Subtraction and Division of Neutrosophic Numbers

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Abstract

In this paper, we define the subtraction and the division of neutrosophic single-valued numbers. The restrictions for these operations are presented for neutrosophic single-valued numbers and neutrosophic single-valued overnumbers / undernumbers / offnumbers. Afterwards, several numeral examples are presented.

Keywords

neutrosophic calculus, neutrosophic numbers, neutrosophic summation, neutrosophic multiplication, neutrosophic scalar multiplication, neutrosophic power, neutrosophic subtraction, neutrosophic division.

1 Introduction

Let $A = (t_1, i_1, f_1)$ and $B = (t_2, i_2, f_2)$ be two single-valued neutrosophic numbers, where $t_1, i_1, f_1, t_2, i_2, f_2 \in [0, 1]$, and $0 \le t_1, i_1, f_1 \le 3$ and $0 \le t_2, i_2, f_2 \le 3$.

The following operational relations have been defined and mostly used in the neutrosophic scientific literature:

1.1 Neutrosophic Summation

$$A \oplus B = (t_1 + t_2 - t_1 t_2, i_1 i_2, f_1 f_2) \quad (1)$$

1.2 Neutrosophic Multiplication

$$A \otimes B = (t_1 t_2, i_1 + i_2 - i_1 i_2, f_1 + f_2 - f_1 f_2) \quad (2)$$

1.3 Neutrosophic Scalar Multiplication

$$\leftthreetimes A = (1 - (1 - t_1)^{\leftthreetimes}, i_1^{\leftthreetimes}, f_1^{\leftthreetimes}), \quad (3)$$

where $\leftthreetimes \in \mathbb{R}$, and $\leftthreetimes > 0$.

1.4 Neutrosophic Power

$$A^{\curlywedge} = (t_1^{\curlywedge}, 1-(1-i_1)^{\curlywedge}, 1-(1-f_1)^{\curlywedge}), \quad (4)$$

where $\curlywedge \in \mathbb{R}$, and $\curlywedge > 0$.

2 Remarks

Actually, the neutrosophic scalar multiplication is an extension of neutrosophic summation; in the last, one has $\curlywedge = 2$.

Similarly, the neutrosophic power is an extension of neutrosophic multiplication; in the last, one has $\curlywedge = 2$.

Neutrosophic summation of numbers is equivalent to neutrosophic union of sets, and neutrosophic multiplication of numbers is equivalent to neutrosophic intersection of sets.

That's why, both the neutrosophic summation and neutrosophic multiplication (and implicitly their extensions neutrosophic scalar multiplication and neutrosophic power) can be defined in many ways, i.e. equivalently to their neutrosophic union operators and respectively neutrosophic intersection operators.

In general:

$$A \oplus B = (t_1 \vee t_2, i_1 \wedge i_2, f_1 \wedge f_2), \quad (5)$$

or

$$A \oplus B = (t_1 \vee t_2, i_1 \vee i_2, f_1 \vee f_2), \quad (6)$$

and analogously:

$$A \otimes B = (t_1 \wedge t_2, i_1 \vee i_2, f_1 \vee f_2) \quad (7)$$

or

$$A \otimes B = (t_1 \wedge t_2, i_1 \wedge i_2, f_1 \vee f_2), \quad (8)$$

where "$\vee$" is the fuzzy OR (fuzzy union) operator, defined, for $\alpha, \beta \in [0,1]$, in three different ways, as:

$$\alpha \overset{1}{\vee} \beta = \alpha + \beta - \alpha\beta, \quad (9)$$

or

$$\alpha \overset{2}{\vee} \beta = max\{\alpha, \beta\}, \quad (10)$$

or

$$\alpha \overset{3}{\vee} \beta = min\{x + y, 1\}, \quad (11)$$

etc.

While "∧" is the fuzzy AND (fuzzy intersection) operator, defined, for $\alpha, \beta \in [0,1]$, in three different ways, as:

$$\alpha \underset{1}{\wedge} \beta = \alpha\beta, \tag{12}$$

or

$$\alpha \underset{2}{\wedge} \beta = min\{\alpha, \beta\}, \tag{13}$$

or

$$\alpha \underset{3}{\wedge} \beta = max\{x + y - 1, 0\}, \tag{14}$$

etc.

Into the definitions of $A \oplus B$ and $A \otimes B$ it's better if one associates $\overset{1}{\vee}$ with $\underset{1}{\wedge}$, since $\overset{1}{\vee}$ is opposed to $\underset{1}{\wedge}$, and $\overset{2}{\vee}$ with $\underset{2}{\wedge}$, and $\overset{3}{\vee}$ with $\underset{3}{\wedge}$, for the same reason. But other associations can also be considered.

For examples:

$$A \oplus B = (t_1 + t_2 - t_1t_2, i_1 + i_2 - i_1i_2, f_1f_2), \tag{15}$$

or

$$A \oplus B = (max\{t_1, t_2\}, min\{i_1, i_2\}, min\{f_1, f_2\}), \tag{16}$$

or

$$A \oplus B = (max\{t_1, t_2\}, max\{i_1, i_2\}, min\{f_1, f_2\}), \tag{17}$$

or

$$A \oplus B = (min\{t_1 + t_2, 1\}, max\{i_1 + i_2 - 1, 0\}, max\{f_1 + f_2 - 1, 0\}). \tag{18}$$

where we have associated $\overset{1}{\vee}$ with $\underset{1}{\wedge}$, and $\overset{2}{\vee}$ with $\underset{2}{\wedge}$, and $\overset{3}{\vee}$ with $\underset{3}{\wedge}$.

Let's associate them in different ways:

$$A \oplus B = (t_1 + t_2 - t_1t_2, min\{i_1, i_2\}, min\{f_1, f_2\}), \tag{19}$$

where $\overset{1}{\vee}$ was associated with $\underset{2}{\wedge}$ and $\underset{3}{\wedge}$; or:

$$A \oplus B = (max\{t_1, t_2\}, i_1, i_2, max\{f_1 + f_2 - 1, 0\}), \tag{20}$$

where $\overset{2}{\vee}$ was associated with $\underset{1}{\wedge}$ and $\underset{3}{\wedge}$; and so on.

Similar examples can be constructed for $A \otimes B$.

3 Neutrosophic Subtraction

We define now, for the first time, the subtraction of neutrosophic number:

$$A \ominus B = (t_1, i_1, f_1) \ominus (t_2, i_2, f_2) = \left(\frac{t_1 - t_2}{1 - t_2}, \frac{i_1}{i_2}, \frac{f_1}{f_2}\right) = C, \tag{21}$$

for all $t_1, i_1, f_1, t_2, i_2, f_2 \in [0, 1]$, with the restrictions that: $t_2 \neq 1$, $i_2 \neq 0$, and $f_2 \neq 0$.

So, the neutrosophic subtraction only partially works, i.e. when $t_2 \neq 1$, $i_2 \neq 0$, and $f_2 \neq 0$.

The restriction that

$$\left(\frac{t_1-t_2}{1-t_2}, \frac{i_1}{i_2}, \frac{f_1}{f_2}\right) \in ([0, 1], [0, 1], [0, 1]) \tag{22}$$

is set when the classical case when the neutrosophic number components t, i, f are in the interval [0, 1].

But, for the general case, when dealing with neutrosophic overset / underset /offset [1], or the neutrosophic number components are in the interval $[\Psi, \Omega]$, where Ψ is called *underlimit* and Ω is called *overlimit*, with $\Psi \leq 0 < 1 \leq \Omega$, i.e. one has *neutrosophic overnumbers / undernumbers / offnumbers*, then the restriction (22) becomes:

$$\left(\frac{t_1-t_2}{1-t_2}, \frac{i_1}{i_2}, \frac{f_1}{f_2}\right) \in ([\Psi, \Omega], [\Psi, \Omega], [\Psi, \Omega]). \tag{23}$$

3.1 Proof

The formula for the subtraction was obtained from the attempt to be consistent with the neutrosophic addition.

One considers the most used neutrosophic addition:

$$(a_1, b_1, c_1) \oplus (a_2, b_2, c_2) = (a_1 + a_2 - a_1 a_2, b_1 b_2, c_1 c_2), \tag{24}$$

We consider the $\ominus$ neutrosophic operation the opposite of the $\oplus$ neutrosophic operation, as in the set of real numbers the classical subtraction $-$ is the opposite of the classical addition $+$.

Therefore, let's consider:

$$(t_1, i_1, f_1) \ominus (t_2, i_2, f_2) = (x, y, z), \tag{25}$$

$$\oplus\ (t_2, i_2, f_2) \qquad\qquad \oplus\ (t_2, i_2, f_2)$$

where $x, y, z \in \mathbb{R}$.

We neutrosophically add $\oplus\ (t_2, i_2, f_2)$ on both sides of the equation. We get:

$$(t_1, i_1, f_1) = (x, y, z) \oplus (t_2, i_2, f_2) = (x + t_2 - x t_2, y i_2, z f_2). \tag{26}$$

Or,

$$\begin{cases} t_1 = x + t_2 - x t_2, \text{whence } x = \frac{t_1 - t_1}{1 - t_2}\ ; \\ i_1 = y i_2, \text{whence } y = \frac{i_1}{i_2}\ ; \\ f_1 = z f_2, \text{whence } z = \frac{f_1}{f_2}\ . \end{cases} \tag{27}$$

3.2 Checking the Subtraction

With $A = (t_1, i_1, f_1)$, $B = (t_2, i_2, f_2)$, and $C = \left(\frac{t_1-t_2}{1-t_2}, \frac{i_1}{i_2}, \frac{f_1}{f_2}\right)$,
where $t_1, i_1, f_1, t_2, i_2, f_2 \in [0,1]$, and $t_2 \neq 1$, $i_2 \neq 0$, and $f_2 \neq 0$, we have:

$$A \ominus B = C. \tag{28}$$

Then:

$$B \oplus C = (t_2, i_2, f_2) \oplus \left(\frac{t_1-t_2}{1-t_2}, \frac{i_1}{i_2}, \frac{f_1}{f_2}\right) = \left(t_2 + \frac{t_1-t_2}{1-t_2} - t_2 \cdot \frac{t_1-t_2}{1-t_2}, i_2 \cdot \frac{i_1}{i_2}, f_2 \cdot \frac{f_1}{f_2}\right) = \left(\frac{t_2-t_2^2+t_1-t_2-t_1t_2+t_2}{1-t_2}, i_1, f_1\right) = \left(\frac{t_1(1-t_2)}{1-t_2}, i_1, f_1\right) = (t_1, i_1, f_1). \tag{29}$$

$$A \ominus C = (t_1, i_1, f_1) \ominus \left(\frac{t_1-t_2}{1-t_2}, \frac{i_1}{i_2}, \frac{f_1}{f_2}\right) = \left(\frac{t_1-\frac{t_1-t_2}{1-t_2}}{1-\frac{t_1-t_2}{1-t_2}}, \frac{i_1}{\frac{i_1}{i_2}}, \frac{f_1}{\frac{f_1}{f_2}}\right) = \left(\frac{\frac{t_1-t_1t_2-t_1+t_2}{1-t_2}}{\frac{1-t_2-t_1+t_2}{1-t_2}}, i_2, f_2\right) = \left(\frac{-t_1t_2+t_2}{1-t_2}, i_2, f_2\right) = \left(\frac{t_2(-t_1+1)}{1-t_2}, i_2, f_2\right) = (t_2, i_2, f_2). \tag{30}$$

4 Division of Neutrosophic Numbers

We define for the first time the division of neutrosophic numbers:

$$A \oslash B = (t_1, i_1, f_1) \oslash (t_2, i_2, f_2) = \left(\frac{t_1}{t_2}, \frac{i_1-i_2}{1-i_2}, \frac{f_1-f_2}{1-f_2}\right) = D, \tag{31}$$

where $t_1, i_1, f_1, t_2, i_2, f_2 \in [0,1]$, with the restriction that $t_2 \neq 0$, $i_2 \neq 1$, and $f_2 \neq 1$.
Similarly, the division of neutrosophic numbers only partially works, i.e. when $t_2 \neq 0$, $i_2 \neq 1$, and $f_2 \neq 1$.
In the same way, the restriction that

$$\left(\frac{t_1}{t_2}, \frac{i_1-i_2}{1-i_2}, \frac{f_1-f_2}{1-f_2}\right) \in ([0,1], [0,1], [0,1]) \tag{32}$$

is set when the traditional case occurs, when the neutrosophic number components t, i, f are in the interval [0, 1].
But, for the case when dealing with neutrosophic overset / underset /offset [1], when the neutrosophic number components are in the interval $[\Psi, \Omega]$, where Ψ is called *underlimit* and Ω is called *overlimit*, with $\Psi \leq 0 < 1 \leq \Omega$, i.e. one has *neutrosophic overnumbers / undernumbers / offnumbers*, then the restriction (31) becomes:

$$\left(\frac{t_1}{t_2}, \frac{i_1-i_2}{1-i_2}, \frac{f_1-f_2}{1-f_2}\right) \in ([\Psi, \Omega], [\Psi, \Omega], [\Psi, \Omega]). \quad (33)$$

4.1 Proof

In the same way, the formula for division $\oslash$ of neutrosophic numbers was obtained from the attempt to be consistent with the neutrosophic multiplication.

We consider the $\oslash$ neutrosophic operation the opposite of the $\otimes$ neutrosophic operation, as in the set of real numbers the classical division $\div$ is the opposite of the classical multiplication $\times$.

One considers the most used neutrosophic multiplication:

$$(a_1, b_1, c_1) \otimes (a_2, b_2, c_2)$$
$$= (a_1 a_2, b_1 + b_2 - b_1 b_2, c_1 + c_2 - c_1 c_2), \quad (34)$$

Thus, let's consider:

$$(t_1, i_1, f_1) \oslash (t_2, i_2, f_2) = (x, y, z), \quad (35)$$
$$\otimes(t_2, i_2, f_2) \qquad\qquad \otimes(t_2, i_2, f_2)$$

where $x, y, z \in \mathbb{R}$.

We neutrosophically multiply $\otimes$ both sides by (t_2, i_2, f_2). We get

$$(t_1, i_1, f_1) = (x, y, z) \otimes (t_2, i_2, f_2)$$
$$= (x t_2, y + i_2 - y i_2, z + f_2 - z f_2). \quad (36)$$

Or,

$$\begin{cases} t_1 = x t_2, \text{whence } x = \frac{t_1}{t_2};: \\ i_1 = y + i_2 - y i_2, \text{whence } y = \frac{i_1-i_2}{1-i_2}; \\ f_1 = z + f_2 - z f_2, \text{whence } z = \frac{f_1-f_2}{1-f_2}. \end{cases} \quad (37)$$

4.2 Checking the Division

With $A = (t_1, i_1, f_1)$, $B = (t_2, i_2, f_2)$, and $D = \left(\frac{t_1}{t_2}, \frac{i_1-i_2}{1-i_2}, \frac{f_1-f_2}{1-f_2}\right)$,

where $t_1, i_1, f_1, t_2, i_2, f_2 \in [0, 1]$, and $t_2 \neq 0$, $i_2 \neq 1$, and $f_2 \neq 1$, one has:

$$A * B = D. \quad (38)$$

Then:

$$\frac{B}{D} = (t_2, i_2, f_2) \times \left(\frac{t_1}{t_2}, \frac{i_1-i_2}{1-i_2}, \frac{f_1-f_2}{1-f_2}\right) = \left(t_2 \cdot \frac{t_1}{t_2}, i_2 + \frac{i_1-i_2}{1-i_2} - i_2 \cdot \frac{i_1-i_2}{1-i_2}, f_2 + \frac{f_1-f_2}{1-f_2} - f_2 \cdot \frac{f_1-f_2}{1-f_2}\right) =$$

$$\left(t_1, \frac{i_2 - i_2^2 + i_1 - i_2 - i_1 i_2 + i_2^2}{1 - i_2}, \frac{f_2 - f_2^2 + f_1 - f_2 - f_1 f_2 + f_2^2}{1 - f_2}\right) = \left(t_1, \frac{i_1(1 - i_2)}{1 - i_2}, \frac{f_1(1 - f_2)}{1 - f_2}\right) = (t_1, i_1, f_1) = A. \quad (39)$$

Also:

$$\frac{A}{D} = \frac{(t_1, i_1, f_1)}{\left(\frac{t_1}{t_2}, \frac{i_1 - i_2}{1 - i_2}, \frac{f_1 - f_2}{1 - f_2}\right)} = \left(\frac{t_1}{\frac{t_1}{t_2}}, \frac{i_1 - \frac{i_1 - i_2}{1 - i_2}}{1 - \frac{i_1 - i_2}{1 - i_2}}, \frac{f_1 - \frac{f_1 - f_2}{1 - f_2}}{1 - \frac{f_1 - f_2}{1 - f_2}}\right) = \left(t_2, \frac{\frac{i_1 - i_1 i_2 - i_1 + i_2}{1 - i_2}}{\frac{1 - i_2 - i_1 + i_2}{1 - i_2}}, \frac{\frac{f_1 - f_1 f_2 - f_1 + f_2}{1 - f_2}}{\frac{1 - f_2 - f_1 + f_2}{1 - f_2}}\right) = \left(t_2, \frac{\frac{i_2(-i_1 + 1)}{1 - i_2}}{\frac{1 - i_1}{1 - i_2}}, \frac{\frac{f_2(-f_1 + 1)}{1 - f_2}}{\frac{1 - f_1}{1 - f_2}}\right) = \left(t_2, \frac{i_2(1 - i_1)}{1 - i_1}, \frac{f_2(1 - f_1)}{1 - f_1}\right) = (t_2, i_2, f_2) = B. \quad (40)$$

5 Conclusion

We have obtained the formula for the subtraction of neutrosophic numbers ⊖ going backwords from the formula of addition of neutrosophic numbers ⊕.

Similarly, we have defined the formula for division of neutrosophic numbers ⊘ and we obtained it backwords from the neutrosophic multiplication ⊗.

We also have taken into account the case when one deals with classical neutrosophic numbers (i.e. the neutrosophic components t, i, f belong to [0, 1]) as well as the general case when t, i, f belong to $[\Psi, \Omega]$, where the underlimit $\Psi \leq 0$ and the overlimit $\Omega \geq 1$.

6 References

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