The prognostic equation for biogeochemical tracers has no unique solution.

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Abstract. In this paper, a tracer prognostic differential equation related to the marine chemistry HAMOC model is studied. Recently the present author found that the Navier - Stokes equation has no unique solution [*Geurdes*, 2017]. The following question can therefore be justified. Do numerical solutions, found from prognostic equations akin to the Navier Stokes equation, provide information as though there is a unique solution.

1. Introduction.

Recent insights into biogeochemical cycles, like the key limiting role of iron as a nutrient [*Martin and Fitzwater*, 1988], justify a computational model search for the distribution of iron in the Oceans. Other elements like phosphorus also play a key role in the growth of chlorophyll. In addition factors such as light are important [*Popova, Ryabchenko and Fasham*, 2000].

In the present paper, we concentrate on chemical tracers that can be described in a prognostic equation. The prognostic is the kernel element of the numerical computations. One thing that catches the eye when looking at prognostic equations for tracers is that their structure resembles somewhat the Navier Stokes equation (NSE). There are numerical models such as HAMOC [*Ilynia, Six, Segschneider, Maier-Reimer, Li and Nunez-Riboni*, 2013] that contain those types of kernel equations. We note that in the field of bio-geochemical cycle study, numerical studies based on prognostic equations can fill in the gaps where data is sparse [*Manizza, Follows, Dutkiewicz, McClelland, Menemenlis, Hill, Townsend-Small and Peterson*, 2009].

Recently, the present author showed that the NSE in three dimensions does not have a solution. Such a conclusion leaves room for so called weak solutions. However, the weak solutions and their numerical equivalents are most likely not unique in the absence of an exact solution. Therefore it is methodologically interesting to turn the attention to tracer prognostics equations as a "NSE next of kin" type of equation. We are allowed to ask if models based on that type of equation can have unique numerical solutions. If there is no unique solution associated to such an equation, the validation of the global cycle models perhaps do not allow the conclusions attached to them presently. Most likely more empirical validation in sampling is then necessary to support the computational modeling.

2. Mathematical model prognostic.

The prognostic equation used by *Dutkiewicz, Follows and Parekh* [2005] can be written down as,

$$\frac{\partial A(x)}{\partial t} + \nabla \cdot (\mathbf{u}(x)A(x)) - \nabla \cdot (\mathbf{K}(x)A(x)) = S_A(x) \tag{1}$$

This equation resembles the kernel of the HAMOC model equation. Accordingly, this is equation (1) in *Ilynia, Six, Segschneider, Maier-Reimer, Li and Nunez-Riboni* [2013]. In our equation (1), the abbreviation, $x = (\mathbf{x}, t) \in \mathbb{R}^3 \times \mathbb{R}^+$, represents the space and time coordinates. Let us take t in a finite interval $[0, T]$ with $0 < T < \infty$. Furthermore, $\mathbf{u}(x)$, is the "transformed mean Eulerian" circulation, $\mathbf{K}(x)$ the mixing tensor and $S_A(x)$ the sources and sinks of the tracer, which concentration is denoted by $A(x)$.

2.1. Approximate interval

In this sub section we look at an approximation which appears to be valid given the context of oceanic bio-cycle studies. Let us concentrate on a certain fully connected finite space and time set $\Omega \subset \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$. Suppose $\mathbf{x} = (x_1, x_2, x_3)$ and $t \in [0, T]$, then Ω is the cartesian product of intervals $\Omega = [x_{1,0}, x_{1,1}] \times [x_{2,0}, x_{2,1}] \times [x_{3,0}, x_{3,1}] \times [t_0, t_1]$. Here, $x_{m,0} < x_{m,1}$, for $m = 1, 2, 3$, and $t_0 < t_1$. Now for, $x \in \Omega$ we may approximate the tracer equation in (1) with $\mathbf{u}(x) \approx \mathbf{u}^0$ a constant vector in $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ approximating the Eulerian circulation in Ω. Subsequently, $\mathbf{K}(x) \approx \mathbf{K}^0$ the approximate constant mixing tensor and approximate constant source-sink $S_A(x) \approx S_A^0$. Those approximate constant

$\mathbf{u}, \mathbf{K}$ and S_A are reasonable given the physical problem at hand. We suppose symmetrical mixing $K^0_{m,j} = K^0_{j,m}$. If we then look at $A^0 = A - tS^0_A$, then the prognostic equation can be written like

$$\frac{\partial A^0(x)}{\partial t} + \left(\mathbf{u}^0 \cdot \nabla\right) A^0(x) - \nabla \cdot \left(\mathbf{K}^0 A^0(x)\right) = 0 \tag{2}$$

Here the right hand approximation is replaced with an equal sign for ease of notation. In fact $A^0 \approx A - tS^0_A$ which is, very likely, close to an equal sign given the physical situation. Let us, further, introduce a spectral parameter breakdown of the solution such as in [*Quispel*, 1983]. E.g. we can have a (real) measure ν and real parameter $k \in \mathbb{R}$ such that

$$A^0(x) = \int_{\mathbb{R}} d\nu(k) A^0_k(x) \tag{3}$$

Subsequently let us look at functions carrying a spectral parameter, k. One would certainly agree that if we find two different functions, $A^0_{1,k}(x)$ and $A^0_{2,k}(x)$ that solve (2), then, via the integration over the measure ν in (3), two different functions $A^0_1(x)$ and $A^0_2(x)$ are obtained. Before introducing such two function let us first introduce the plane wave [*Quispel*, 1983], factor for $x \in \Omega$,

$$\rho_{k,\mathbf{b}}(x) = \exp\left[-||\mathbf{b}||^2 k^2 t - ik\left(\mathbf{b} \cdot \mathbf{Lx}\right)\right] \tag{4}$$

In the definition of the plane wave the $||\mathbf{b}||^2$ represents the Euclidean norm of the constants vector $\mathbf{b}$. Furthermore, $\mathbf{L}$ is a real 3×3 constant-in-x tensor (matrix) such that, $p = 1,2,3,\ n = 1,2,3$

$$\mathbf{b} \cdot \mathbf{Lx} = \sum_p b_p \sum_n L_{p,n} x_n \tag{5}$$

Hence we find that, for $m = 1,2,3$,

$$\frac{\partial}{\partial x_m}\left(\mathbf{b} \cdot \mathbf{Lx}\right) = \sum_p b_p L_{p,m} \tag{6}$$

2.2. Integral equation

Starting from the $x \in \Omega$, it is possible to write down an integral equation that resembles the ones that are employed for linearizion of nonlinear partial differential equations [*Nijhoff, Quispel, van der Linden and Capel*, 1983]

$$A^0_{1,k}(x) + iA^0_{2,k}(x) = \sum_q r_{q,k}\rho_{iq,\mathbf{a}}(x) + i\sum_q s_{q,k}\rho_{iq,\mathbf{b}}(x) \tag{7}$$

In this equation, $q \in \mathbb{Z}^+ - \{0\}$, together with, $r_{q,k} \in \mathbb{C}$ and, $s_{q,k} \in \mathbb{R}$. Note that **a** and **b** are different vectors. Moreover, the $A^0_{2,k}(x)$ in equation (7) is defined by

$$A^0_{2,k}(x) = \iint_{\mathbb{R}^2} d\lambda(\ell)d\lambda(\ell')\frac{\rho_{\ell-\ell',\mathbf{b}}(x)}{\ell+\ell'}A^0_{1,k+\ell+\ell'}(x) \tag{8}$$

Hence, equation(7) is an integral equation comparable to what is employed by *Quispel* [1983]. In the equation (8) the λ is a real measure and ℓ and ℓ' are real numbers. If, for arbitrary k, we have $A^0_{1,k}(x) \in \mathbb{R}$ then it follows, because of symmetry in exchange of ℓ and ℓ', while noting $\{\rho_{\ell-\ell',\mathbf{b}}(x)\}^* = \rho_{-\ell+\ell',\mathbf{b}}(x)$ and $*$ is complex conjugation, that $A^0_{2,k}(x)$ is a real number too. Moreover, it is noted that if a real (domain and co-domain) operator $\mathcal{O}$ exists such that the right-hand side of the integral equation (5) vanishes, then we may obtain that, $\mathcal{O}A^0_{1,k}(x) + i\mathcal{O}A^0_{2,k}(x) = 0$. If $\mathcal{O}A^0_{j,k}(x) \in \mathbb{R}$, for, $j = 1, 2$, then $\mathcal{O}A^0_{j,k}(x) = 0$, for, $j = 1, 2$.

2.3. Operations and parameters for linearization of the approximate prognostic in $x \in \Omega$.

2.3.1. The $\mathbf{u^0} \cdot \boldsymbol{\nabla}$ operator

Let us start the investigation of the approximate prognostic by noting that the operation $\mathbf{u}^0 \cdot \nabla$ can be written in a sum format $\sum_m u^0_m \frac{\partial}{\partial x_m}$, with $m = 1, 2, 3$. Hence applying this operation to the left and right hand side of the integral equation in (7) we find

$$\sum_m u_m^0 \frac{\partial A_{1,k}^0(x)}{\partial x_m} + i \sum_m u_m^0 \frac{\partial A_{2,k}^0(x)}{\partial x_m} = \sum_m u_m^0 \frac{\partial}{\partial x_m} \left[\sum_q r_{q,k} \rho_{iq,\mathbf{a}}(x) + i \sum_q s_{q,k} \rho_{iq,\mathbf{b}}(x) \right] \quad (9)$$

If we for the moment write $\mathbf{y}$ for $\mathbf{a}$ or $\mathbf{b}$ it can be acknowledged that

$$\frac{\partial}{\partial x_m} \rho_{iq,\mathbf{y}}(x) = q \rho_{iq,\mathbf{y}}(x) \frac{\partial}{\partial x_m} (\mathbf{y} \cdot \mathbf{L}\mathbf{x}) \quad (10)$$

With the use of the expression in (6) it follows

$$\frac{\partial}{\partial x_m} \rho_{iq,\mathbf{y}}(x) = q \rho_{iq,\mathbf{y}}(x) \sum_p y_p L_{p,m} \quad (11)$$

with, for completeness, $q \in \mathbb{Z}^+ - \{0\}$ and $p, m = 1, 2, 3$. Remembering that $\mathbf{y}$ can represent $\mathbf{a}$ or $\mathbf{b}$ the following observation can be made. We are in $\mathbb{R}^3$. Hence it is possible to have three mutual orthogonal vectors. Hence, looking at $\mathbf{u}^0$, it is possible to select $\mathbf{a}$ and $\mathbf{b}$ such that in combination with $\mathbf{L}$

$$\sum_m u_m^0 \sum_p a_p L_{p,m} = \sum_m u_m^0 \sum_p b_p L_{p,m} = 0 \quad (12)$$

Returning to equation (9) and bearing in mind the above exercise with the operator $\mathbf{u}^0 \cdot \nabla$, it can be concluded that $\mathbf{u}^0 \cdot \nabla$ is a $\mathcal{O}$-type operator aluded to in section-2.2. Hence, $\mathbf{u}^0 \cdot \nabla A_{1,k}^0(x) = \mathbf{u}^0 \cdot \nabla A_{2,k}^0(x) = 0$.

2.3.2. The $\frac{\partial}{\partial t} - \nabla \cdot (\mathbf{K}^0 \nabla)$ operator

With the result of section-2.3.1 in mind let us turn the attention to the rest of the operator employed in the approximate prognostic equation. The operator $\frac{\partial}{\partial t} - \nabla \cdot (\mathbf{K}^0 \nabla)$ is complicated. Let us start with looking at $\frac{\partial}{\partial t}$. This operation is applied to the right hand side of the integral equation (7). The tactics of how to approach the operator $\frac{\partial}{\partial t}$ on the right hand side of the integral equation resembles that of the section-2.3.1. So we may look at

$$\frac{\partial}{\partial t}\rho_{iq,\mathbf{y}}(x) = ||\mathbf{y}||^2 q^2 \rho_{iq,\mathbf{y}}(x) \tag{13}$$

Applying the operator $\nabla \cdot (\mathbf{K}^0 \nabla)$ to the right hand of the integral equation (7) implies

$$\sum_{m,j} \frac{\partial}{\partial x_m} K^0_{m,j} \frac{\partial}{\partial x_j} \rho_{iq,\mathbf{y}}(x) = q \sum_{m,j} \frac{\partial}{\partial x_m} \rho_{iq,\mathbf{y}}(x) K^0_{m,j} \sum_p y_p L_{p,j} = \\ q^2 \rho_{iq,\mathbf{y}}(x) \sum_{m,j} K^0_{m,j} \sum_{p'} y_{p'} L_{p',m} \sum_p y_p L_{p,j} \tag{14}$$

Subsequent reshuffling of the sums gives

$$\sum_{m,j} \frac{\partial}{\partial x_m} K^0_{m,j} \frac{\partial}{\partial x_j} \rho_{iq,\mathbf{y}}(x) = q^2 \rho_{iq,\mathbf{y}}(x) \sum_{p'} \sum_p y_{p'} y_p \sum_{m,j} K^0_{m,j} L_{p',m} L_{p,j} = \\ q^2 \rho_{iq,\mathbf{y}}(x) \sum_{p'} \sum_p y_{p'} y_p \delta_{p,p'} = q^2 \rho_{iq,\mathbf{y}}(x) ||\mathbf{y}||^2 \tag{15}$$

Here, the $\mathbf{L}$ matrix is such that

$$\sum_{m,j} K^0_{m,j} L_{p',m} L_{p,j} = \delta_{p,p'} \tag{16}$$

This is the condition for (15), $m, j, p, p' = 1, 2, 3$ and $q \in \mathbb{Z}^+ - \{0\}$. Subsequently, looking at (13) and the result of (15) then

$$\left(\frac{\partial}{\partial t} - \nabla \cdot \left(\mathbf{K}^0 \nabla \right) \right) \rho_{iq,\mathbf{y}}(x) = 0 \tag{17}$$

Hence, the operator $\frac{\partial}{\partial t} - \nabla \cdot (\mathbf{K}^0 \nabla)$ is similarly of the type of $\mathcal{O}$ alluded to in section-2.2. Looking at the integral equation (7) this implies that

$$\left(\frac{\partial}{\partial t} - \nabla \cdot \left(\mathbf{K}^0 \nabla \right) \right) A^0_{n,k}(x) = 0 \tag{18}$$

and here, $n = 1, 2$. Hence, combining this result with $\mathbf{u}^0 \cdot \nabla A^0_{1,k}(x) = \mathbf{u}^0 \cdot \nabla A^0_{2,k}(x) = 0$, it must be possible for the approximate prognostic equation to have two solutions at least. One solution is derived from $A^0_{1,k}(x)$ the other from $A^0_{2,k}(x)$, which in fact is an integral transformation, given in (8), of $A^0_{1,k}(x)$.

2.4. Consistency for $A^0_{2,k}$ under $\frac{\partial}{\partial t} - \nabla \cdot (\mathbf{K}^0 \nabla)$

In order to study the consistency we need to look at the transformation in (8). The two x dependent terms are of the greatest importance in the evaluation. Therefore let us introduce the abbreviation $\langle \cdot \rangle$ to denote the weighted integration in the transformation (8). Hence,

$$A^0_{2,k}(x) = \left\langle \rho_{\ell-\ell',\mathbf{b}}(x) A^0_{1,g}(x) \right\rangle \tag{19}$$

with the abbreviation, $g = k + \ell + \ell'$. Looking at the operator form we find

$$\frac{\partial}{\partial t} A^0_{2,k}(x) = \langle -(\ell - \ell')^2 ||\mathbf{b}||^2 \rho_{\ell-\ell',\mathbf{b}}(x) A^0_{1,g}(x) \rangle + \left\langle \rho_{\ell-\ell',\mathbf{b}}(x) \frac{\partial}{\partial t} A^0_{1,g}(x) \right\rangle \tag{20}$$

The second part of the operator

$$\nabla \cdot \left(\mathbf{K}^0 \nabla\right) A^0_{2,k}(x) = \left\langle \sum_{m,j} \frac{\partial}{\partial x_m} K^0_{m,j} \frac{\partial}{\partial x_j} \rho_{\ell-\ell',\mathbf{b}}(x) A^0_{1,g}(x) \right\rangle =$$
$$\left\langle \sum_{m,j} \frac{\partial}{\partial x_m} K^0_{m,j} \rho_{\ell-\ell',\mathbf{b}}(x) \left\{ (-i)(\ell - \ell') \sum_p b_p L_{p,j} A^0_{1,g}(x) + \frac{\partial}{\partial x_j} A^0_{1,g}(x) \right\} \right\rangle \tag{21}$$

So introducing the $\frac{\partial}{\partial x_m}$ gives, suppressing x in the notation for convenience,

$$\nabla \cdot \left(\mathbf{K}^0 \nabla\right) A^0_{2,k} =$$
$$\left\langle \rho_{\ell-\ell',\mathbf{b}} \sum_{m,j} K^0_{m,j} (-i)(\ell - \ell') \sum_{p'} b_{p'} L_{p',m} \left\{ (-i)(\ell - \ell') \sum_p b_p L_{p,j} A^0_{1,g} + \frac{\partial A^0_{1,g}}{\partial x_j} \right\} \right\rangle +$$
$$\left\langle \rho_{\ell-\ell',\mathbf{b}} \sum_{m,j} K^0_{m,j} \left\{ (-i)(\ell - \ell') \sum_p b_p L_{p,j} \frac{\partial A^0_{1,g}}{\partial x_m} + \frac{\partial^2 A^0_{1,g}}{\partial x_j \partial x_m} \right\} \right\rangle \tag{22}$$

In the previous equation the form of equation (16) resides. This gives $\delta_{p,p'}$. Hence ,

$$W_1 = \left\langle (-1)(\ell - \ell')^2 ||\mathbf{b}||^2 \rho_{\ell-\ell',\mathbf{b}}(x) A^0_{1,g}(x) \right\rangle \tag{23}$$

Because $K^0_{m,j}$ is symmetric $m, j = 1, 2, 3$, the second term in (22) is

$$W_2 = \left\langle \rho_{\ell-\ell',\mathbf{b}}(x) \sum_{m,j} K^0_{m,j} \left\{ (-2i)(\ell - \ell') \sum_p b_p L_{p,j} \frac{\partial A^0_{1,g}(x)}{\partial x_m} \right\} \right\rangle \tag{24}$$

The third term in fact is equal to

$$W_3 = \left\langle \rho_{\ell-\ell',\mathbf{b}}(x) \nabla \cdot \left(\mathbf{K}^0 \nabla\right) A^0_{1g}(x) \right\rangle \tag{25}$$

The previous three forms allow the following conclusion

$$\left(\frac{\partial}{\partial t} - \nabla \cdot \left(\mathbf{K}^0 \nabla\right)\right) A^0_{2,k}(x) = 2i \left\langle \rho_{\ell-\ell',\mathbf{b}}(x) \sum_{m,j} K^0_{m,j} \left\{ (\ell - \ell') \sum_p b_p L_{p,j} \frac{\partial A^0_{1,g}(x)}{\partial x_m} \right\} \right\rangle \tag{26}$$

when we accept that $\left(\frac{\partial}{\partial t} - \nabla \cdot (\mathbf{K}^0 \nabla)\right) A^0_{1,k}(x) = 0$, for arbitrary k. Looking at (20), the first term here drops off against W_1 in (23). The second term of (20) cancels against W_3 of (25), because we have $\forall_{k \in \mathbb{R}} \left(\frac{\partial}{\partial t} - \nabla \cdot (\mathbf{K}^0 \nabla)\right) A^0_{1,k}(x) = 0$.

This implies that for consistency we need to have, $W_2 = 0$ from (24)

$$\left\langle (\ell - \ell') \rho_{\ell-\ell',\mathbf{b}}(x) \sum_{m,j,p} b_p K^0_{m,j} L_{p,j} \frac{\partial A^0_{1,g}(x)}{\partial x_m} \right\rangle = 0 \tag{27}$$

2.4.1. The operator $\sum_{m,j,p} b_p K^0_{m,j} L_{p,j} \frac{\partial}{\partial x_m}$.

The result in (27) and therefore the consistency, $\left(\frac{\partial}{\partial t} - \nabla \cdot (\mathbf{K}^0 \nabla)\right) A^0_{2,k}(x) = 0$ can be studied with applying the operator, $\sum_{m,j,p} b_p K^0_{m,j} L_{p,j} \frac{\partial}{\partial x_m}$ to the integral equation in (7). Hence,

$$\begin{aligned} &\sum_{m,j,p} b_p K^0_{m,j} L_{p,j} \frac{\partial}{\partial x_m} A^0_{1,k}(x) + i \sum_{m,j,p} b_p K^0_{m,j} L_{p,j} \frac{\partial}{\partial x_m} A^0_{2,k}(x) = \\ &\qquad \sum_{m,j,p} b_p K^0_{m,j} L_{p,j} \frac{\partial}{\partial x_m} \left(\sum_q r_{q,k} \rho_{iq,\mathbf{a}}(x) + i \sum_q s_{q,k} \rho_{iq,\mathbf{b}}(x) \right) \end{aligned} \tag{28}$$

Similar to e.g. section-2.3.1, the use of $\rho_{iq,\mathbf{y}}(x)$ is justified looking at (28). So, it follows that, ignoring the q sums and coefficients for the moment,

$$\sum_{m,j,p} b_p K^0_{m,j} L_{p,j} \frac{\partial}{\partial x_m} \rho_{iq,\mathbf{y}}(x) = q \rho_{iq,\mathbf{y}}(x) \sum_{m,j,p} b_p K^0_{m,j} L_{p,j} \sum_{p'} y_{p'} L_{p',m} \tag{29}$$

In this equation we may recognize, $\sum_{m,j} K^0_{m,j} L_{p',m} L_{p,j} = \delta_{p,p'}$ from (16). This implies that

$$\sum_{m,j,p} b_p K^0_{m,j} L_{p,j} \frac{\partial}{\partial x_m} \rho_{iq,\mathbf{y}}(x) = q \rho_{iq,\mathbf{y}}(x) \sum_p b_p y_p \tag{30}$$

We note again that $\mathbf{y} = \mathbf{a}$ can be taken looking at the rigt hand side of (28). Hence if $\mathbf{a}$ and $\mathbf{b}$ are orthogonal, the first term in (28) on the right hand side vanishes. Then, because we work with real functions and coefficients, it follows that

$$\sum_{m,j,p} b_p K^0_{m,j} L_{p,j} \frac{\partial}{\partial x_m} A^0_{1,k}(x) = 0 \tag{31}$$

If this is combined with the result from (27) the consistency, $\left(\frac{\partial}{\partial t} - \nabla \cdot (\mathbf{K}^0 \nabla)\right) A^0_{2,k}(x) = 0$ applies. Comparing the imaginary parts on left and right hand of (28) it is also found that, using $\mathbf{y} = \mathbf{b}$ in (30), the consistency requires that

$$\sum_{m,j,p} b_p K^0_{m,j} L_{p,j} \frac{\partial}{\partial x_m} A^0_{2,k}(x) = \sum_q q ||\mathbf{b}||^2 s_{q,k} \rho_{iq,\mathbf{b}}(x) \tag{32}$$

Let us return to the use of the form in (8) and observe the truth of (31). It then clearly follows that the requirement in (32), in terms of $A^0_{1,k}$, is

$$\iint_{\mathbb{R}^2} d\lambda(\ell) d\lambda(\ell') (-i)(\ell - \ell') \frac{\rho_{\ell-\ell',\mathbf{b}}(x)}{\ell + \ell'} A^0_{1,k+\ell+\ell'}(x) = \sum_q q s_{q,k} \rho_{iq,\mathbf{b}}(x) \tag{33}$$

Here equation (16) is employed to the result of the $\frac{\partial}{\partial x_m}$ differentiation on the left hand side, such that the $||\mathbf{b}||^2$ drops off. It can be easily verified, taking the complex conjugate and interchange the ℓ and the ℓ', that the left hand of (33) is real. So, this requirement can be met in principle for $x \in \Omega$. We also note that from $\left(\frac{\partial}{\partial t} - \nabla \cdot (\mathbf{K}^0 \nabla)\right) \rho_{iq,\mathbf{y}}(x) = 0$ in (17) and from equation (31) together with $\left(\frac{\partial}{\partial t} - \nabla \cdot (\mathbf{K}^0 \nabla)\right) A^0_{1,k}(x) = 0$, the equation in (33) is consistent.

3. Conclusion

It was demonstrated that the solution of an approximation of the tracer prognostic equation of *Dutkiewicz, Follows and Parekh* [2005] has no unique solution. This agrees with another finding that the Navier Stokes equation (NSE) in $d = 3$ spatial dimensions has no exact solution [*Geurdes*, 2017]. The form of the prognostic of e.g. *Dutkiewicz, Follows and Parekh* [2005] resembles NSE.

The question can therefore be raised what the numerical solution of [*Dutkiewicz, Follows and Parekh*, 2005] and similar studies is telling us. Of course it has to be acknowledged that the infinite sequence that can be derived from the prognostic equation is approximative. The marine chemistry, however, seems to allow the approximation. Moreover, it also has to be acknowledged that the present infinite sequence of solutions is not tested numerically. Subsequently, we also acknowledge the following. The infinite series of solution has a particular form where the operators $\mathbf{u}^0 \cdot \nabla$ and $\frac{\partial}{\partial t} - \nabla \cdot (\mathbf{K}^0 \nabla)$ working on the start-point solution, $A^0 = A^0_1$, identically vanish. We note that if $A^0 \approx A - tS^0_A$, this is an approximate solution to the prognostic in $x \in \Omega$. Although it was not studied here, it is most likely simple to extend the present analysis to linear integral equations where, for constant c, we have $\mathbf{u}^0 \cdot \nabla A = -c$ and $\left(\frac{\partial}{\partial t} - \nabla \cdot (\mathbf{K}^0 \nabla)\right) A = c$. This justifies the claim that our analysis holds more general value.

The question then is how does this result affect the conclusions of HAMOC model computations. It appears to make sense to claim that when there is no unique solution to the kernel equation of the model then the verification of validity of the model computations cannot come from numerics. At its very least the argument that computation can supplement sparse data, such as claimed by *Manizza, Follows, Dutkiewicz, McClelland,* *Menemenlis, Hill, Townsend-Small and Peterson* [2009], can be doubted. The present author believes that strong evidence was presented for the possibility that prognostic equations and the numeric computations derived thereof have no unique solution.

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