectively*

Type 2: $A \,\tilde{\bullet}\, B = \langle A^1 \bullet B^1, A^2 \circ B^2, A^3 \circ B^3 \rangle$

$$A^1 \bullet B^1 = (A^1 \ominus B^1) \oplus B^1$$

$$A^3 \circ B^3 = (A^3 \oplus B^3) \ominus B^3$$

$$A^3 \circ B^3 = (A^3 \Theta B^3) \oplus B^3$$

Fig.13(ii): *Neutrosophic Crisp closing components in type2* $\langle A^1, A^2, A^3 \rangle$ *respectively*

6. Algebraic Properties in Neutrosophic Crisp:

In this section, we investigate some of the algebraic properties of the neutrosophic crisp erosion and dilation, as well as the neutrosophic crisp opening and closing operator [15].

6.1 Properties of the Neutrosophic Crisp Erosion Operator:

6.1.1 Proposition:

The Neutrosophic erosion satisfies the monotonicity for all A, B ∈ $\mathcal{NC}(Z^2)$.

$Type1: a) A \subseteq B \Rightarrow \langle A^1\Theta C^1, A^2\Theta C^2, A^3\Theta C^3 \rangle$

$\subseteq \langle B^1\Theta C^1, B^2\Theta C^2, B^3\Theta C^3 \rangle$

$A^1\Theta C^1 \subseteq B^1\Theta C^1, A^2\Theta C^2 \subseteq B^2\Theta C^2$

$A^3\Theta C^3 \supseteq A^3\Theta C^3$

$b) A \subseteq B \Rightarrow \langle C^1\Theta A^1, C^2\Theta A^2, C^3\Theta A^3 \rangle$

$\subseteq \langle C^1\Theta B^1, C^2\Theta B^2, C^3\Theta B^3 \rangle$

$C^1\Theta A^1 \subseteq C^1\Theta B^1, C^2\Theta A^2 \subseteq C^2\Theta B^2$

$C^3\Theta A^3 \supseteq C^3\Theta A^3$

$Type2: a) A \subseteq B \Rightarrow \langle A^1\Theta C^1, A^2\Theta C^2, A^3\Theta C^3 \rangle$

$\subseteq \langle B^1\Theta C^1, B^2\Theta C^2, B^3\Theta C^3 \rangle$

$A^1\Theta C^1 \subseteq B^1\Theta C^1, A^2\Theta C^2 \supseteq B^2\Theta C^2$

$A^3\Theta C^3 \supseteq A^3\Theta C^3$

$b) A \subseteq B \Rightarrow \langle C^1\Theta A^1, C^2\Theta A^2, C^3\Theta A^3 \rangle$

$\subseteq \langle C^1\Theta B^1, C^2\Theta B^2, C^3\Theta B^3 \rangle$

$C^1\Theta A^1 \subseteq C^1\Theta B^1, C^2\Theta A^2 \supseteq C^2\Theta B^2$

$C^3\Theta A^3 \supseteq C^3\Theta A^3$

Note that: Dislike the Neutrosophic crisp dilation operator, the Neutrosophic crisp erosion does not satisfy commutativity and the associativity properties.

6.1.2 Proposition: for any family $(A_i | i \in I)$ in $\mathcal{NC}(Z^2)$ $(A_i | i \in I)$ *in* $\mathcal{N}(Z^2)$ and B $\mathcal{NC}(Z^2)$.

Type1: a) $\cap_{i \in I} A_i \,\tilde{\ominus}\, B)$ $\cap_{i \in I} A_i \,\tilde{\ominus}\, B = \cap_{i \in I}(A_i \,\tilde{\ominus}\, B)$

$$= \left\langle \bigcap_{i\in I} A_i^1\Theta B^1, \bigcap_{i\in I} A_i^2\Theta B^2, \bigcap_{i\in I} A_i^3 \oplus B^3 \right\rangle$$

$$= \left\langle \bigcap_{i\in I}(A_i^1\Theta B^1), \bigcap_{i\in I}(A_i^2\Theta B^2), \bigcap_{i\in I}(A_i^3 \oplus B^3) \right\rangle$$

b) $B\tilde{\Theta}\bigcap_{i\in I} A_i = \bigcap_{i\in I}(B\Theta A_i)$ $B \,\tilde{\ominus}\, \cap_{i \in I} A_i$

$$= \left\langle B^1\Theta\bigcap_{i\in I} A_i^1, B^2\Theta\bigcap_{i\in I} A_i^2, B^3 \oplus \bigcap_{i\in I} A_i^3 \right\rangle$$

$$= \left\langle \bigcap_{i\in I}(B^1\Theta A_i^1), \bigcap_{i\in I}(B^2\Theta A_i^2), \bigcap_{i\in I}(B^3 \oplus A_i^3) \right\rangle$$

Type2: a) $\cap_{i \in I} A_i \,\tilde{\ominus}\, B)$ $\cap_{i \in I} A_i \,\tilde{\ominus}\, B = \cap_{i \in I}(A_i \,\tilde{\ominus}\, B)$

$$= \left\langle \bigcap_{i\in I} A_i^1\Theta B^1, \bigcap_{i\in I} A_i^2 \oplus B^2, \bigcap_{i\in I} A_i^3 \oplus B^3 \right\rangle$$

$$= \left\langle \bigcap_{i\in I}(A_i^1\Theta B^1), \bigcap_{i\in I}(A_i^2 \oplus B^2), \bigcap_{i\in I}(A_i^3 \oplus B^3) \right\rangle$$

b) $B\tilde{\Theta}\bigcap_{i\in I} A_i = \bigcap_{i\in I}(B\Theta A_i)$ $B \,\tilde{\ominus}\, \cap_{i \in I} A_i$

$$= \left\langle B^1\Theta\bigcap_{i\in I} A_i^1, B^2 \oplus \bigcap_{i\in I} A_i^2, B^3 \oplus \bigcap_{i\in I} A_i^3 \right\rangle$$

$$= \left\langle \bigcap_{i\in I}(B^1\Theta A_i^1), \bigcap_{i\in I}(B^2 \oplus A_i^2), \bigcap_{i\in I}(B^3 \oplus A_i^3) \right\rangle$$

Proof: a) in two type:

Type 1: $\cap_{i\in I} A_i \oplus B$ $\bigcap_{i\in I} A_i \Theta B$

$$=\left\langle \bigcap_{b\in B}(\bigcap_{i\in I} A^1_{ib}), \bigcap_{b\in B}(\bigcap_{i\in I} A^2_{ib}), \bigcup_{b\in B}(\bigcap_{i\in I} A^3_{i(-b)})\right\rangle$$

$\cap_{i\in I} A_i \oplus B =$
$\langle \cup (\cap A^1_{ib}), \cup (\cap A^2_{ib}), \cap (\cap A^3_{i(-b)})\rangle$

$$=\left\langle \bigcap_{i\in I}(\bigcap_{b\in B} A^1_{i(-b)}), \bigcap_{i\in I}(\bigcup_{b\in B} A^2_{i(-b)}), \bigcap_{i\in I}(\bigcup_{b\in B} A^3_{ib})\right\rangle$$

$\langle \cap_{i\in I}(\cup_{b\in B} A^1_{ib}), \cap_{i\in I}(\cup_{b\in B} A^2_{ib}), \cap_{i\in I}(\cap_{b\in B} A^3_{i(-b)})\rangle = \cap_{i\in I}(A_i \oplus B)$.

$$=\bigcap_{i\in I}(A_i\Theta B)$$

Type 2: similarity, we can show that it is true in type 2,

b) The proof is similar to point a).

6.1.3 Proposition: for any family $(A_i | i\in I)$ in $\mathcal{NC}(Z^2)$ and $B\in \mathcal{NC}(Z^2)$

Type 1: $a)\bigcup_{i\in I} A_i \tilde\Theta B = \bigcup_{i\in I}(A_i\tilde\Theta B)$

$$=\left\langle \bigcup_{i\in I} A_i^1\Theta B^1, \bigcup_{i\in I} A_i^2 \Theta B^2, \bigcup_{i\in I} A_i^3\oplus B^3\right\rangle$$

$$=\left\langle \bigcup_{i\in I}(A_i^1\Theta B^1), \bigcup_{i\in I}(A_i^2\Theta B^2), \bigcup_{i\in I}(A_i^3\oplus B^3)\right\rangle$$

$$b)B\tilde\Theta \bigcup_{i\in I} A_i = \bigcup_{i\in I}(B\tilde\Theta A_i)$$

$$=\left\langle B^1\Theta\bigcup_{i\in I} A_i^1, B^2\Theta\bigcup_{i\in I} A_i^2, B^3\oplus\bigcup_{i\in I} A_i^3\right\rangle$$

$$=\left\langle \bigcup_{i\in I}(B^1\Theta A_i^1), \bigcup_{i\in I}(B^2\Theta A_i^2), \bigcup_{i\in I}(B^3\oplus A_i^3)\right\rangle$$

Type 2: $a)\bigcup_{i\in I} A_i \tilde\Theta B = \bigcup_{i\in I}(A_i\tilde\Theta B)$

$$=\left\langle \bigcup_{i\in I} A_i^1\Theta B^1, \bigcup_{i\in I} A_i^2\oplus B^2, \bigcup_{i\in I} A_i^3\oplus B^3\right\rangle$$

$A\subseteq B \Rightarrow$
$\langle A^1\oplus C^1, A^2\oplus C^2, A^3\oplus C^3\rangle \subseteq$
$\langle B^1\oplus C^1, B^2\oplus C^2, B^3\oplus C^3\rangle$

$$=\left\langle \bigcup_{i\in I}(A_i^1\Theta B^1), \bigcup_{i\in I}(A_i^2\oplus B^2), \bigcup_{i\in I}(A_i^3\oplus B^3)\right\rangle$$

$$b)B\tilde\Theta\bigcup_{i\in I} A_i = \bigcup_{i\in I}(B\tilde\Theta A_i)$$

$$=\left\langle B^1\Theta\bigcup_{i\in I} A_i^1, B^2\oplus\bigcup_{i\in I} A_i^2, B^3\oplus\bigcup_{i\in I} A_i^3\right\rangle$$

$$=\left\langle\bigcup_{i\in I}(B^1\Theta A_i^1), \bigcup_{i\in I}(B^2\oplus A_i^2), \bigcup_{i\in I}(B^3\oplus A_i^3)\right\rangle$$

Proof: a)

Type1: $\bigcup_{i\in I} A_i\tilde\Theta B =$

$$\left\langle \bigcap_{b\in B}(\bigcup_{i\in I} A^1_{i(-b)}), \bigcap_{b\in B}(\bigcup_{i\in I} A^1_{i(-b)}), \bigcup_{b\in B}(\bigcup_{i\in I} A^1_i)\right\rangle$$

$$\left\langle \bigcup_{i\in I}(\bigcap_{b\in B} A^1_{i(-b)}), \bigcup_{i\in I}(\bigcap_{b\in B} A^2_{i(-b)}), \bigcup_{i\in I}(\bigcup_{b\in B} A^3_{ib})\right\rangle = \bigcup_{i\in I}(A_i\tilde\Theta B)$$

Type 2: can be verified in a similar way as in type 1.

b) The proof is similar to point a)

6.2 Proposition: (Properties of the Neutrosophic Crisp Dilation Operator):

6.2.1 Proposition:

The neutrosophic Dilation satisfies the following properties: $\forall A,B\in\mathcal{NC}(Z^2)$ $\forall A, B \in \mathcal{N}(Z^2)$

i) Commutativity: $A\tilde\oplus B = B\tilde\oplus A$
$A\oplus B = B\oplus A$

ii) Associativity: $(A\tilde\oplus B)\tilde\oplus B = A\tilde\oplus(B\tilde\oplus B)$
$(A\oplus B)\oplus C = A\oplus(B\oplus C)$.

iii) Monotonicity: (increasing in both arguments):

Type1:

a) $A\subseteq B \Rightarrow \langle A^1\oplus C^1, A^2\oplus C^2, A^3\oplus C^3\rangle$
$\subseteq \langle B^1\oplus C^1, B^2\oplus C^2, B^3\oplus C^3\rangle$

$A^1\oplus C^1\subseteq B^1\oplus C^1, A^2\oplus C^2\subseteq B^2\oplus C^2$ and
$A^3\oplus C^3 \supseteq B^3\oplus C^3$

$C^1\oplus A^1 \subseteq C^1\oplus B^1$, $C^2\oplus A^2 \subseteq C^2\oplus B^2$ *and* $C^3\oplus A^3 \supseteq C^3\oplus B^3$

$b) A\subseteq B\Rightarrow \langle C^1\oplus A^1, C^2\oplus A^2, C^3\oplus A^3\rangle$

$\subseteq \langle C^1\oplus B^1, C^2\oplus B^2, C^3\oplus B^3\rangle$

$C^1\oplus A^1\subseteq C^1\oplus B^1, C^2\oplus A^2\subseteq C^2\oplus B^2$ and
$C^3\oplus A^3\supseteq C^3\oplus B^3$

$A\subseteq B\Rightarrow$
$\langle C^1\oplus A^1, C^2\oplus A^2, C^3\oplus A^3\rangle \subseteq$
$\langle C^1\oplus B^1, C^2\oplus B^2, C^3\oplus B^3\rangle$

Type2:

$a) A\subseteq B\Rightarrow \langle A^1\oplus C^1, A^2\oplus C^2, A^3\oplus C^3\rangle$

$$\subseteq \left\langle B^1 \oplus C^1, B^2 \oplus C^2, B^3 \oplus C^3 \right\rangle$$

$A^1 \oplus C^1 \subseteq B^1 \oplus C^1, A^2 \oplus C^2 \supseteq B^2 \oplus C^2$ and $A^3 \oplus C^3 \supseteq B^3 \oplus C^3$

$C^1 \oplus A^1 \subseteq C^1 \oplus B^1$, $C^2 \oplus A^2 \subseteq C^2 \oplus B^2$ and $C^3 \oplus A^3 \supseteq C^3 \oplus B^3$

$$b) A \subseteq B \Rightarrow \left\langle C^1 \oplus A^1, C^2 \oplus A^2, C^3 \oplus A^3 \right\rangle$$

$$\subseteq \left\langle C^1 \oplus B^1, C^2 \oplus B^2, C^3 \oplus B^3 \right\rangle$$

and $C^1 \oplus A^1 \subseteq C^1 \oplus B^1, C^2 \oplus A^2 \supseteq C^2 \oplus B^2$

$C^3 \oplus A^3 \supseteq C^3 \oplus B^3$

6.2.2 Proposition: for any family $(A_i | i \in I)$ in $\mathcal{NC}(Z^2)$ and B $\in \mathcal{NC}(Z^2)$ and $B \in \mathcal{N}(Z^2)$

Type 1: a) $\cap_{i\in I} A_i \tilde{\oplus} B = \cap_{i\in I}(A_i \tilde{\oplus} B)$

$$= \left\langle \bigcap_{i\in I} A_i^1 \oplus B^1, \bigcap_{i\in I} A_i^2 \oplus B^2, \bigcap_{i\in I} A_i^3 \Theta B^3 \right\rangle$$

$$= \left\langle \bigcap_{i\in I}(A_i^1 \oplus B^1), \bigcap_{i\in I}(A_i^2 \oplus B^2), \bigcap_{i\in I}(A_i^3 \Theta B^3) \right\rangle$$

b) $B \tilde{\oplus} \cap_{i\in I} A_i = \cap_{i\in I}(B \tilde{\oplus} A_i)$

$$= \left\langle B^1 \oplus \bigcap_{i\in I} A_i^1, B^2 \oplus \bigcap_{i\in I} A_i^2, B^3 \Theta \bigcap_{i\in I} A_i^3 \right\rangle$$

$$= \left\langle \bigcap_{i\in I}(B^1 \oplus A_i^1), \bigcap_{i\in I}(B^2 \oplus A_i^2), \bigcap_{i\in I}(B^3 \Theta A_i^3) \right\rangle$$

$\langle B^1 \oplus \cap_{i\in I} A^1{}_i, B^2 \oplus \cap_{i\in I} A^2{}_i, B^3 \ominus \cap_{i\in I} A^3{}_i \rangle = \langle \cap_{i\in I} (B^1 \oplus A^1{}_i), \cap_{i\in I} (B^2 \oplus A^2{}_i), \cap_{i\in I} (B^3 \ominus A^3{}_i) \rangle$

Type 2: a) $\cap_{i\in I} A_i \tilde{\oplus} B = \cap_{i\in I}(A_i \tilde{\oplus} B)$

$$= \left\langle \bigcap_{i\in I} A_i^1 \oplus B^1, \bigcap_{i\in I} A_i^2 \Theta B^2, \bigcap_{i\in I} A_i^3 \Theta B^3 \right\rangle$$

$$= \left\langle \bigcap_{i\in I}(A_i^1 \oplus B^1), \bigcap_{i\in I}(A_i^2 \Theta B^2), \bigcap_{i\in I}(A_i^3 \Theta B^3) \right\rangle$$

b) $B \tilde{\oplus} \cap_{i\in I} A_i = \cap_{i\in I}(B \tilde{\oplus} A_i)$

$$= \left\langle B^1 \oplus \bigcap_{i\in I} A_i^1, B^2 \Theta \bigcap_{i\in I} A_i^2, B^3 \Theta \bigcap_{i\in I} A_i^3 \right\rangle$$

$$= \left\langle \bigcap_{i\in I}(B^1 \oplus A_i^1), \bigcap_{i\in I}(B^2 \Theta A_i^2), \bigcap_{i\in I}(B^3 \Theta A_i^3) \right\rangle$$

Proof: we will prove this property for the two types of the neutrosophic crisp intersection operator:

Type 1: $\cap_{i\in I} A_i \oplus B \bigcap_{i\in I} A_i \oplus B$

$$= \left\langle \bigcup_{b\in B}(\bigcap_{i\in I} A_{ib}^1), \bigcup_{b\in B}(\bigcap_{i\in I} A_{ib}^2), \bigcap_{b\in B}(\bigcap_{i\in I} A_{i(-b)}^3) \right\rangle$$

$\cap_{i\in I} A_i \oplus B = \langle \cup (\cap A^1{}_{ib}), \cup (\cap A^2{}_{ib}), \cap (\cap A^3{}_{i(-b)}) \rangle$

$$= \left\langle \bigcap_{i\in I}(\bigcup_{b\in B} A_{ib}^1), \bigcap_{i\in I}(\bigcup_{b\in B} A_{ib}^2), \bigcap_{i\in I}(\bigcap_{b\in B} A_{i(-b)}^3) \right\rangle$$

$\langle \cap_{i\in I} (\cup_{b\in B} A^1{}_{ib}), \cap_{i\in I} (\cup_{b\in B} A^2{}_{ib}), \cap_{i\in I} (\cap_{b\in B} A^3{}_{i(-b)}) \rangle = \cap_{i\in I}(A_i \oplus B)$.

$$= \bigcap_{i\in I} (A_i \oplus B)$$

Type 2: $\cap_{i\in I} A_i \oplus B$

$$\left\langle \quad MCGDM(\quad \sigma) \; base(\quad Si \right\rangle$$

$$= \left\langle \bigcap_{i\in I}(\bigcup_{b\in B} A_{ib}^1), \bigcap_{i\in I}(\bigcap_{b\in B} A_{ib}^2), \bigcap_{i\in I}(\bigcap_{b\in B} A_{i(-b)}^3) \right\rangle$$

$\cap_{i\in I} A_i \oplus B = \langle \cup (\cap A^1{}_{ib}), \cap (\cap A^2{}_{i(-b)}), \cap (\cap A^3{}_{i(-b)}) \rangle$ $=$ $\bigcap_{i\in I} (A_i \oplus B)$

b) The proof is similar to a)

6.2.3 Proposition: for any family $(A_i | i \in I)$ in $\mathcal{NC}(Z^2)$ and B $\in \mathcal{NC}(Z^2)$

Type 1: $a) \bigcup_{i\in I} A_i \tilde{\oplus} B = \bigcup_{i\in I}(A_i \tilde{\oplus} B)$

$$\left\langle \bigcup_{i\in I} A_i^1 \oplus B^1, \bigcup_{i\in I} A_i^2 \oplus B^2, \bigcup_{i\in I} A_i^3 \Theta B^3 \right\rangle$$

$$= \left\langle \bigcup_{i\in I}(A_i^1 \oplus B^1), \bigcup_{i\in I}(A_i^2 \oplus B^2), \bigcup_{i\in I}(A_i^3 \Theta B^3) \right\rangle$$

$$b) B \tilde{\oplus} \bigcup_{i\in I} A_i = \bigcup_{i\in I}(B \tilde{\oplus} A_i)$$

$$\left\langle B^1 \oplus \bigcup_{i\in I} A_i^1, B^2 \oplus \bigcup_{i\in I} A_i^2, B^3 \Theta \bigcup_{i\in I} A_i^3 \right\rangle$$

$$= \left\langle \bigcup_{i\in I}(B^1 \oplus A_i^1), \bigcup_{i\in I}(B^2 \oplus A_i^2), \bigcup_{i\in I}(B^3 \Theta A_i^3) \right\rangle$$

Type 2: $a) \bigcup_{i\in I} A_i \tilde{\oplus} B = \bigcup_{i\in I}(A_i \tilde{\oplus} B)$

$$\left\langle \bigcup_{i\in I} A_i^1 \oplus B^1, \bigcup_{i\in I} A_i^2 \Theta B^2, \bigcup_{i\in I} A_i^3 \Theta B^3 \right\rangle$$

$$= \left\langle \bigcup_{i\in I}(A_i^1 \oplus B^1), \bigcup_{i\in I}(A_i^2 \Theta B^2), \bigcup_{i\in I}(A_i^3 \Theta B^3) \right\rangle$$

$$b) B \tilde{\oplus} \bigcup_{i\in I} A_i = \bigcup_{i\in I}(B \tilde{\oplus} A_i)$$

$$= \left\langle B^1 \oplus \bigcup_{i\in I} A_i^1, B^2 \Theta \bigcup_{i\in I} A_i^2, B^3 \Theta \bigcup_{i\in I} A_i^3 \right\rangle$$

$$=\left\langle \bigcup_{i\in I}(B^1\oplus A_i^1),\bigcup_{i\in I}(B^2\Theta A_i^2),\bigcup_{i\in I}(B^3\Theta A_i^3)\right\rangle$$

Proof: a) we will prove this property for the two types of the neutrosophic crisp union operator:

$$Type1:\bigcup_{i\in I}A_i\tilde{\oplus}B=\left\langle \bigcup_{b\in B}(\bigcup_{i\in I}A_{ib}^1),\bigcup_{b\in B}(\bigcup_{i\in I}A_{ib}^2),\bigcap_{b\in B}(\bigcup_{i\in I}A_{i(-b)}^3)\right\rangle$$

$$=\bigcup_{i\in I}(A_i\tilde{\oplus}B)=\left\langle \bigcup_{i\in I}(\bigcup_{b\in B}A_i^1),\bigcup_{i\in I}(\bigcup_{b\in B}A_{ib}^2),\bigcup_{i\in I}(\bigcap_{b\in B}A_{i(-b)}^3)\right\rangle$$

$$Type\ 2:\ \bigcup_{i\in I}A_i\tilde{\oplus}B=\left\langle \bigcup_{b\in B}(\bigcup_{i\in I}A_i^1),\bigcap_{b\in B}(\bigcup_{i\in I}A_{i(-b)}^2),\bigcap_{b\in B}(\bigcup_{i\in I}A_{i(-b)}^3)\right\rangle$$

$$=\bigcup_{i\in I}(A_i\tilde{\oplus}B)=\left\langle \bigcup_{i\in I}(\bigcup_{b\in B}A_i^1),\bigcup_{i\in I}(\bigcap_{b\in B}A_{i(-b)}^2),\bigcup_{i\in I}(\bigcap_{b\in B}A_{i(-b)}^3)\right\rangle$$

b) **The proof** is similar to (a)

6.2.4 Proposition (*Duality Theorem of Neutrosophic Crisp Dilation*):

let A, B $\in\mathcal{NC}(Z^2)$. Neutrosophic crisp Erosion and Dilation are dual operations i.e.

Type1:

$co(coA\oplus B)=\langle co(coA^1\oplus B^1),co(coA^2\oplus B^2),co(coA^3\ominus B^3)\rangle$

$$co(coA\tilde{\oplus}B)=\left\langle co(coA^1\oplus B^1),co(coA^2\oplus B^2),co(coA^3\Theta B^3)\right\rangle$$

$$=\left\langle A^1\Theta B^1,A^2\Theta B^2,A^3\oplus B^3\right\rangle$$

$\langle A^1\ominus B^1,A^2\ominus B^2,A^3\oplus B^3\rangle$ $A\tilde{\Theta}B=$

Type2:

$co(coA\oplus B)=\langle co(coA^1\oplus B^1),co(coA^2\ominus B^2),co(coA^3\ominus B^3)\rangle$

=

$$\left\langle A^1\Theta B^1,A^2\oplus B^2,A^3\oplus B^3\right\rangle$$

$\langle A^1\ominus B^1,A^2\oplus B^2,A^3\oplus B^3\rangle$

$\langle A^1\ominus B^1,A^2\ominus B^2,A^3\oplus B^3\rangle$ $A\tilde{\Theta}B=$

6.3 Properties of the Neutrosophic Crisp Opening Operator:

6.3.1 Proposition:

The neutrosophic opening satisfies the monotonicity $\forall A,B\in\mathcal{NC}(Z^2)$

$$Type1:\ A\subseteq B\Rightarrow\left\langle A^1\circ C^1,A^2\circ C^2,A^3\circ C^3\right\rangle$$

$$\subseteq\left\langle B^1\circ C^1,B^2\circ C^2,B^3\circ C^3\right\rangle$$

$$A^1\circ C^1\subseteq B^1\circ C^1,A^2\circ C^2\subseteq B^2\circ C^2,$$

$$A^3\circ C^3\supseteq B^3\circ C^3$$

$$Type2:\ A\subseteq B\Rightarrow\left\langle A^1\circ C^1,A^2\circ C^2,A^3\circ C^3\right\rangle$$

$$\subseteq\left\langle B^1\circ C^1,B^2\circ C^2,B^3\circ C^3\right\rangle$$

$$A^1\circ C^1\subseteq B^1\circ C^1,A^2\circ C^2\supseteq B^2\circ C^2$$

$$A^3\circ C^3\supseteq B^3\circ C^3$$

6.3.2 Proposition: for any family in $(A_i|i\in I)$ $(A_i|i\in I)$ $\mathcal{NC}(Z^2)$ and $B\in\mathcal{NC}(Z^2)$

$$Type1:\ \bigcap_{i\in I}A_i\tilde{\circ}B=\bigcap_{i\in I}(A_i\tilde{\circ}B)$$

) $\cap_{i\in I}A_i\tilde{\ominus}B$ $\cap_{i\in I}(A_i\tilde{\ominus}B)$

$$\left\langle \bigcap_{i\in I}A_i^1\circ B^1,\bigcap_{i\in I}A_i^2\circ B^2,\bigcap_{i\in I}A_i^3\bullet B^3\right\rangle$$

$$=\left\langle \bigcap_{i\in I}(A_i^1\circ B^1),\bigcap_{i\in I}(A_i^2\circ B^2),\bigcap_{i\in I}(A_i^3\bullet B^3)\right\rangle$$

$$Type2:\ \bigcap_{i\in I}A_i\tilde{\circ}B=\bigcap_{i\in I}(A_i\tilde{\circ}B)$$

) $\cap_{i\in I}A_i\tilde{\ominus}B$ $\cap_{i\in I}(A_i\tilde{\ominus}B)$

$$\left\langle \bigcap_{i\in I}A_i^1\circ B^1,\bigcap_{i\in I}A_i^2\bullet B^2,\bigcap_{i\in I}A_i^3\bullet B^3\right\rangle$$

$$=\left\langle \bigcap_{i\in I}(A_i^1\circ B^1),\bigcap_{i\in I}(A_i^2\bullet B^2),\bigcap_{i\in I}(A_i^3\bullet B^3)\right\rangle$$

6.3.3 Proposition: for any family $(A_i|i\in I)$ in $\mathcal{NC}(Z^2)$ and B $\in$ $\mathcal{NC}(Z^2)$

$$Type1:\ \bigcup_{i\in I}A_i\tilde{\circ}B=\bigcup_{i\in I}(A_i\tilde{\circ}B)$$

) $\cap_{i\in I}A_i\tilde{\ominus}B$ $\cap_{i\in I}(A_i\tilde{\ominus}B)$

$$=\left\langle \bigcup_{i\in I}A_i^1\circ B^1,\bigcup_{i\in I}A_i^2\circ B^2,\bigcup_{i\in I}A_i^3\bullet B^3\right\rangle$$

$$=\left\langle \bigcup_{i\in I}(A_i^1\circ B^1),\bigcup_{i\in I}(A_i^2\circ B^2),\bigcup_{i\in I}(A_i^3\bullet B^3)\right\rangle$$

$$Type2:\bigcup_{i\in I}A_i\tilde{\circ}B=\bigcup_{i\in I}(A_i\tilde{\circ}B)$$

$$=\left\langle \bigcup_{i\in I}A_i^1\circ B^1,\bigcup_{i\in I}A_i^2\bullet B^2,\bigcup_{i\in I}A_i^3\bullet B^3\right\rangle$$

$$=\left\langle \bigcup_{i\in I}(A_i^1\circ B^1),\bigcup_{i\in I}(A_i^2\bullet B^2),\bigcup_{i\in I}(A_i^3\bullet B^3)\right\rangle$$

Proof: Is similar to the procedure used to prove the propositions given in § 6.1.3 and § 6.2.3.

6.4 Properties of the Neutrosophic Crisp Closing

6.4.1 Proposition:

The neutrosophic closing satisfies the monotonicity

$\forall A, B \in \mathcal{NC}(Z^2)$

Type1:

a) $A \subseteq B \Rightarrow \langle A^1 \bullet C^1, A^2 \bullet C^2, A^3 \bullet C^3 \rangle$

$\subseteq \langle B^1 \bullet C^1, B^2 \bullet C^2, B^3 \bullet C^3 \rangle$

$A^1 \bullet C^1 \subseteq B^1 \bullet C^1, A^2 \bullet C^2 \subseteq B^2 \bullet C^2,$

$A^3 \bullet C^3 \supseteq B^3 \bullet C^3$

Type2:

a) $A \subseteq B \Rightarrow \langle A^1 \bullet C^1, A^2 \bullet C^2, A^3 \bullet C^3 \rangle$

$\subseteq \langle B^1 \bullet C^1, B^2 \bullet C^2, B^3 \bullet C^3 \rangle$

$A^1 \bullet C^1 \subseteq B^1 \bullet C^1, A^2 \bullet C^2 \supseteq B^2 \bullet C^2,$

$A^3 \bullet C^3 \supseteq B^3 \bullet C^3$

6.4.2 Proposition: for any family $(A_i | i \in I)$ $(A_i | i \in I)$

In $\mathcal{NC}(Z^2)$ and $B \in \mathcal{NC}(Z^2)$

Type1: $\bigcap_{i \in I} A_i \tilde{\bullet} B = \bigcap_{i \in I} (A_i \tilde{\bullet} B)$) $\cap_{i \in I} A_i \tilde{\ominus} B$

$\cap_{i \in I} (A_i \tilde{\ominus} B)$

$$\left\langle \bigcap_{i \in I} A_i^1 \bullet B^1, \bigcap_{i \in I} A_i^2 \bullet B^2, \bigcap_{i \in I} A_i^3 \circ B^3 \right\rangle$$

$$= \left\langle \bigcap_{i \in I} (A_i^1 \bullet B^1), \bigcap_{i \in I} (A_i^2 \bullet B^2), \bigcap_{i \in I} (A_i^3 \circ B^3) \right\rangle$$

Type2: $\bigcap_{i \in I} A_i \tilde{\bullet} B = \bigcap_{i \in I} (A_i \tilde{\bullet} B)$) $\cap_{i \in I} A_i \tilde{\ominus} B$

$\cap_{i \in I} (A_i \tilde{\ominus} B)$

$$\left\langle \bigcap_{i \in I} A_i^1 \bullet B^1, \bigcap_{i \in I} A_i^2 \circ B^2, \bigcap_{i \in I} A_i^3 \circ B^3 \right\rangle$$

$$= \left\langle \bigcap_{i \in I} (A_i^1 \bullet B^1), \bigcap_{i \in I} (A_i^2 \circ B^2), \bigcap_{i \in I} (A_i^3 \circ B^3) \right\rangle$$

6.4.3 Proposition: for any family $(A_i | i \in I)$ $(A_i | i \in I)$

in $\mathcal{NC}(Z^2)$ and $B \in \mathcal{NC}(Z^2)$

Type1: $\bigcup_{i \in I} A_i \tilde{\bullet} B = \bigcup_{i \in I} (A_i \tilde{\bullet} B)$) $\cap_{i \in I} A_i \tilde{\ominus} B$

$\cap_{i \in I} (A_i \tilde{\ominus} B)$

$$= \left\langle \bigcup_{i \in I} A_i^1 \bullet B^1, \bigcup_{i \in I} A_i^2 \bullet B^2, \bigcup_{i \in I} A_i^3 \circ B^3 \right\rangle$$

$$= \left\langle \bigcup_{i \in I} (A_i^1 \bullet B^1), \bigcup_{i \in I} (A_i^2 \bullet B^2), \bigcup_{i \in I} (A_i^3 \circ B^3) \right\rangle$$

Type2: $\bigcup_{i \in I} A_i \tilde{\bullet} B = \bigcup_{i \in I} (A_i \tilde{\bullet} B)$

$$= \left\langle \bigcup_{i \in I} A_i^1 \bullet B^1, \bigcup_{i \in I} A_i^2 \circ B^2, \bigcup_{i \in I} A_i^3 \circ B^3 \right\rangle$$

$$= \left\langle \bigcup_{i \in I} (A_i^1 \bullet B^1), \bigcup_{i \in I} (A_i^2 \circ B^2), \bigcup_{i \in I} (A_i^3 \circ B^3) \right\rangle$$

Proof: Is similar to the procedure used to prove the propositions given in § 6.1.3.

6.4.4 Proposition (*Duality theorem of Closing*)**:**

let A, $B \in \mathcal{NC}(Z^2)$; Neutrosophic erosion and dilation are dual operations i.e.

Type1:

$co(coA \tilde{\bullet} B) = \langle co(coA^1 \bullet B^1), co(coA^2 \bullet B^2), co(coA^3 \circ B^3) \rangle$

$co(coA \tilde{\bullet} B) = \langle co(coA^1 \bullet B^1), co(coA^2 \bullet B^2), co(coA^3 \circ B^3) \rangle$

$$= \langle A^1 \circ B^1, A^2 \circ B^2, A^3 \bullet B^3 \rangle = A \tilde{\circ} B$$

Type2:

$co(coA \tilde{\bullet} B) = \langle co(coA^1 \bullet B^1), co(coA^2 \circ B^2), co(coA^3 \circ B^3) \rangle$

$$= \langle A^1 \circ B^1, A^2 \bullet B^2, A^3 \bullet B^3 \rangle = A \tilde{\circ} B$$

7. Neutrosophic Crisp Mathematical Morphological Filters:

7.1 Neutrosophic Crisp External Boundary:

Where A^1 A^1 is the set of all pixels that belong to the foreground of the picture, A^3 A^3 contains the pixels that belong to the background whilecontains those A^2 A^2 pixel which do not belong to neither. A^3 nor A^1 A^1 A^3

Let A, $B \in \mathcal{NC}(Z^2)$, such that $A = \langle A^1, A^2, A^3 \rangle$ and B is some structure element of the form $B = \langle B^1, B^2, B^3 \rangle$; then the NC boundary extraction filter is defined to be:

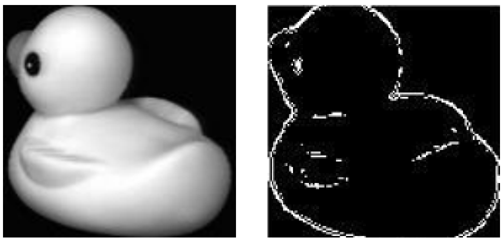
$$\partial_1 A^1 = A^1 - (A^1 \Theta B^1)$$

$$\partial_3 A^3 = (A^3 \Theta B^3) - A^3,$$

$$\partial(A) = A^2 - (\partial_1 A^1 \cup \partial_3 A^3)$$

$$\partial^*(A) = A^2 - [(A^3 \oplus B^3) - (A^1 \Theta B^1)]$$

$$b(A) = \partial^*(A) \cap \partial(A)$$

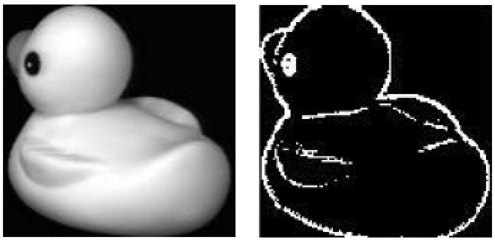


a) b)

Fig. 14: *Applying the neutrosophic crisp External boundary: a) the Original image b) Neutrosophic crisp boundary.*

7.2 Neutrosophic Crisp Top-hat Filter:

$$B_1(A^1) = A^1 - (A^1 \circ B^1)$$
$$B_3(A^3) = (A^3 \bullet B^3) - A^3$$
$$B(A) \;= A^2 - (B_1(A^1) \cup B_3(A^3))$$
$$B^*(A) \;= A^2 - \left[(A^1 \circ B^1) - (A^3 \bullet B^3)\right]$$
$$\widetilde{Top}_{hat}(A) = B(A) \cap B^*(A)$$

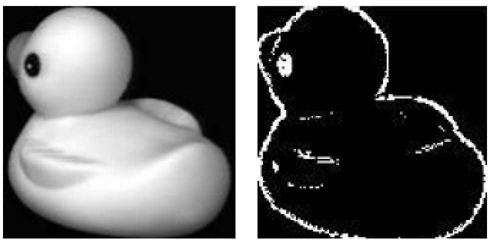


a) b)

Fig. 15: *Applying the Neutrosophic crisp top-hat filter: a) Original image b) Neutrosophic Crisp components $\left\langle A^1, A^2, A^3 \right\rangle$ respectively*

7.3 Bottom-hat filter:

$$B_1(A^1) = (A^1 \bullet B^1) - A^1$$
$$B_3(A^3) = A^3 - (A^3 \circ B^3)$$
$$B(A) \;= A^2 - (B_1(A^1) \cup B_3(A^3))$$
$$B^*(A) \;= A^2 - \left[(A^1 \bullet B^1) - (A^3 \circ B^3)\right]$$
$$\widetilde{Bottom}_{hat}(A) = B(A) \cap B^*(A)$$



a) b)

Fig. 16: *Applying the Neutrosophic crisp Bottom-hat filter: Neutrosophic Crisp components $\left\langle A^1, A^2, A^3 \right\rangle$ respectively*

8 Conclusion:

In this paper we established a foundation for what we called "Neutrosophic Crisp Mathematical Morphology". Our aim was to generalize the concepts of the classical mathematical morphology.

For this purpose, we developed serval neutrosophic crisp morphological operators; namley, the neutrosophic crisp dilation, the neutrosophic crisp erosion, the neutrosophic crisp opening and the neutrosophic crisp closing operators. These operators were presented in two different types, each type is determined according to the behaviour of the seconed component of the triple strucure of the operator.

Furthermore, we developed three neutrosophic crisp morphological filters; namely, the neutrosophic crisp boundary extraction, the neutrosophic crisp Top-hat and the neutrosophic crisp Bottom-hat filters.

Some promising expermintal results were presented to visualise the effect of the new introduced operators and filters on the image in the neutrosophic domain instead of the spatial domain.

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