On the Riemann Hypothesis, Complex Scalings and Logarithmic Time Reversal

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May 2017

Abstract

An approach to solving the Riemann Hypothesis is revisited within the framework of the special properties of Θ (theta) functions, and the notion of $\mathcal{CT}$ invariance. The conjugation operation $\mathcal{C}$ amounts to complex scaling transformations, and the $\mathcal{T}$ operation $t \to (1/t)$ amounts to the reversal $log(t) \to -log(t)$. A judicious scaling-like operator is constructed whose spectrum $E_s = s(1-s)$ is real-valued, leading to $s = \frac{1}{2} + i\rho$, and/or $s =$ real. These values are the location of the non-trivial and trivial zeta zeros, respectively. A thorough analysis of the one-to-one correspondence among the zeta zeros, and the orthogonality conditions among pairs of eigenfunctions, reveals that *no* zeros exist off the critical line. The role of the $\mathcal{C}, \mathcal{T}$ transformations, and the properties of the Mellin transform of Θ functions were essential in our construction.

Keywords: Quantum Mechanics, Dirac Operators, Riemann Hypothesis, Hilbert-Polya conjecture, Modularity.

Riemann's outstanding hypothesis (RH) [1] that the non-trivial complex zeros of the zeta-function $\zeta(s)$ must be of the form $s_n = 1/2 \pm i\rho_n$, is one of most important open problems in pure mathematics. The zeta-function has a relation with the number of prime numbers less than a given quantity and the zeros of zeta are deeply connected with the distribution of primes [1]. References [2] are devoted to the mathematical properties of the zeta-function. The RH has also been studied from the point of view of mathematics and physics by [7], [13], [19], [23], [10], [24], [8], [9], [25], [12], [22], [27], [18], [28], [30], among many others . And most recently by [29], [14] and [15].

Let us begin with the one-dimensional differential operators [21], [16]

$$D_1 = -\frac{d}{d\ln t} + \frac{dV}{d\ln t} + k. \tag{1.1}$$

where k is an arbitrary parameter. The eigenvalues s can be complex-valued in general, and its eigenfunctions are

$$\psi_s(t) = t^{-s+k} e^{V(t)}. \tag{1.2a}$$

D_1 is *not* self-adjoint since it is an operator that does not admit an adjoint extension to the whole real line characterized by the *real* variable t. The parameter k is also real-valued. For this reason the eigenvalues of D_1 can be complex valued numbers s. The conjugation operation $\mathcal{C}$ acting on the eigenfunctions is defined as

$$\psi_s(t) \ = \ t^{-s+k} e^{V(t)} \ \rightarrow \ \psi_{s^*}(t) \ = \ t^{-s^*+k} e^{V(t)} \ = \ t^{-s^*+s} \ \psi_s(t). \tag{1.2b}$$

one learns that the conjugation operation $\mathcal{C}$ can also be recast as a *scaling* transformation of $\psi_s(t)$ by t-dependent (local) scaling factors

$$t^{-s^*+s} \ = \ e^{(-s^*+s) \ (ln \ t)} \ = e^{\ 2 \ i \ \mathcal{I}m(s) \ (ln \ t)} \ = \ e^{i\theta_s(t)} \tag{1.2c}$$

which amounts to a t-dependent phase rotation $\theta_s(t)$ which is proportional to the imaginary part of s and to $ln\ t$.

We also define the "mirror" operator to D_1 as follows,

$$D_2 = \frac{d}{d\ln t} - \frac{dV(1/t)}{d\ln t} + k. \tag{1.3}$$

that is related to D_1 by the substitution $t \rightarrow 1/t$ and by noticing that

$$\frac{dV(1/t)}{d\ln(1/t)} = -\frac{dV(1/t)}{d\ln t}. \tag{1.4}$$

where $V(1/t)$ is not equal to $V(t)$ and D_2 is *not* self-adjoint either. The eigenfunctions of the D_2 operator are $\Psi_s(\frac{1}{t})$, with the same eigenvalue s

$$D_2 \ \Psi_s(\frac{1}{t}) \ = \ s \ \Psi_s(\frac{1}{t}) \tag{1.5}$$

A "Wick rotation" of variables $t = iz$ furnishes $z \rightarrow -(1/z)$ which is a truly modular $SL(2,Z)$ transformation $z \rightarrow (az + b/cz + d)$ with unit determinant $ad - bc = 1$.

Out of the infinity of possible choices for $V(t)$, one may choose $V(t)$ which is related to the Bernoulli string spectral counting function, and given by the Jacobi theta series as follows

$$e^{2V(t)} \ = \ \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t^l} \ = \ 2\omega(t^l) + 1. \tag{1.6}$$

where l is another real parameter introduced corresponding to the scaling exponent t^l in eq-(1.6).

The related theta function defined by Gauss is given by

$$G(1/x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2/x} = 2\omega(1/x) + 1. \tag{1.7}$$

where $\omega(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$. The Gauss-Jacobi series obeys the relation

$$G(\frac{1}{x}) \ = \ \sqrt{x}\ G(x). \tag{1.8}$$

resulting from the Poisson re-summation formula.

After setting $e^{2V(t)} = G(t^l) = G(x)$, where $x \equiv t^l$, by recurring to the properties of the Gauss-Jacobi theta series under the $x \to 1/x$ transformations (1.8), and when the parameters l, k are *constrained* to obey the condition $l = 4(2k-1)$, one can show that the eigenfunctions of the D_2 operator $\Psi_s(\frac{1}{t})$, satisfy the key relation $\Psi_{1-s}(t) = \Psi_s(\frac{1}{t})$ [16], [21].

A little algebra reveals that the pair of mirror "Hamiltonians" $H_A = D_2 D_1$ and $H_B = D_1 D_2$, when $l = 4(2k-1)$ have for eigenvalues and eigenfunctions the following

$$H_A\ \Psi_s(t) = s(1-s)\Psi_s(t). \quad H_B\ \Psi_s(\frac{1}{t}) = s(1-s)\Psi_s(\frac{1}{t}). \tag{1.9}$$

due to the relation $\Psi_s(1/t) = \Psi_{1-s}(t)$ based on the modular properties of the Gauss-Jacobi series, $G(\frac{1}{x}) \ = \ \sqrt{x}\ G(x)$. Therefore, despite that H_A, H_B are *not* Hermitian they have the same spectrum $s(1-s)$ which is *real*-valued only in the critical line *and* in the real line. Eq-(1.9) is the one-dimensional version of the eigenfunctions of the two-dimensional hyperbolic Laplacian given in terms of the Eisenstein's series.

The inner product is defined as follows [1]

$$\langle f|g\rangle = \int_0^\infty f^* g \frac{dt}{t}.$$

Based on this definition, the inner product of two eigenfunctions of D_1 is

$$\langle \psi_{s_1}|\psi_{s_2}\rangle \ = \ \int_0^\infty \ e^{2V}\ t^{-s_{12}+2k-1}\ dt; \quad s_{12}\ \equiv\ s_1^* + s_2 \tag{1.10a}$$

A regularization of the integral (1.10a) can be attained by removing the zero $n = 0$ mode of the Gauss-Jacobi series. Upon doing so and performing the change of variables $x = t^l$, it leads to

$$\frac{2}{l}\int_0^\infty \ e^{2V}\ x^{\frac{2(-s_{12}+2k)}{2l}-1}\ dx \ = \ \frac{2}{l} Z\left[\frac{2}{l}(2k - s_{12})\right] \ = \ \frac{2}{l}\ Z[s] \tag{1.10b}$$

[1] At the moment we are not concerned if one has a Banach or a Hilbert space

where we have denoted $s_{12} = s_1^* + s_2 = x_1 + x_2 + i(y_2 - y_1)$, and $s = \frac{2}{l}(-s_{12} + 2k)$. The completed zeta function $Z[s]$ results from the evaluation of the Mellin transform as shown next. It is known that the completed zeta function $Z[s]$ can be expressed in terms of the Jacobi theta series, $\omega(x)$ defined by eqs-(1.6, 1.7) as the integral [2]

$$\begin{aligned}
\int_0^\infty \sum_{n=1}^\infty e^{-\pi n^2 x} x^{s/2-1} dx &= \\
& \int_0^\infty x^{s/2-1} \omega(x) dx \qquad . \\
&= \frac{1}{s(s-1)} + \int_1^\infty [x^{s/2-1} + x^{(1-s)/2-1}] \, \omega(x) dx \\
&= Z(s) = Z(1-s),
\end{aligned} \tag{1.11}$$

where the completed zeta function is defined as

$$Z(s) \equiv \pi^{-s/2} \, \Gamma(\frac{s}{2}) \, \zeta(s). \tag{1.12}$$

and which obeys the functional relation $Z(s) = Z(1-s)$.

To sum up, a family of scaling-like operators D_1, D_2 in one dimension allows to evaluate the inner product of their eigenfunctions $\Psi_s(t)$ (after removing the zero mode of the Gauss-Jacobi theta series) giving $(2/l) Z \, [\frac{2}{l}(2k - s^* - s)]$, where $Z(s)$ is the Riemann completed zeta function and the l, k parameters are constrained to obey $l = 4(2k-1)$ in order to have the relation $\Psi_s(1/t) = \Psi_{1-s}(t)$. Hence, by using the properties of the Gauss-Jacobi series $G(\frac{1}{x}) \; = \; \sqrt{x} \; G(x)$ it follows that under the log-time reversal $\mathcal{T}$ operation $lnt \to -lnt$ (equivalent to $t \to \frac{1}{t}$) the eigenfunctions $\Psi_s(t)$ behave as

$$\mathcal{T} \; \Psi_s(t) \; = \; \Psi_s(\frac{1}{t}) \; = \; \Psi_{1-s}(t). \tag{1.13}$$

In order to avoid the removal of the zero mode $n = 0$ of the Gauss-Jacobi theta series and evaluate the integrals appearing in the inner products, in [16] we proposed a family of theta series where *no* regularization is needed in the construction of the inner products. There is a two-parameter family of theta series $\Theta_{j,m}(t)$ that yield well defined inner products *without* the need to extract the zero mode $n = 0$ divergent contribution. We found that the two parameter family of theta series related to a different choice for $V(t)$ is given by

$$e^{2V_{j,m}(t)} \; = \; \Theta_{j,m}(t^l) \; \equiv \; \sum_{n=-\infty}^{n=\infty} n^{2m} \; H_{2j}(n\sqrt{2\pi t^l}) \; e^{-\pi n^2 t^l}, \;\; m = 1, 2, \ldots \; ; \; j = 0, 1, 2, \ldots \tag{1.14}$$

Due to the *weighted* theta series in eq-(1.14) the zero mode $n = 0$ does *not* contribute to the sum in eq-(1.14), since m is a positive integer, and the Mellin transform of $\Theta_{j,m}(t^l) = \Theta_{j,m}(x)$ $(x = t^l)$ after exploiting the symmetry of the even-degree Hermite polynomials, is given by [5], [6]

$$\int_0^\infty \ [\ \frac{2}{l} \sum_{n=1}^{n=\infty} \ n^{2m} \ H_{2j}(n\sqrt{2\pi x}) \ e^{-\pi n^2 x} \ ] \ x^{s/2-1} \ dx \ =$$

$$-\ \frac{2}{l} \ (8\pi)^j \ P_j(s) \ \pi^{-s/2} \ \Gamma(\frac{s}{2}) \ \zeta(s-2m); \quad Re \ s \ > \ 1+2m, \quad m = 1, 2, ... \tag{1.15}$$

The polynomial pre-factor in (1.15) is given in terms of a terminating Hypergeometric series [6]

$$P_j(s) = (8\pi)^{-j}(-1)^j \frac{(2j)!}{j!} \ {}_2F_1(-j, \frac{s}{2}; \frac{1}{2}; 2). \tag{1.16}$$

The polynomial $P_j(s)$ has simple zeros on the critical line $Re\ s = \frac{1}{2}$, obeys the functional relation $P_j(s) = (-1)^j P_j(1-s)$ and in particular $P_j(s = \frac{1}{2}) = 0$ when $j = odd$ [6]. It is only when $j = even$ that $P_j(s = \frac{1}{2}) \neq 0$. In order to find the analytical continuation of the Mellin transform (1.15) for *all* values of s in the complex plane we must use the analytical continuation of $\zeta(s)$ as found by Riemann in his celebrated paper. A Poisson re-summation formula for $\Theta_{j,m}(x)$ (1.14) leads to the important relation

$$\frac{(-1)^j}{\sqrt{x}} \ \Theta_{j,m}(\frac{1}{x}) \ = \ \Theta_{j,m}(x). \tag{1.17}$$

which allows us to show that only when $j = even$ one can implement the $\mathcal{T}$ transformations of the new eigenfunctions $\Psi_s^{j,m}(t) = t^{-s+k} \ e^{V_{j,m}(t)}$ of D_1, and corresponding to the weighted theta series $\Theta_{j,m}(t^l)$ of eq-(1.14), giving

$$\mathcal{T} \ \Psi_s^{j,m}(t) \ = \ \Psi_s^{j,m}(\frac{1}{t}) \ = \ \Psi_{1-s}^{j,m}(t) \tag{1.18}$$

this relationship requires that one must have

$$j = even, \quad l \ = \ 4(2k-1) \tag{1.19}$$

Therefore, the eigenfunctions and eigenvalues of the pair of "Hamiltonians" is

$$H_A \Psi_s^{j,m}(t) \ = \ s(1-s) \ \Psi_s^{j,m}(t), \quad H_B \Psi_s^{j,m}(\frac{1}{t}) = s(1-s)\Psi_s^{j,m}(\frac{1}{t}) \tag{1.20}$$

subjected to the conditions in eq-(1.18).

We explicitly inserted the superscripts j, m in $\Psi_s^{j,m}(t) = t^{-s+k} \ e^{V_{j,m}(t)}$ to denote the j, m dependence in the definition of $V(t)$ given by eq-(1.14).

In what follows we shall omit the superscripts j, m for convenience. Defining

$$\Psi_s^{\mathcal{CT}}(t) \ \equiv \ \mathcal{C} \ \mathcal{T} \ \Psi_s(t) \ = \ \mathcal{C} \ \Psi_s(\frac{1}{t}) \ = \ \mathcal{C} \ \Psi_{1-s}(t) \ = \ \Psi_{1-s^*}(t) \qquad (1.21)$$

one finds that it is also an eigenfunction of H_A with an eigenvalue $s^*(1-s^*)$:

$$H_A \mid \Psi_s^{\mathcal{CT}}(t) \ > \ = \ H_A \ \mathcal{CT} \mid \Psi_s(t) \ > \ = \ H_A \mid \Psi_{1-s^*}(t) \ > \ =$$

$$s^*(1-s^*) \mid \Psi_{1-s^*}(t) \ > \ = \ s^*(1-s^*) \ \mathcal{CT} \mid \Psi_s(t) \ > \ = \ (E_s)^* \mid \Psi_s^{\mathcal{CT}}(t) \ > . \qquad (1.22)$$

where we have defined $(E_s)^* = s^*(1-s^*)$.

If the $\mathcal{CT}$ action on $s(1-s)\ \Psi_s$ is *linear* : $\mathcal{CT}\ s(1-s)\ \Psi_s \ = \ s(1-s)\ \mathcal{CT}\ \Psi_s$, instead of *antilinear* : $\mathcal{CT}\ s(1-s)\ \Psi_s \ = \ s^*(1-s^*)\ \mathcal{CT}\ \Psi_s$, and if

$$< \Psi_s \mid [H_A, \mathcal{CT}] \mid \Psi_s > \ = \ 0 \ \Rightarrow$$

$$< \Psi_s \mid H_A \ \mathcal{CT} \mid \Psi_s > \ - \ < \Psi_s \mid \mathcal{CT} \ H_A \mid \Psi_s > \ =$$

$$(E_s)^* \ < \Psi_s \mid \mathcal{CT} \mid \Psi_s > \ - \ E_s \ < \Psi_s \mid \mathcal{CT} \mid \Psi_s > \ =$$

$$(E_s^* \ - \ E_s) \ < \Psi_s \mid \mathcal{CT} \mid \Psi_s > \ = \ 0. \qquad (1.23)$$

Similar results follow for the H_B operator. From eq-(1.23) one has two cases to consider.

• Case A : If the pseudo-norm is null

$$< \Psi_s \mid \mathcal{CT} \mid \Psi_s > \ = \ 0 \ \Rightarrow \ (E_s \ - \ E_s^*) \ \neq \ 0 \qquad (1.24)$$

then the *complex* eigenvalues $E_s = s(1-s)$ and $E_s^* = s^*(1-s^*)$ are *complex* conjugates of each other. In this case one cannot prove the RH, and there exists the possibility that there are quartets of non-trivial Riemann zeta zeros (off the critical line) given by $s_n,\ 1-s_n,\ s_n^*,\ 1-s_n^*$.

• Case B : If the pseudo-norm is *not* null :

$$< \Psi_s \mid \mathcal{CT} \mid \Psi_s > \ \neq \ 0 \ \Rightarrow \ (E_s \ - \ E_s^*) \ = \ 0 \qquad (1.25)$$

then the eigenvalues are *real* given by $E_s = s(1-s) = E_s^* = s^*(1-s^*)$ and which implies that $s = real$ (location of the trivial zeta zeros) and/or $s = \frac{1}{2} + i\rho$ (location of the non-trivial zeta zeros). In this case the RH would be true and the non-trivial Riemann zeta zeros are given by $s_n = \frac{1}{2} + i\rho_n$ and $1 - s_n = s_n^* = \frac{1}{2} - i\rho_n$. We are going to prove next why Case **A** does and cannot occur, therefore the RH is true because we are left with case **B**. [2]

Therefore one has now at our disposal a well defined inner product of the states $\Psi_s^{j,m}(t)$ (*without* the need to regularize the integrals by extracting out the zero $n=0$ mode of the theta series). In particular, from eq-(1.15) one learns

[2] The authors [4] were the first to my knowledge who explored the possibility that PT symmetry might be relevant to the RH. It was their work which inspired us.

that the inner product of the states $\Psi_s^{j,m}(t)$ with the states $\Psi_{\frac{1}{2}+2m}(t)$, where $m = 1, 2, ...$, is given by

$$< \Psi^{j,m}_{\frac{1}{2}+2m}(t) \mid \Psi^{j,m}_s(t) > \ = \ - \frac{2}{l} \ (8\pi)^j \ P_j(s+2m) \ \pi^{-(s+2m)/2} \ \Gamma(\frac{s+2m}{2}) \ \zeta(s). \tag{1.26}$$

to arrive at this result above requires performing the change of variables $t^l = x$, and *fixing* uniquely the values $l = -2; k = \frac{1}{4}$ obeying the required constraint $l = 4(2k-1)$ in eq-(1.18).

Therefore, for each given value of j, m, the non-trivial zeta zeros $s_n = \frac{1}{2} \pm \rho_n$ are in a one-to-one correspondence with the states $\Psi^{j,m}_{s_n}(t)$ *orthogonal* to the states $\Psi^{j,m}_{\frac{1}{2}+2m}(t)$ in eq-(1.26) :

$$< \Psi^{j,m}_{\frac{1}{2}+2m}(t) \mid \Psi^{j,m}_{s_n}(t) > \ =$$

$$- \frac{2}{l} \ (8\pi)^j \ P_j(s_n+2m) \ \pi^{-(s_n+2m)/2} \ \Gamma(\frac{s_n+2m}{2}) \ \zeta(s_n) = 0; \quad m = 1, 2, 3, ... \tag{1.27}$$

There is no summation over the j, m indices in (1.27), and $n = 1, 2, 3, \cdots$ label the non-trivial zeta zeros $s_n = \frac{1}{2} + i\rho_n$ in the critical line. The complex conjugate of eq-(1.27) yields the $(s_n)^* = \frac{1}{2} - i\rho_n$ complex conjugate zeros. The "energy" eigenvalues corresponding to the zeta zeros in the critical line are given by $E_{s_n} = s_n(1-s_n) = \frac{1}{4} + (\rho_n)^2 = (E_{s_n})^*$.

The trivial zeta zeros $\zeta(-2n) = 0$ for $n = 1, 2, 3, \cdots$ appear when one takes the inner products $< \Psi^{j,m}_{s_1}(t) | \Psi^{j,m}_{\frac{1}{2}-s_1}(t) > \ \sim \ \zeta(-2m) = 0$, when s_1 is *real* valued. It remains to prove, when $l = -2, k = \frac{1}{4}$, $t^l = x$, and $s_{12} = s_1^* + s_2 = s_1^* + (1 - s_1^*) = 1$, that

$$< \Psi^{j,m}_s(t) \mid \mathcal{CT} \mid \Psi^{j,m}_s(t) > \ = \ < \Psi^{j,m}_s(t) \mid\mid \Psi^{j,m}_{1-s^*}(t) > \ =$$

$$\int_0^\infty \ [\ \frac{2}{l} \ \sum_{n=1}^{n=\infty} \ n^{2m} \ H_{2j}(n\sqrt{2\pi x}) \ e^{-\pi n^2 x} \ ] \ x^{\frac{2(-s_{12}+2k)}{2l} - 1} \ dx \ =$$

$$- \frac{2}{l} \ (8\pi)^j \ P_j(s = \frac{1}{2}) \ \pi^{-1/4} \ \Gamma(\frac{1}{4}) \ \zeta(\frac{1}{2} - 2m) \ \neq 0; \quad j = even, \ m = 1, 2, 3, ... \tag{1.28}$$

Hence, one arrives at a definite solid conclusion in eq-(1.28). Because $\zeta(\frac{1}{2} - 2m) \neq 0$ when $m = 1, 2, ...$, and $P_j(\frac{1}{2}) \neq 0$ when $j = even$ in eq-(1.28), then $< \Psi_s \mid \mathcal{CT} \mid \Psi_s > \ \neq 0$, so this rules out case **A** in eq-(1.24), and singles out case **B** in eq-(1.25) leading to the conclusion that $E_s = s(1-s) = real \Rightarrow s = \frac{1}{2} + i\rho$ (and/or s = real), which is the *location* of the non-trivial zeta zeros (if the RH is true) and trivial zeta zeros, respectively.

Armed with these findings that the eigenvalues s which define the eigenfunctions $\Psi_s(t)$ must be real and/or reside in the critical line, we can proceed further than we did back in [16] and gain more information about the location of the zeta zeros. Let us analyze the scenario in case the RH were *not* true.

Given any real number $s' = \frac{1}{2} + \xi \in \mathcal{R}$, such that $\xi > 2m$, and $s = \frac{1}{2} + i\lambda \in \mathcal{L}$ residing in the critical line, let us imagine that the inner product

$$< \Psi^{j,m}_{s'}(t) \mid \Psi^{j,m}_{s}(t) > \; = \; < \Psi^{j,m}_{\frac{1}{2}+\xi}(t) \mid \Psi^{j,m}_{s}(t) > \; =$$

$$- \frac{2}{l} (8\pi)^j \; P_j(s+\xi) \; \pi^{-(s+\xi)/2} \; \Gamma(\frac{s+\xi}{2}) \; \zeta(s+\xi-2m) \; = \; 0 \qquad (1.29)$$

and its complex conjugate

$$< \Psi^{j,m}_{s'}(t) \mid \Psi^{j,m}_{s^*}(t) > \; = \; < \Psi^{j,m}_{\frac{1}{2}+\xi}(t) \mid \Psi^{j,m}_{s^*}(t) > \; =$$

$$- \frac{2}{l} (8\pi)^j \; P_j(s^*+\xi) \; \pi^{-(s^*+\xi)/2} \; \Gamma(\frac{s^*+\xi}{2}) \; \zeta(s^*+\xi-2m) \; = \; 0 \qquad (1.30)$$

generate other nontrivial zeros off the critical line given by $z \equiv s+\xi-2m$; $z^* \equiv s^*+\xi-2m$, respectively. Due to the Vallee de la Poussin theorem that there are *no* zeros when the real part of s is $0, 1$, one learns that $2m < \xi < 2m + \frac{1}{2}$.

By symmetry, $1 - z^* = s - \xi + 2m$, and $1 - z = s^* - \xi + 2m$ should also be another pair of complex conjugate (putative) zeros off the critical line since the number of zeros off the critical line must appear in *quartets* resulting from the symmetry property of the completed zeta function $Z(s) = Z(1-s)$. However, due to the fact that the numbers m are *positive* integers, and from inspection of the fundamental integral in eq-(1.15), one can infer that this latter pair of complex conjugate zeros $s - \xi + 2m$, and $s^* - \xi + 2m$, *cannot* be obtained from an orthogonality condition among the $\Psi^{jm}_s(t)$ and $\Psi^{jm}_{s'}(t)$, for any s located in the critical line, and $s' = \frac{1}{2} + \xi$ located in the real line ($\xi > 2m$). Consequently, if there were zeros off the critical line, only $half$ of those could be obtained from imposing the orthogonality conditions. The only way one could generate all the (non-trivial) zeros from the orthogonality conditions is when *all* of them reside in the critical line and which is consistent with the RH.

It is true that one could have the following inner products

$$< \Psi^{j,m}_{\frac{1}{2}+\xi'}(t) \mid \Psi^{j,m}_{s}(t) > \; = \; - \frac{2}{l} (8\pi)^j \; P_j(s+\xi') \; \pi^{-(s+\xi')/2} \; \Gamma(\frac{s+\xi'}{2}) \zeta(s+\xi'-2m) \qquad (1.31)$$

of the states $\Psi^{j,m}_s(t)$ with another state $\Psi^{j,m}_{\frac{1}{2}+\xi'}(t)$ associated to a $different$ value of $\xi' \neq \xi$, such that

$$\zeta(s+\xi'-2m) \; = \; \zeta(s-\xi+2m) \; = \; 0 \qquad (1.32)$$

namely, one could perform the identification

$$s+\xi'-2m \; = \; s-\xi+2m \; \Rightarrow \; \xi \; + \; \xi' \; = \; 4m, \;\; \xi > 2m > 0, \;\; 0 < \xi' < 2m \qquad (1.33)$$

and claim that one has found the sought-after pair of orthogonal states $\Psi^{jm}_{\frac{1}{2}+\xi'}(t)$, $\Psi^{jm}_s(t)$ which generates the putative zero off the critical line given by $s - \xi + 2m$.

But in this case one would have to choose *two* different "ground" states, $\Psi^{jm}_{\frac{1}{2}+\xi}(t)$, $\Psi^{jm}_{\frac{1}{2}+\xi'}(t)$ in order to evaluate the inner products with the states $\Psi^{j,m}_s(t)$. The "energy" eigenvalues $E_s = s(1-s)$ associated with the two "ground" states are $\frac{1}{4}-\xi^2$, $\frac{1}{4}-(\xi')^2$, respectively. However, the fact that these "energy" eigenvalues are *not* the same is very problematic if one wishes to label these states as two degenerate "ground" states which are both orthogonal to the states $\Psi^{jm}_s(t)$. Also, from the Vallee de la Poussin theorem one can infer that

$$2m-\frac{1}{2} \ < \ \xi' \ < \ 2m \ < \ \xi < \ 2m+\frac{1}{2} \ \Rightarrow \ \frac{1}{4}-\xi^2 \ < \ \frac{1}{4}-(\xi')^2 \tag{1.34}$$

meaning that the state $\Psi^{jm}_{\frac{1}{2}+\xi}(t)$ is the true "ground" state since it has has *lower* energy than the state $\Psi^{jm}_{\frac{1}{2}+\xi'}(t)$. Consequently, one can only obtain *half* of the quartet of putative zeros off the critical line, and

$$< \Psi_{\frac{1}{2}+\xi'}(t)^{jm}|\Psi^{jm}_{\frac{1}{2}+\xi}(t) > \ \sim \ P_j(\frac{1}{2}+\xi+\xi')\ \zeta(\frac{1}{2}+\xi+\xi'-2m) \ \neq \ 0 \tag{1.35}$$

so that the states $\Psi^{jm}_{\frac{1}{2}+\xi}(t)$, $\Psi^{jm}_{\frac{1}{2}+\xi'}(t)$ are *not* orthogonal.

Another possibility is to look at the inner products $\Psi^{jm'}_s(t)$ with $\Psi^{jm'}_{\frac{1}{2}+\xi'}(t)$. The orthogonality condition yields in this case the relation

$$s+\xi'-2m' \ = \ s-\xi+2m \ \Rightarrow \ \xi + \xi' \ = \ 2m + 2m'; \ \ \xi > 2m > 0, \ \ 0 < \xi' < 2m' \tag{1.36}$$

However, in this case one would be choosing eigenfunctions $\Psi^{jm'}_s(t), \Psi^{jm'}_{\frac{1}{2}+\xi'}(t)$ of another *different* operator D'_1 resulting from the different value of $m' \neq m$, and which leads to a different weighted theta series in eq-(1.14).

Therefore, in order to generate the *quartets* of putative zeros off the critical line one would be forced to look at the orthogonality conditions of $\Psi^{jm}_s(t)$ with respect to *two* different "ground" states, or involving eigenfunctions of many *different* operators $D_1, D'_1, D''_1, \cdots$ associated with many *different* functions $V_{j,m}(t), V_{j,m'}(t), V_{j,m''}(t), \cdots$, instead of focusing on the orthogonality conditions involving eigenfunctions $\Psi^{j,m}_s(t)$ of only *one* operator D_1, associated to only *one* function $V_{j,m}(t)$, and with respect to only *one* "ground" state.

The above arguments are reminiscent of our prior physical interpretation of the location of the nontrivial Riemann zeta zeros. These locations corresponded to the presence of tachyonic-resonances/tachyonic-condensates in bosonic string theory [11]. We found that if there were zeros *off* the critical line violating the RH these zeros do *not* correspond to any *poles* of the string scattering amplitude.

In this work we have found that one *cannot* properly obtain all the quartets of putative zeros off the critical line from the orthogonality conditions of $\Psi^{jm}_s(t)$

with only "ground" state, for each fixed values of j, m. The zeros in the critical line could be properly obtained from the orthogonality conditions of $\Psi_s^{jm}(t)$ with $\Psi_{\frac{1}{2}+2m}(t)$, for each fixed values of j, m. We believe that complex scalings and logarithmic time reversal transformations hold important clues as to why the Riemann hypothesis is true. The role of the $\mathcal{C}, \mathcal{T}$ transformations, and the properties of the Mellin transform [6] were essential in our construction.

To finalize we should add that when s_1, s_2 reside in the critical line, the inner products

$$< \Psi_{s_1}^{j,m}(t) \mid \Psi_{s_2}^{j,m}(t) > \ \sim \ P_j(\frac{1}{2} + i(\rho_2 - \rho_1))\ \zeta(\frac{1}{2} + i(\rho_2 - \rho_1) - 2m) \quad (1.37)$$

could be zero when $P_j(\frac{1}{2} + i(\rho_2 - \rho_1)) = 0$, since the polynomial $P_j(s)$ has simple zeros on the critical line [6]. Whereas the "norm" of these states is *not* null

$$< \Psi_{s_1}^{j,m}(t)|\Psi_{s_1}^{j,m}(t) > \ = \ < \Psi_{s_2}^{j,m}(t)|\Psi_{s_2}^{j,m}(t) > \ \sim \ P_j(\frac{1}{2})\ \zeta(\frac{1}{2} - 2m) \ \neq \ 0 \quad (1.38)$$

since $P_j(\frac{1}{2}) \neq 0$ when $j = even$. This is very relevant if one wishes the states to belong to Banach, Hilbert spaces. This should impose constraints on the values of j, m in order to have positive-definite norms.

To summarize the whole construction : we saw earlier that a *regularization* of the integral (1.10a) (after removing the zero $n = 0$ mode of the Gauss-Jacobi series) would have led to to eq-(1.10b) $\langle \psi_{\frac{1}{2}} | \psi_s \rangle \sim Z[\frac{1}{2} + s - \frac{1}{2}] = Z[s]$, when $l = -2; k = \frac{1}{4}$. When s resides in the critical line, the orthogonal states ψ_{s_n} to the "ground" state $\psi_{\frac{1}{2}}$ are in a one-to-one correspondence with the non-trivial zeta zeros $Z[s_n] = 0$ in the critical line, because the completed zeta function is proportional to $\zeta(s)$.

The quartet of putative zeros off the critical line will now have a one-to-one correspondence to the states $\psi_{\tilde{s}_n}$ orthogonal to the two degenerate ground states $\psi_{\frac{1}{2}+\xi}; \psi_{\frac{1}{2}-\xi}$, where now both of these states have the *same* "energy" $E = \frac{1}{4} - \xi^2$. Consequently, both are now valid (degenerate) ground sates. Upon taking the inner products leads to $\langle \psi_{\frac{1}{2}\pm\xi} | \psi_{\tilde{s}_n} \rangle \sim Z[\tilde{s}_n \pm \xi] = 0$, and these zeros would be symmetrically distributed with respect to the critical line, $\tilde{s}_n \pm \xi$. To this pair of zeros one may add their complex conjugates, completing the quartet of zeros. However, we do have a blessing in disguise when we had to dismiss and bypass the states in eqs-(1.10a, 1.10b) because in order to evaluate the inner products one had to *remove* the zero mode.

This is what directed us to choose the eigenfunctions $\Psi_s^{j,m}$ derived from the weighted theta series $\Theta_{j,m}$ in eq-(1.14) because *no* regularization of the inner products was needed. And, in doing so, one could no longer have two legitimate "ground" states $\Psi_{\frac{1}{2}+\xi}^{j,m}$, $\Psi_{\frac{1}{2}+\xi'}^{j,m}$ ($\xi > 2m; \xi' < 2m$) with the *same* energy. As a result, this would force us to choose the state $\Psi_{\frac{1}{2}+\xi}^{j,m}$ with lower energy, and consequently, one would only be able to recapture $half$ of the zeros in each quartet of putative zeros off the critical line. Thus, the only way to recapture all the nontrivial zeros from the orthogonality conditions involving well defined inner products occurs when $\xi = \xi' = 2m$, leading then to the RH $< \Psi^{j,m}_{\frac{1}{2}+2m}(t) \mid \Psi^{j,m}_{s_n}(t) > \ \sim \ \zeta(s_n) = 0$ with $s_n = \frac{1}{2} \pm i\rho_n$.

Acknowledgements

We thank to M. Bowers for her hospitality and assistance.

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