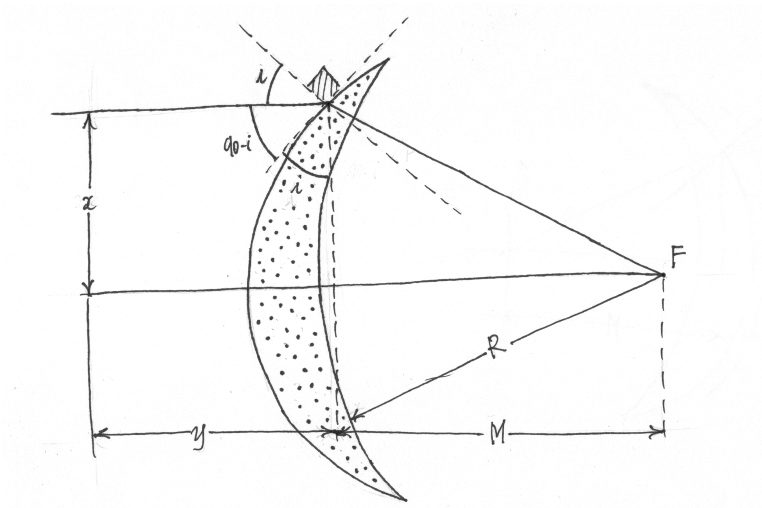
I search to obtain a optimal lens, that work as a Newtonian parabolic reflective lens, using a polynomial surface that deflects the rays in a single focal point




I obtain an optimal lens with an index of refraction n (one polynomial lens for each n).

The geometry of the lens is polynomial:

$$y = \sum_j \alpha_j x^j$$
$$\dot{y} = \sum_j j\alpha_j x^{j-1} = \tan(i)$$
$$\frac{\sin(i)}{\sin(r)} = n$$
$$\tan(i) = \frac{\sin(i)}{\cos(i)} = \frac{cos(90-i)}{\sin(90-i)} = \frac{1}{\tan(90-i)} = \dot{y}$$
$$\sin(i) = \frac{\tan(i)}{\sqrt{1+\tan^2(i)}} = \frac{\dot{y}}{\sqrt{1+\dot{y}^2}}$$
$$\tan(r) = \frac{\sin(r)}{\cos(r)} = \frac{\sin(i)}{\sqrt{n^2-\sin^2(i)}} = \frac{\dot{y}}{\sqrt{(n^2-1)\dot{y}^2+n^2}}$$
$$\frac{M}{x} = \tan(90-i+r) = \frac{\tan(90-i)+\tan(r)}{1-\tan(90-i)\tan(r)} = \frac{\frac{1}{\dot{y}}+\frac{\dot{y}}{\sqrt{(n^2-1)\dot{y}^2+n^2}}}{1-\frac{1}{\sqrt{(n^2-1)\dot{y}^2+n^2}}} = \frac{\sqrt{(n^2-1)\dot{y}^2+n^2}+\dot{y}^2}{\dot{y}\sqrt{(n^2-1)\dot{y}^2+n^2}-\dot{y}}$$

for each ray $y + M = F$, so each ray converge in a focus F:

$$E = \frac{1}{2}\sum_s E_s^2 = \frac{1}{2}\sum_s \left[y + x\frac{\sqrt{(n^2-1)\dot{y}^2+n^2}+\dot{y}^2}{\dot{y}\sqrt{(n^2-1)\dot{y}^2+n^2}-\dot{y}} - F\right]^2$$

$$\frac{\partial y}{\partial \alpha_j} = x^j$$

$$\frac{\partial \dot{y}}{\partial \alpha_j} = jx^{j-1}$$

$$\frac{\partial E}{\partial \alpha_j} = \sum_s E_s \left[x_s^j \frac{\partial E_s}{\partial y} + jx_s^{j-1}\frac{\partial E_s}{\partial \dot{y}}\right] =$$

$$= \sum_s E_s \{x_s^j +$$

$$+ jx_s^{j-1}\left[x_s \frac{\left(\frac{(n^2-1)\dot{y}}{\sqrt{(n^2-1)\dot{y}^2+n^2}}+2\dot{y}\right)\left(\dot{y}\sqrt{(n^2-1)\dot{y}^2+n^2}-\dot{y}\right)-\left(\sqrt{(n^2-1)\dot{y}^2+n^2}+\dot{y}^2\right)\left(\sqrt{(n^2-1)\dot{y}^2+n^2}+\frac{(n^2-1)\dot{y}^2}{\sqrt{(n^2-1)\dot{y}^2+n^2}}-1\right)}{\left(\dot{y}\sqrt{(n^2-1)\dot{y}^2+n^2}-\dot{y}\right)^2}\right]\}$$

the local minimum of the focusing error can be obtained by optimizing each parameter, with Gradient descent, conjugate gradient or other algorithms. Each ray near other ray is focused, and the optimal lens is obtained by increasing the number of the parameters (I think that it is possible to built a metallic form to melt the lens material, or milling the lenses with numerical control machines).

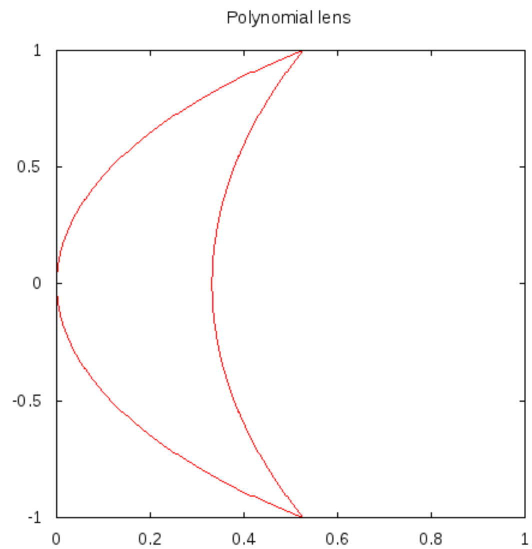
The deflected ray (by the polynomial surface) is normal to the spherical surface that does not deflect the ray, so that the focus does not change.

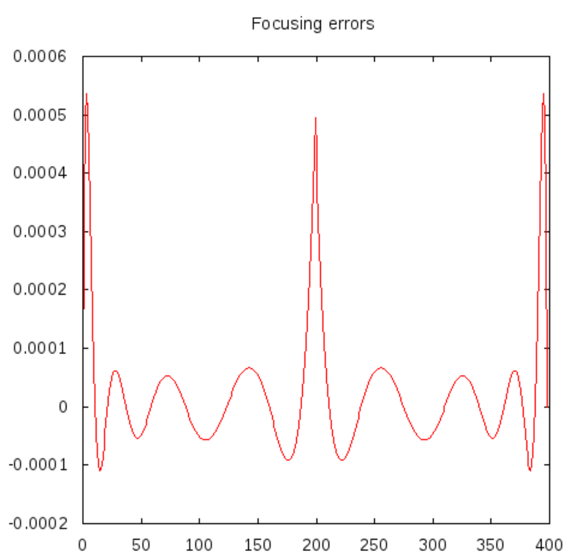
The material used is little (because there are two surface with the same curvature).

The idea is to built mathematically a lens using an infinity of little refractive plane in the space, where each plane deflect the ray in the focus: the movement of the plane in the space cause a rotation of the plane, and the surface obtained by the bring near the lens is the optimal lens.

The polynomial lens have this approximate solution:

$$
\begin{aligned}
y = {} & 4.509960156302713814 \cdot 10^{-01} x^{2} + 1.411690780195129557 \cdot 10^{-03} x^{3} + \\
& + 4.766722756582915931 \cdot 10^{-02} x^{4} + 1.673369609310476898 \cdot 10^{-02} x^{5} + \\
& + 6.409001929098001066 \cdot 10^{-04} x^{6} - 3.575231627039537301 \cdot 10^{-04} x^{7} + \\
& + 2.346697739730467916 \cdot 10^{-03} x^{8} + 3.839708723570891638 \cdot 10^{-03} x^{9} + \\
& + 3.524094973499152114 \cdot 10^{-03} x^{10} + 2.182457511548385185 \cdot 10^{-03} x^{11} + \\
& + 6.415248487680839153 \cdot 10^{-04} x^{12} - 5.942704474358375196 \cdot 10^{-04} x^{13} + \\
& - 1.330361289664431217 \cdot 10^{-03} x^{14} - 1.573844644684032574 \cdot 10^{-03} x^{15} + \\
& - 1.432307459799034336 \cdot 10^{-03} x^{16} - 1.044714309323199766 \cdot 10^{-03} x^{17} + \\
& - 5.421795186805168477 \cdot 10^{-04} x^{18} - 2.987468533779639541 \cdot 10^{-05} x^{19} + \\
& + 4.181536532802970766 \cdot 10^{-04} x^{20} + 7.572263750603347831 \cdot 10^{-04} x^{21} + \\
& + 9.671502126585796699 \cdot 10^{-04} x^{22} + 1.046431895078397894 \cdot 10^{-03} x^{23} + \\
& + 1.006755618206826407 \cdot 10^{-03} x^{24} + 8.682503235851485097 \cdot 10^{-04} x^{25} + \\
& + 6.557088804871227115 \cdot 10^{-04} x^{26} + 3.957307643732150231 \cdot 10^{-04} x^{27} + \\
& + 1.146727039615480282 \cdot 10^{-04} x^{28} - 1.627372161619605002 \cdot 10^{-04} x^{29} + \\
& - 4.142650526063397241 \cdot 10^{-04} x^{30} - 6.206435015405732928 \cdot 10^{-04} x^{31} + \\
& - 7.657580519643944473 \cdot 10^{-04} x^{32} - 8.366323979123896658 \cdot 10^{-04} x^{33} + \\
& - 8.232796449235066787 \cdot 10^{-04} x^{34} - 7.184696345866222623 \cdot 10^{-04} x^{35} + \\
& - 5.174494867621585340 \cdot 10^{-04} x^{36} - 2.176439895309207589 \cdot 10^{-04} x^{37} + \\
& + 1.816456282227110978 \cdot 10^{-04} x^{38} + 6.795321401355409245 \cdot 10^{-04} x^{39} + \\
& + 1.273814139957624426 \cdot 10^{-03} x^{40}
\end{aligned}
$$


Polynomial lens
1
0.5
0
-0.5
-1
0
0.2
0.4
0.6
0.8
1


Focusing errors
0.0006
0.0005
0.0004
0.0003
0.0002
0.0001
0
-0.0001
-0.0002
0
50
100
150
200
250
300
350
400