Rough Standard Neutrosophic Sets

An Application on Standard Neutrosophic Information Systems

Nguyen Xuan Thao
Faculty of Information Technology
Vietnam National University of Agriculture
Hanoi, Vietnam
nxthao2000@gmail.com

Bui Cong Cuong
Institute of Mathematics
Vietnam Academy of Science and Technology
Hanoi, Vietnam
bccuong@gmail.com

Florentin Smarandache
Department of Mathematics
University of New Mexico
Gallup, NM, USA
smarand@unm.edu

***Abstract*—A rough fuzzy set is the result of approximation of a fuzzy set with respect to a crisp approximation space. It is a mathematical tool for the knowledge discovery in the fuzzy information systems. In this paper, we introduce the concepts of rough standard neutrosophic sets, standard neutrosophic information system and give some results of the knowledge discovery on standard neutrosophic information system based on rough standard neutrosophic sets.**

Keywords*—*rough set, standard neutrosophic set, rough standard neutrosophic set, standard neutrosophic information systems

I. INTRODUCTION

Rough set theory was introduced by Z. Pawlak in 1980s [1]. It becomes a usefully mathematical tool for data mining, especially for redundant and uncertain data. At first, the establishment of the rough set theory is based on equivalence relation. The set of equivalence classes of the universal set, obtained by an equivalence relation, is the basis for the construction of upper and lower approximation of the subset of universal set.

Fuzzy set theory was introduced by L. Zadeh since 1965 [2]. Immediately, it became a useful method to study in the problems of imprecision and uncertainty. Since, a lot of new theories treating imprecision and uncertainty have been introduced. For instance, Intuitionistic fuzzy sets were introduced in1986, by K. Atanassov [3], which is a generalization of the notion of a fuzzy set. When fuzzy set give the degree of membership of an element in a given set, Intuitionistic fuzzy set give a degree of membership and a degree of non-membership of an element in a given set. In 1999 [17], F. Smarandache gave the concept of neutrosophic set which generalized fuzzy set and intuitionistic fuzzy set. It is a set in which each proposition is estimated to have a degree of truth (T), adegree of indeterminacy (I) and a degree of falsity (F). Over time, many subclasses of neutrosophic sets were proposed. They are also more advantageous in the practical application. Wang et al. [18] proposed interval neutrosophic sets and some operators of them. Smarandache [17] and Wang et al. [19] proposed a single valued neutrosophic set as an instance of the neutrosophic set accompanied with various set theoretic operators and properties. Ye [20] defined the concept of simplified neutrosophic sets. It is a set where each element of the universe has a degree of truth, indeterminacy and falsity respectively and which lies between [0, 1] and some operational laws for simplified neutrosophic sets and to propose two aggregation operators, including a simplified neutrosophic weighted arithmetic average operator and a simplified neutrosophic weighted geometric average operator. In 2013, B.C. Cuong and V. Kreinovich introduced the concept of picture fuzzy set [4,5], as a particular case of neutrosophic set, in which a given element has three memberships: a degree of positive membership, a degree of negative membership, and a degree of neutral membership of an element in this set. After that, L. H. Son has given the application of the picture fuzzy set in the clustering problems [7,8]. We also regard picture fuzzy sets as a particular case of the standard neutrosophic sets [6].

In addition, combining rough set and fuzzy set has also many interesting results. The approximation of rough (or fuzzy) sets in fuzzy approximation space give us the fuzzy rough set [9,10,11]; and the approximation of fuzzy sets in crisp approximation space give us the rough fuzzy set [9,10]. W.Z. Wu et al, [11] present a general framework for the study of fuzzy rough sets in both constructive and axiomatic approaches. By the same, W. Z. Wu and Y. H. Xu were investigated the fuzzy topological structures on the rough fuzzy sets [12], in which both constructive and axiomatic approaches are used. In 2012, Y. H. Xu and W. Z. Wu were also investigated the rough intuitionistic fuzzy set and the intuitionistic fuzzy topologies in crisp approximation spaces [13]. In 2013 B. Davvaz and M. Jafarzadeh study the rough intuitionistic fuzzy information system [14]. In 2014, X.T. Nguyen introduces the rough picture fuzzy sets.It is the result of approximation of a picture fuzzy set with respect to a crisp approximation space [15].

II. BASIC NOTIONS OF STANDARD NEUTROSOPHIC SET AND ROUGH SET

In this paper, we denote U be a nonempty set called the universe of discourse. The class of all subsets of U will be denoted by $P(U)$ and the class of all fuzzy subsets of U will be denoted by $F(U)$.

Definition 1. [6]. A standard neutrosophic (PF) set A on the universe U is an object of the form

$$A = \{(x, \mu_A(x), \eta_A(x), \gamma_A(x)) \mid x \in U\}$$

where $\mu_A(x)(\in [0,1])$ is called the "degree of positive membership of x in A", $\eta_A(x)(\in [0,1])$ is called the "degree of neutral membership of x in A" and $\gamma_A(x)(\in[0,1])\gamma_A(x)(\in [0,1])$ is called the "degree of negative membership of x in A", and where μ_A, η_A μ_A, γ_A and γ_A are dependent components alltogether (see [24]) and therefore they η_A satisfy the following condition:

$$\mu_A(x) + \eta_A(x) + \gamma_A(x) \le 1, (\forall x \in X)$$

$$\mu_A(x) + \gamma_A(x) + \eta_A(x)) \le 1, (\forall x \in X).$$

The family of all standard neutrosophic set in U is denoted by $PFS(U)$. The complement of a picture fuzzy set A is

$$\sim A = \{(x, \gamma_A(x), \eta_A(x), \mu_A(x)) \mid \forall x \in U\}.$$

Obviously, any intuitionistic fuzzy set $A = \{(x, \mu_A(x), \gamma_A(x))\}$ may be identified with the standard neutrosophic set in the form

$$A = \{(x, \mu_A(x), 0, \gamma_A(x)X) \mid x \in U\}$$

$A = \{(x, \mu_A(x), \gamma_A(x), 0) | x \in U\}$.

The operators on $PFS(U)$: $A \subseteq B$, $A \cap B$ $A \cup B$ were introduced [4]:

Now we define some special PF sets: a constant PF set is the PF set $(\widehat{\alpha, \beta, \theta}) = \{(x, \alpha, \beta, \theta) | x \in U\}$; the PF universe set is $U = 1_U = (\widehat{1,0,0}) = \{(x, 1,0,0) | x \in U\}$ and the PF empty set is
$\emptyset = 0_U = (\widehat{0,0,1}) = \{(x, 0,0,1) | x \in U\}$
$\emptyset = 0_U = (\widehat{0,1,0}) = \{(x, 0,1,0) | x \in U\}$.

For any $x \in U$, standard neutrosophic set 1_x and $1_{U-\{x\}}$ are, respectively, defined by: for all $y \in U$

$$\mu_{1_x}(y) = \begin{cases} 1 \text{ if } y = x \\ 0 \text{ if } y \ne x \end{cases}, \quad \eta_{1_x}(y) = \begin{cases} 0 \text{ if } y = x \\ 0 \text{ if } y \ne x \end{cases},$$

$$\gamma_{1_x}(y) = \begin{cases} 0 \text{ if } y = x \\ 1 \text{ if } y \ne x \end{cases}; \quad \mu_{1_{U-\{x\}}}(y) = \begin{cases} 0 \text{ if } y = x \\ 1 \text{ if } y \ne x \end{cases},$$

$$\eta_{1_{U-\{x\}}}(y) = \begin{cases} 0 \text{ if } y = x \\ 0 \text{ if } y \ne x \end{cases}, \quad \gamma_{1_{U-\{x\}}}(y) = \begin{cases} 1 \text{ if } y = x \\ 0 \text{ if } y \ne x \end{cases}$$

Definition 2. (Lattice $(D^*, \le_{D^*})$). Let
$D^* = \{(x_1, x_2, x_3) \in [0,1]^3 : x_1 + x_2 + x_3 \le 1\}$.

We define a relation $\le_{D^*}$ on D^* as follows: $\forall (x_1, x_2, x_3), (y_1, y_2, y_3) \in D^*$
then
$(x_1, x_2, x_3) \le_{D^*} (y_1, y_2, y_3)$ $(x_1, x_2, x_3) \le_{D^*} (y_1, y_2, y_3)$
if only if (or $(x_1 < y_1, x_3 \ge y_3)$ $(x_1 < y_1, x_3 \ge y_3)$ or
$(x_1 = y_1, x_3 > y_3)(x = x', y > y')$ or
$(x_1 = y_1, x_3 = y_3, x_2 \le y_2)(x = x', y = y', z \le z'))$ and
$(x_1, x_2, x_3) =_{D^*} (y_1, y_2, y_3) \Leftrightarrow (x_1 = y_1, x_2 = y_2, x_3 = y_3)$.

We have $(D^*, \le_{D^*})$ is a lattice. Denote $0_{D^*} = (0,0,1)$, $1_{D^*} = (1,0,0)$ Now, we define some operators on D^*.

Definition 3.

(i) Negative of $x = (x_1, x_2, x_3) \in D^*$ is $\overline{x} = (x_3, x_2, x_1)$

(ii) For all $x = (x_1, x_2, x_3) \in D^*$ we have

$$x \wedge y = (x_1 \wedge y_1, x_2 \wedge y_2, x_3 \vee y_3)$$

$$x \vee y = (x_1 \vee y_1, x_2 \wedge y_2, x_3 \wedge y_3)$$

We have some properties of those operators.

Lemma 1.

(a) For all $x = (x_1, x_2, x_3) \in D^*$ we have

(b1) $\overline{x \wedge y} = \overline{x} \vee \overline{y}$ $\overline{x \wedge y} = \overline{x} \vee \overline{y}$

(b2) $\overline{x \vee y} = \overline{x} \wedge \overline{y}$ $\overline{x \vee y} = \overline{x} \wedge \overline{y}$

(b) For all $x, y, u, v \in D^*$ and $x \le_{D^*} u, y \le_{D^*} v$ we have

(c1) $x \wedge y \le_{D^*} u \wedge v$

(c2) $x \vee y \le_{D^*} u \vee v$

Proof.

(a) We have
$\overline{x \wedge y} = (x_3 \vee y_3, x_2 \wedge y_2, x_1 \wedge y_1) =$
$(x_3, x_2, x_1) \vee (y_3, y_2, y_1) = \overline{x} \vee \overline{y}$

Similary
$\overline{x \vee y} = (x_3 \wedge y_3, x_2 \wedge y_2, x_1 \vee y_1) =$
$(x_3, x_2, x_1) \vee (y_3, y_2, y_1) = \overline{x} \vee \overline{y}$

(b) For $a, b, c, d \in [0,1]$, if $a \le b, c \le d$ then $a \wedge c \le b \wedge d$ and. From definition 2, definition 3 we have the result to prove. □

Now, we mention the level sets of the standard neutrosophic sets. Where $(\alpha, \beta, \theta) \in D^*$, we define:

- $(\alpha, \beta, \theta) -$ level cut set of the standard neutrosophic set $A = \{(x, \mu_A(x), \eta_A(x), \gamma_A(x)) \mid x \in U\}$
$A = \{(x, \mu_A(x), \gamma_A(x), \eta_A(x)) | x \in U\}$ as follows:

$$A_{\theta}^{\alpha,\beta} = \{x \in U | (\mu_A(x), \eta_A(x), \gamma_A(x)) \geq (\alpha, \beta, \theta)\}$$
$$= \{x \in U | (\mu_A(x), \eta_A(x), \gamma_A(x)) \geq (\alpha, \beta, \theta)\}$$

- strong $(\alpha, \beta, \theta)-$ level cut set of the standard neutrosophic set A as follows:

$$A_{\theta^+}^{\alpha^+,\beta^+} = \{x \in U | (\mu_A(x), \eta_A(x), \gamma_A(x)) > (\alpha, \beta, \theta)\}$$

- $(\alpha^+, \beta, \theta)$ –- level cut set of the standard neutrosophic set A as

$$A_{\theta}^{\alpha^+,\beta} = \{x \in U | \mu_A(x) > \alpha, \gamma_A(x) \leq \theta\}$$

- $(\alpha, \beta, \theta^+)$ – level cut set of the standard neutrosophic set A as

$$A_{\theta^+}^{\alpha,\beta} = \{x \in U | \mu_A(x) \geq \alpha, \ \gamma_A(x) < \theta\}$$

When $\beta = 0$ we denoted

$$A_{\theta}^{\alpha} = A_{\theta}^{\alpha,0} = \{x \in U | (\mu_A(x), \eta_A(x), \gamma_A(x)) \geq (\alpha, 0, \theta)\}$$

- (α^+, θ^+) – level cut set of the standard neutrosophic set A as

$$A_{\theta^+}^{\alpha^+} = \{x \in U | \mu_A(x) > \alpha, \gamma_A(x) < \theta\}$$

- $\alpha -$ level cut set of the degree of positive membership of x in A as

$$A^{\alpha} = \{x \in U | \mu_A(x) \geq \alpha\}$$

the strong $\alpha -$ level cut set of the degree of positive membership of x in A as

$$A^{\alpha^+} = \{x \in U | \mu_A(x) > \alpha\}$$

- $\theta -$ level low cut set of the degree of negative membership of x in A as

$$A_{\theta} = \{x \in U | \gamma_A(x) \leq \theta\}$$

the strong $\theta -$ level low cut set of the degree of negative membership of x in A as

$$A_{\theta^+} = \{x \in U | \gamma_A(x) < \theta\}.$$

Example 1.

Given the universe $U = \{u_1, u_2, u_3\}$. Then $A = ((u_1, 0.8, 0.05, 0.1), (u_2, 0.7, 0.1, 0.2), (u_3, 0.5, 0.01, 0.4))$ is a standard neutrosophic set on U. Then $A_{0.1}^{0.7,0.2} = \{u_1, u_2\}$ but $A_{0.1}^{0.7,0.1} = \{u_1\}$ and $A_{0.1^+}^{0.7,0.2} = \{u_1\}$, $A_{0.1}^{0.7} = \{u_1\}$, $A_{0.1^+}^{0.7} = \emptyset$, $A^{0.5} = \{u_1, u_2, u_3\}$, $A^{0.5^+} = \{u_1, u_2\}$, $A_{0.2^+} = \{u_1\}, A_{0.2} = \{u_1, u_2\}$.

Definition 3.

Let U be a nonempty universe of discourse which many be infinite. A subset $R \in P(U \times U)$ is referred to as a (crisp) binary relation on U. The relation R is referred to as:

- Reflexive: if for all $x \in U$, $(x, x) \in R$.
- Symmetric: if for all $x, y \in U$, $(x, y) \in R$ $x, y \in U, (x, y) \in R$ then $(y, x) \in R$.
- Transitive: if for all $x, y, z \in U$, $(x, y) \in R, (y, z) \in R$ $x, y, z \in U, (x, y) \in R, (y, z) \in R$ then $(x, z) \in R$
- Similarity: if R is reflexive and symmetric
- Preorder: if R is reflexive and transitive
- Equivalence: if R is reflexive and symmetric, transitive.

A crisp approximation space is a pair (U, R). For an arbitrary crisp relation R on U, we can define a set-valued mapping $R_s : U \to P(U)$ by:

$$R_s(x) = \{y \in U | (x, y) \in R\}, \ x \in U.$$

Then, $R_s(x)$ is called the successor neighborhood of x x with respect to (w.r.t) R.

Definition 4. [9]

Let (U, R) be a crisp approximation space. For each crisp set $A \subseteq U$, we define the upper and lower approximations of A (w.r.t) (U, R) denoted by $\overline{R}(A)$ and $\underline{R}(A)$, respectively, are defined as follows

$$\overline{R}(A) = \{x \in U : R_s(x) \cap A \neq \emptyset\},$$
$$\underline{R}(A) = \{x \in U : R_s(x) \subseteq A\}$$
$$\underline{R}(A) = \{x \in U : R_s(x) \subseteq A\}.$$

Remark 2.1.

Let (U, R) be a Pawlak approximation space, i.e. R is an equivalence relation. Then $R_s(x) = [x]_R$ holds. For each crisp set $A \subseteq U$, the upper and lower approximations of A (w.r.t) (U, R) denoted by $\overline{R}(A)$ and $\underline{R}(A)$, respectively, are defined as follows

$$\overline{R}(A) = \{x \in U : [x]_R \cap A \neq \emptyset\}$$
$$\underline{R}(A) = \{x \in U : [x]_R \subseteq A\}.$$

Definition 5 [16]

Let (U, R) be a crisp approximation space. For each fuzzy set $A \subseteq U$, we define the upper and lower approximations of A (w.r.t) (U, R) denoted by $\overline{R}(A)$ and $\underline{R}(A)$, respectively, are defined as follows

$$\overline{R}(A) = \{x \in U : R_s(x) \cap A \neq \emptyset\},$$
$$\underline{R}(A) = \{x \in U : R_s(x) \subseteq A\}$$

where

$\mu_{\overline{R}(A)}(x) = max\{\mu_A(y) \mid y \in R_s(x)\}$,

$\mu_{\underline{R}A}(x) = min\{\mu_A(y) \mid y \in R_s(x)\}$

Remark 2.2.

Let (U, R) be a Pawlak approximation space, i.e. R is an equivalence relation. Then $R_s(x) = [x]_R$ holds. For each fuzzy set $A \subseteq U$, the upper and lower approximations of A (w.r.t) (U, R) denoted by $\overline{R}(A)$ and $\underline{R}(A)$, respectively, are defined as follows

$\overline{R}(A) = \{x \in U: [x]_R \cap A \neq \emptyset\}$, $\underline{R}(A) = \{x \in U: [x]_R \subseteq A\}$

This is the rough fuzzy set in [6].

III. ROUGH STANDARD NEUTROSOPHIC SET

A rough standard neutrosophic set is the approximation of a standard neutrosophic set w. r. t a crisp approximation space. Here, we consider the upper and lower approximations of a standard neutrosophic set in the crisp approximation spaces together with their membership functions, respectively.

Definition 5:

Let (U, R) be a crisp approximation space. For $A \in PFS(U)$, the upper and lower approximations of A (w.r.t) (U, R) denoted by $\overline{RP}(A)$ $\overline{RP}(A)$ and $\underline{RP}(A)$, respectively, are defined as follows:

$\overline{RP}(A) = \{(x, \mu_{\overline{RP}(A)}(x), \eta_{\overline{RP}(A)}(x), \gamma_{\overline{RP}(A)}(x)) \mid x \in U\}$

$\underline{RP}(A) = \{(x, \mu_{\underline{RP}(A)}(x), \eta_{\underline{RP}(A)}(x), \gamma_{\underline{RP}(A)}(x)) \mid x \in U\}$

where

$\mu_{\overline{RP}(A)}(x) = \bigvee_{y\in R_s(x)} \mu_A(y)$,

$\eta_{\overline{RP}(A)}(x) = \bigwedge_{y\in R_s(x)} \eta_A(y)$,

$\underline{RP}(A) = \{(x, \mu_{\underline{RP}(A)}(x), \gamma_{\underline{RP}(A)}(x), \eta_{\underline{RP}(A)}(x)) \mid x \in U\}$;

and

$\underline{RP}(A) = \{(x, \mu_{\underline{RP}(A)}(x), \gamma_{\underline{RP}(A)}(x), \eta_{\underline{RP}(A)}(x)) \mid x \in U\}$,

$\eta_{\underline{RP}(A)}(x) = \bigwedge_{y\in R_s(x)} \eta_A(y)$,

$\gamma_{\underline{RP}(A)}(x) = \bigvee_{y\in R_s(x)} \gamma_A(y)$.

We have $\overline{RP}(A)$ and $\underline{RP}(A)$ are two standard neutrosophic sets in U. Indeed, for each $x \in U$, for all $\epsilon > 0$, it exists $y_0 \in U$ $y_0 \in U$ such that

$\mu_{\overline{RP}(A)}(x) - \epsilon \leq \mu_A(y_0) \leq \mu_{\overline{RP}(A)}(x)$,

$\eta_{\overline{RP}(A)}(x) \leq \eta_A(y_0)$,

$\gamma_{\overline{RP}(A)}(x) \leq \gamma_A(y_0)$

so that $\mu_{\overline{RP}(A)}(x) - \epsilon + \eta_{\overline{RP}(A)}(x) + \gamma_{\overline{RP}(A)}(x)$

$\leq \mu_A(y_0) + \eta_A(y_0) + \gamma_A(y_0) \leq 1$

$\mu_{\overline{RP}(A)}(x) - \epsilon + \eta_{\overline{RP}(A)}(x) + \gamma_{\overline{RP}(A)}(x) \leq$.

Hence $\mu_{\overline{RP}(A)}(x) + \eta_{\overline{RP}(A)}(x) + \gamma_{\overline{RP}(A)}(x) \leq 1 + \epsilon$, for all $\epsilon > 0$. It means, i.e, $\overline{RP}(A)$ is a standard neutrosophic set. By the same way, we obtain $\underline{RP}(A)$ is a standard neutrosophic set. Moreover, $\underline{RP}(A) \subset \overline{RP}(A)$.

Thus the standard neutrosophic mappings $\overline{RP}$, $\underline{RP}: PFS(U) \to PFS(U)$are referred to as the upper and lower PF approximation operators, respectively, and the pair $PR(A) = (\underline{PR}(A), \overline{RP}(A))$ $RP(A) = (\underline{RP}(A), \overline{RP}(A))$ is called the rough standard neutrosophic set of A w.r.t the approximation space. The picture fuzzy set denoted by $\sim RP(A)$ and is defined by

$\sim PR(A) = (\sim \underline{PR}(A), \sim \overline{RP}(A))$

$\sim RP(A) = (\sim\underline{RP}(A), \sim\overline{RP}(A))$ where $\sim\underline{RP}(A)$ and $\sim\overline{RP}(A)$ are the complements of the PF sets $\overline{RP}(A)$ and $\underline{RP}(A)$ respectively.

Example 2.

We consider the universe set $U = \{u_1, u_2, u_3, u_4, u_5\}$ and a binary relation R on U in Table 1. Here, if $u_i R u_j$ then cell (i, j) takes a value of 1, else cell (i, j) takes a value of 0 (i, j = 1, 2, 3, 4, 5). A standard neutrosophic

$A = \{(u_1, 0.7, 0.1, 0.2), (u_2, 0.6, 0.2, 0.1), (u_3, 0.6, 0.2, 0.05),$

$(u_2, 0.6, 0.2, 0.1), (u_3, 0.6, 0.2, 0.05)\}$

TABLE I.

R	u_1	u_2	u_3	u_4	u_5
u_1	1	0	1	0	0
u_2	0	1	0	1	1
u_3	1	0	1	0	1
u_4	0	1	0	1	0
u_5	0	0	1	1	1

Binary relation R on U

We have $R_s(u_1) = \{u_1, u_3\}$, $R_s(u_2) = \{u_2, u_4, u_5\}$, $R_s(u_3) = \{u_1, u_3, u_5\}$, $R_s(u_4) = \{u_2, u_4\}$,

$R_s(u_5) = \{u_3, u_4, u_5\}$ $R_s(u_5) = \{u_3, u_4, u_5\}$.

So that, we obtain results

$\mu_{\overline{RP}(A)}(u_1) = \vee_{y\in R_s(u_1)} \mu_A(y)$

$\mu_{\overline{RP}(A)}(u_1) = \vee_{y\in R_s(u_1)} \mu_A(y) = \max\{\mu_A(u_1), \mu_A(u_3)\}$

$= \max\{0.7, 0.6\} = 0.7$,

$\eta_{\underline{RP}(A)}(u_1) = \wedge_{y\in R_s(u_1)} \eta_A(y) = \min\{\eta_A(u_1), \eta_A(u_3)\}$

$= \max\{0.7, 0.6\} = 0.7$,

$\gamma_{\underline{RP}(A)}(u_1) = \wedge_{y\in R_s(u_1)} \gamma_A(y) = \min\{\gamma_A(u_1), \gamma_A(u_3)\}$

$\gamma_{RP(A)}(u_1) = \wedge_{y\in R_s(u_1)} \gamma_A(y) = \min\{\gamma_A(u_1), \gamma_A(u_3)\} =$ $\max\{0.7, 0.6\} = 0.7 \min\{0.2, 0.05\} = 0.05$

Similar calculations for other elements of U, we have upper approximations of A is

$\overline{RP}(A)=\{(u_1,0.7,0.1,0.05),(u_2,0.6,0.2,0.1),$
$(u_3,0.7,0.1,0.05),(u_4,0.6,0.2,0.1),(u_5,0.6,0.2,0.05)\}$
and lower approximations of A is

$\underline{RP}(A)=\{(u_1,0.6,0.1,0.2),(u_2,0.4,0.2,0.2),$
$(u_3,0.4,0.1,0.2),(u_4,0.5,0.2,0.15),(u_5,0.4,0.2,0.2)\}$.

Some basic properties of rough standard neutrosophic set approximation operators represent in the following theorem:

Theorem 1.

Let (U,R) be a crisp approximation space, then the upper and lower rough standard neutrosophic approximation operators satisfy the following properties: $\forall A,B,A_j \in PFS(U), j\in J, J$ is an index set,

(PL1) $\underline{PR}(\sim A)=\sim\overline{RP}(A)$

(PL2)
$\underline{RP}(A\cup(\alpha,\beta,\theta))=\underline{RP}(A)\cup(\alpha,\beta,\theta)$

(PL3) $\underline{RP}(U)=U\,\underline{RP}(U)=U$

$\eta_{\underline{RP}(A)}(x)=\wedge_{y\in R_s(x)}\eta_A(y)$

(PL5) $\underline{RP}(A\cup B)\supseteq\underline{RP}(A)\cup\underline{RP}(B)$

(PL6) $A\subseteq B\Rightarrow\underline{RP}(A)\subseteq\underline{RP}(B)$

(PU1) $\overline{RP}(\sim A)=\sim\underline{RP}(A)\,\overline{RP}(\sim A)=\sim\underline{PR}(A)$
(PU2) $\overline{PR}(A\cap(\widetilde{\alpha,\beta,\theta}))=\overline{PR}(A)\cap(\widetilde{\alpha,\beta,\theta})$
(PU3) $\overline{PR}(\emptyset)=\emptyset$

(PU4) $\overline{RP}(\cup_{j\in J}A_j)=\cup_{j\in J}\overline{RP}(A_j)$
(PU5) $\overline{RP}(A\cap B)\subseteq\overline{RP}(A)\cap\overline{RP}(B)$
(PU6) $A\subseteq B\Rightarrow\overline{RP}(A)\subseteq\overline{RP}(B)$

Proof.

(PL1).
$\underline{RP}(\sim A)=\{(x,\ \mu_{\underline{RP}(\sim A)}(x),\eta_{\underline{RP}(\sim A)}(x),\ \gamma_{\underline{RP}(\sim A)}(x))\mid x\in U\}$

In which,

$\mu_{\underline{RP}(\sim A)}(x)=\vee_{y\in R_s(x)}\mu_{\sim A}(y)=\vee_{y\in R_s(x)}\gamma_A(y)=$
$\gamma_{\overline{RP}(A)}(x)$;

$\eta_{\underline{RP}(\sim A)}(x)=\wedge_{y\in R_s(x)}\eta_{\sim A}(y)=\wedge_{y\in R_s(x)}\eta_A(y)=$
$\eta_{\overline{RP}(A)}(x)$

$\gamma_{\underline{RP}(\sim A)}(x)=\wedge_{y\in R_s(x)}\gamma_{\sim A}(y)=\wedge_{y\in R_s(x)}\mu_A(y)=$
$\mu_{\overline{RP}(A)}(x)$

From that and lemma 1, we have $\underline{PR}(\sim A)=\sim\overline{RP}(A)$.

(PL2) Because $(\widetilde{\alpha,\beta,\theta})=\{(x,\alpha,\beta,\theta)|x\in U\}$, we have

$\mu_{\underline{RP}(A\cup(\alpha,\beta,\theta))}(x)=\vee_{y\in R_s(x)}\mu_{\underline{RP}(A\cup(\alpha,\beta,\theta))}(y)$

$\vee_{y\in R_s(x)}\mu_{RP(A\cup(\widetilde{\alpha,\beta,\theta}))}(y)=\vee_{y\in R_s(x)}\max\{\mu_{\underline{RP}(A)}(y),\alpha\}$

$=\max\{\vee_{y\in R_s(x)}\mu_{\underline{RP}(A)}(y),\vee_{y\in R_s(x)}\alpha\}$

$=\ max\{\vee_{y\in R_s(x)}\mu_{RP(A)}(y),\vee_{y\in R_s(x)}\alpha\}$

$max\{\mu_{\underline{RP}(A)}(x),\mu_{((\alpha,\beta,\theta))}(x)\}=\mu_{\underline{RP}(A)\cup(\alpha,\beta,\theta)}(x)$.

By the same way, we have

$\eta_{\underline{RP}(A\cup(\alpha,\beta,\theta))}(x)=\eta_{\underline{RP}A\cup(\alpha,\beta,\theta)}(x)$

and

$\gamma_{\underline{RP}(A\cup(\alpha,\beta,\theta))}(x)=\gamma_{\underline{RP}A\cup(\alpha,\beta,\theta)}(x)$.

It means $\underline{RP}(A\cup(\widetilde{\alpha,\beta,\theta}))=\underline{RP}(A)\cup(\widetilde{\alpha,\beta,\theta})$.

(PL3) Since $U=1_U=(\widetilde{1,0,0})=\{(x,1,0,0)|x\in U\}$, then we can obtain (PL3) $\underline{RP}(U)=U$ by using definition 5.

The results (PL4), (PL5), (PL6) were proved by using the definition of lower and upper approximation spaces (definition 5) and lemma 1. $\mu_{\mu_{RP((\widetilde{\alpha,\beta,\theta}))}}(x)$

Similarly, we have (PU1), (PU2), (PU3), (PU4), (PU5), PU(6). □

Theorem 2.

Let (U,R) be a crisp approximation space. Then

a) $\underline{RP}(U)=U=\overline{RP}(U)$ and $\underline{RP}(\emptyset)=\emptyset=\overline{RP}(\emptyset)\underline{RP}(\emptyset)=\emptyset=\overline{RP}(\emptyset)$.

b) $\underline{RP}(A)\subseteq\overline{RP}(A)$ forall $A\in PFS(U)$.□

Proof.

(a) Using (PL3), (PL6), (PU3), (PU6), we easy prove $\underline{RP}(U)=U=\overline{RP}(U)$ and $\underline{RP}(\emptyset)=\emptyset=\overline{RP}(\emptyset)$.

(b) Based on definition 5, we have

$\mu_{\underline{RP}(A)}(x)=\wedge_{y\in R_s(x)}\mu_A(y)\ \leq\ \mu_{\overline{RP}(A)}(x)=\vee_{y\in R_s(x)}\mu_A(y)$,

$\eta_{\underline{RP}(A)}(x)=\wedge_{y\in R_s(x)}\mu_A(y)=\eta_{\overline{RP}(A)}(x)$,

and

$\gamma_{\underline{RP}(A)}(x)=\vee_{y\in R_s(x)}\gamma_A(y)\geq\ \wedge_{y\in R_s(x)}\gamma_A(y)=\gamma_{\overline{RP}(A)}(x)$

So that $\underline{RP}(A)\subseteq\overline{RP}(A)$ forall $A\in PFS(U)$.□

In the case of connections between special types of crisp relation on U, and properties of rough standard neutrosophic approximation operators, we have the following

Lemma 2.

If R is a symmetric crisp binary relation on U, then for all $A, B \in PFS(U)$,

$$\overline{RP}(A) \subseteq B \Leftrightarrow A \subseteq \underline{RP}(B)$$

Proof.

Let R be a symmetric crisp binary relation on U, i.e, $y \in R_s(x) \Leftrightarrow x \in R_s(y)$, $\forall x, y \in U$. We assume contradiction that $\overline{RP}(A) \subseteq B$ but $A \not\subset \underline{RP}(B)$. For each $x \in U$, we consider all the cases:

+ if $\mu_A(x) > \mu_{\underline{RP}(B)}(x) = \wedge_{y \in R_s(x)} \mu_B(y)$ then it exists $y_0 \in R_s(x)$ such that

$$\mu_A(x) > \mu_B(y_0) \geq \mu_{\overline{RP}(A)}(y_0) = \vee_{z \in R_s(y_0)} \mu_A(z) \geq \mu_A(x)$$

(because $y_0 \in R_s(x)$ then $x \in R_s(y_0)$. This is not true.

+ the cases $\gamma_A(x) < \gamma_{\underline{RP}(B)}(x)$ or $\eta_A(x) > \eta_{\underline{RP}(B)}(x)$ is also not true. □

Theorem 3.

Let (U, R) be a crisp approximation space, and $\overline{RP}$, are the upper and lower PF approximation operators. Then

(a) R is reflexive if and only if at least one of the following conditions are satisfied

(a1) (PLR) $\underline{RP}(A) \subseteq A \quad \forall A \in PFS(U)$

(a2) (PUR) $A \subseteq \overline{RP}(A) \; \forall A \in PFS(U)$

(b) R is symmetric if and only if at least one of the following conditions are satisfied

(b1) (PLR) $\overline{RP}(\underline{RP}(A)) \subseteq A \quad \forall A \in PFS(U)$

(b2) (PUR) $A \subseteq \underline{RP}\left(\overline{RP}(A)\right) \forall A \in PFS(U)$

(c) R is transitive if and only if at least one of the following conditions are satisfied

(c1) (PLT) $\underline{RP}(A) \subseteq \underline{RP}(\underline{RP}(A)) \quad \forall A \in PFS(U)$

(c2) (PUT) $\overline{RP}(A) \subseteq \overline{RP}\left(\overline{RP}(A)\right) \forall A \in PFS(U)$

Proof.

(a). We assume that R is reflexive, i.e., $x \in R_S(x)$, so that $\forall A \in PFS(U)$

we have

$$\mu_{\underline{RP}(A)}(x) = \wedge_{y \in R_s(x)} \mu_A(y) \leq \mu_A(x),$$

$$\eta_{\underline{RP}(A)}(x) = \wedge_{y \in R_s(x)} \mu_A(y) \leq \eta_A(x),$$

and $\gamma_{\underline{RP}(A)}(x) = \vee_{y \in R_s(x)} \gamma_A(y) \geq \gamma_A(x)$.

It means that $\underline{RP}(A) \subseteq A$, $\forall A \in PFS(U)$, i.e., (a1) was verified. Similarly, we consider upper approximation of:

$$\mu_{\overline{RP}(A)}(x) = \vee_{y \in R_s(x)} \mu_A(y) \geq \mu_A(x),$$

$\eta_{\overline{RP}(A)}(x) = \wedge_{y \in R_s(x)} \mu_A(y) = \eta_A(x)$, and $\gamma_{\overline{RP}(A)}(x) = \wedge_{y \in R_s(x)} \gamma_A(y) \leq \gamma_A(x)$.

It means $A \subseteq \overline{RP}(A)$, $\forall A \in PFS(U)$, i.e., (a2) is satisfied. Now, assume that (a1) $\underline{RP}(A) \subseteq A$, $\forall A \in PFS(U)$ we show that R is reflexive. Indeed, We assume contradiction that R is not reflexive, i.e., $x \notin R_S(x)$. We consider $A = 1_{U-\{x\}}$,

i.e., $\mu_{1_{U-\{x\}}}(y) = \begin{cases} 0 \text{ if } y = x \\ 1 \text{ if } y \neq x \end{cases}$,

$$\eta_{1_{U-\{x\}}}(y) = \begin{cases} 0 \text{ if } y = x \\ 0 \text{ if } y \neq x \end{cases}, \quad \gamma_{1_{U-\{x\}}}(y) = \begin{cases} 1 \text{ if } y = x \\ 0 \text{ if } y \neq x \end{cases}.$$

Then $\gamma_{\underline{RP}(A)}(x) = \vee_{y \in R_s(x)} \gamma_A(y) = 0 \geq \gamma_A(x) = 1$.

This is not true. It implies R is reflexive.

Similarly, we assume that (a2) $A \subseteq \overline{RP}(A)$, $\forall A \in PFS(U)$ we show that R is reflexive. Indeed, We assume contradiction that R is not reflexive, i.e., $x \notin R_S(x)$.

We consider $A = 1_x$, i.e., $\mu_{1_x}(y) = \begin{cases} 1 \text{ if } y = x \\ 0 \text{ if } y \neq x \end{cases}$,

$$\eta_{1_x}(y) = \begin{cases} 0 \text{ if } y = x \\ 0 \text{ if } y \neq x \end{cases}, \quad \gamma_{1_x}(y) = \begin{cases} 0 \text{ if } y = x \\ 1 \text{ if } y \neq x \end{cases}.$$

Then $\mu_{\overline{RP}(A)}(x) = \vee_{y \in R_s(x)} \mu_A(y) = 0 \geq \mu_A(x) = 1$.

This is not true. It implies R is reflexive.

(b).

We verify case (b1).

We assume that R is symmetric, i.e., if $x \in R_S(y)$ then $y \in R_S(x)$. For all $A \in PFS(U)$, because $x \in R_S(y)$ then $\wedge_{z \in R_s(y)} \mu_A(z) \leq \mu_A(x)$, $\wedge_{z \in R_s(y)} \mu_A(z) \leq \mu_A(x)$, $\vee_{z \in R_s(y)} \gamma_A(z) \geq \gamma_A(x)$ for all $y \in R_S(x)$, we have

$$\mu_{\overline{RP}(\underline{RP}(A))}(x) = \vee_{y \in R_s(x)}(\wedge_{z \in R_s(y)} \mu_A(z)) \leq \mu_A(x),$$

$\eta_{\overline{RP}(\underline{RP}(A))}(x) = \vee_{y \in R_s(x)}(\wedge_{z \in R_s(y)} \eta_A(z)) \leq \eta_A(x)$; and

$$\gamma_{\overline{RP}(\underline{RP}(A))}(x) = \wedge_{y \in R_s(x)}(\vee_{z \in R_s(y)} \gamma_A(z)) \geq \gamma_A(x).$$

It means that $\overline{RP}(\underline{RP}(A)) \subseteq A \quad \forall A \in PFS(U)$.

Now, we assume contradiction that $\overline{RP}(\underline{RP}(A)) \subseteq A \quad \forall A \in PFS(U)$ but R is not symmetric, i.e., if $x \in R_S(y)$ then $y \notin R_S(x)$ and if $y \in R_S(x)$ then $x \notin R_S(y)$. We consider $A = 1_{U-\{x\}}$. Then, $\mu_{\overline{RP}(\underline{RP}(A))}(x) = \vee_{y \in R_s(x)} (\wedge_{z \in R_s(y)} \mu_A(z)) = 1 > \mu_A(x) = 0$. It is not true, because $\mu_{\overline{RP}(\underline{RP}(A))}(x) \leq \mu_A(x)$, for all $x \in U$. So that R is symmetric.

By the same way, it yields (b2).

(c). R is transitive, i.e., if for all $x, y, z \in U$: $z \in R_S(y), y \in R_S(x)$ then $z \in R_S(x)$. It means that $R_S(y) \subseteq R_S(x)$, so that for all $A \in PFS(U)$ we have $\wedge_{z \in R_s(x)} \mu_A(z) \leq \wedge_{z \in R_s(y)} \mu_A(z)$.

Hence $\wedge_{y \in R_s(x)} (\wedge_{z \in R_s(x)} \mu_A(z)) \leq \wedge_{y \in R_s(x)} (\wedge_{z \in R_s(y)} \mu_A(z))$.

Because $\mu_{\underline{RP}(A)}(x) = \wedge_{y \in R_s(x)} (\wedge_{z \in R_s(x)} \mu_A(z))$ and $\mu_{\underline{RP}(\underline{RP}(A))}(x) = \wedge_{y \in R_s(x)} (\wedge_{z \in R_s(y)} \mu_A(z))$. So that $\mu_{\underline{RP}(A)}(x) \leq \mu_{\underline{RP}(\underline{RP}(A))}(x)$, for all $x \in U, A \in PFS(U)$. It mean that (c1) was varified. Now, we assume contradiction that (c1): $\overline{RP}(A) \subseteq \overline{RP}(\overline{RP}(A)) \forall A \in PFS(U)$, but R is not transitive, i.e., $x, y, z \in U$: $z \in R_S(y), y \in R_S(x)$ then $z \notin R_S(x)$. We consider $A = 1_{U-\{x\}}$, then $\mu_{\underline{RP}(A)}(x) = \wedge_{z \in R_s(x)} \mu_A(z) = 1$, but $\mu_{\underline{RP}(\underline{RP}(A))}(x) = \wedge_{y \in R_s(x)} (\wedge_{z \in R_s(y)} \mu_A(z)) = 0$. It is false. By same way, we show that (c2) is true. Hence, (c) was verified.□

Now, according to Theorem 1, Lemma 1 and Theorem 3, we obtain the following results:

Theorem 4.

Let R be a similarity crisp binary relation on U and $\overline{RP}$, $\underline{RP}: PFS(U) \to PFS(U)$ are the upper and lower PF approximation operators. Then, for all $A \in PFS(U)$

$$A = \underline{RP}(A) - \quad \overline{RP}(A) = A$$

$$- \quad \sim A = \underline{RP}(\sim A) \Leftrightarrow \overline{RP}(\sim A) = \sim A$$

IV. THE STANDARD NEUTROSOPHIC INFORMATION SYSTEMS

In this section, we introduce a new concept: standard neutrosophic information system.

Let (U, A, F) be a classical information system. Here U is the (nonempty) set of objects, i.e., $U = \{u_1, u_2, \dots, u_n\}$, $A = \{a_1, a_2, \dots, a_m\}$ is the attribute set, and F is the relation set of U and A, i.e., $F = \{f_j : U \to V_j, j = 1,2, \dots, m\}$ where V_j is the domain of the attribute a_j, $j = 1, 2, \dots, m$.

We call (U, A, F, D, G) an information system or decision table, where $U, A, F)$ is the classical information system, A is the condition attribute set and D is the decision attribute set, i.e., $D = \{d_1, d_2, \dots, d_p\}$ and G is the relation set of U and D, i.e., $G = \{g_j : U \to V_j', j = 1,2, \dots, p\}$ where V_j' is the domain of the attribute d_j, $j = 1, 2, \dots, p$.

Let (U, A, F, D, G) be the information system. For $B \subseteq A \cup D$, we define a relation, denoted $R_B = IND(B)$, as follows, $\forall x, y \in U$:

$x \; IND(B) \; y \iff f_j(x) = f_j(y)$ for all $j \in \{j: a_j \in B\}$.

The equivalence class of $x \in U$ based on R_B is $[x]_B = \{y \in U: yR_Bx\}$.

Here, we consider $R_A = IND(A)$, $R_D = IND(D)$. If $R_A \subseteq R_D$ $R_A \subseteq R_D$, i.e., for any $[x]_A, x \in U$ there exists $[x]_D$ such that $[x]_A \subseteq [x]_D$, then the information system is called a consistent information system, other called an inconsistent information system.

Let (U, A, F, D, G) be the information system, where (U, A, F) be a classical information system. If $D = \{D_k | k = 1,2, \dots, q\}$, where D_k is a fuzzy subset of U, then (U, A, F, D, G) be the fuzzy information system. If $D = \{D_k | k = 1,2, \dots, q\}$where D_k is an intutionistic fuzzy subset of U, then (U, A, F, D, G) be an intuitionistic fuzzy information system.

Definition 6.

Let (U, A, F, D, G) be the information system or decision table, where (U, A, F) be a classical information system. If $D = \{D_k | k = 1,2, \dots, q\}$ where D_k is a standard neutrosophic subset of U and G is the relation set of U and D, then (U, A, F, D, G) is called a standard neutrosophic information system.

Example 2.

The following table 2 gives a standard neutrosophic information system, where the objects set $U = \{u_1, u_2, \dots, u_{10}\}$, condition attribute set is

$A = \{a_1, a_2, a_3\}$ and the decision attribute set is $D = \{D_1, D_2, D_3\}$, where $D_k (k = 1,2,3)$ is the standard neutrosophic subsets of U.

***Table 2:** A standard neutrosophic information system*

U	a_1	a_2	a_3	D_1	D_2	D_3
u_1	3	2	1	(0.2,0,3,0.5)	(0.15,0.6,0.2)	(0.4,0.05,0.5)
u_2	1	3	2	(0.3,0.1,0.5)	(0.3,0.3,0.3)	(0.35,0.1,0.4)
u_3	3	2	1	(0.6,0,0.4)	(0.3,0.05,0.6)	(0.1,0.45,0.4)
u_4	3	3	1	(0.15,0.1,0.7)	(0.1,0.05,0.8)	(0.2,0.4,0.3)
u_5	2	2	4	(0.05,0,2,0.7)	(0.2,0.4,0.3)	(0.05,0.4,0.5)
u_6	2	3	4	(0.1,0.3,0.5)	(0.2,0.3,0.4)	(1,0,0)
u_7	1	3	2	(0.25,0.3,0.4)	(1,0,0)	(0.3,0.3,0.4)
u_8	2	2	4	(0.1,0.6,0.2)	(0.25,0.3,0.4)	(0.4,0,0.6)
u_9	3	2	1	(0.45,0,1,0.45)	(0.25,0.4,0.3)	(0.2,0.5,0.3)
u_{10}	1	3	2	(0.05,0.05,0.9)	(0.4,0.2,0.3)	(0.05,0.7,0.2)

V. THE KNOWLEDGE DISCOVERY IN THE STANDARD NEUTROSOPHIC INFORMATION SYSTEMS

In this section, we will give some results about the knowledge discovery for a standard neutrosophic information systems by using the basic theory of rough standard neutrosophic set in section 3. Throughout this paper, let (U, A, F, D, G) be the standard neutrosophic information system and $B \subseteq A$, we denote $\underline{RP_B}(D_i)$ is the lower rough standard neutrosophic approximation of $D_i \in PFS(U)$ on approximation space (U, R_B).

Theorem 5.

Let (U, A, F, D, G) be the standard neutrosophic information system and $B \subseteq A$. If for any $x \in U$:

$$\left(\mu_{D_i}(x), \eta_{D_i}(x), \gamma_{D_i}(x)\right) \geq \left(\alpha(x), \beta(x), \theta(x)\right)$$

$$= \underline{RP_B}(D_i)(x) > \underline{RP_B}(D_j)(x)(i \neq j),$$

then $[x]_B \cap (\sim D_j)^{\beta(x),0}_{\alpha(x)} \neq \emptyset$ $[x]_B \cap (\sim D_j)^{\beta(x),0}_{\alpha(x)} \neq \emptyset$ $[x]_B \cap \left(\sim D_j\right)^{\theta(x),0}_{\alpha(x)} \neq \emptyset$ $[x]_B \cap (\sim D_j)^{\theta(x),0}_{\alpha(x)} \neq \emptyset$ and $[x]_B \subseteq (D_i)^{\alpha(x),\beta(x)}_{\theta(x)}$ $[x]_B \cap (\sim D_j)^{\beta(x),0}_{\alpha(x)} \neq \emptyset [x]_B \subseteq (D_i)^{\alpha(x),\theta(x)}_{\beta(x)}$

$[x]_B \cap (\sim D_j)^{\beta(x),0}_{\alpha(x)} \neq \emptyset$

where $\left(\alpha(x), \beta(x), \theta(x)\right) \in D^*$.

Proof.

We have

$(D_i)^{\alpha(x),\beta(x)}_{\theta(x)} = \{y \in U : \left(\mu_{D_i}(y), \eta_{D_i}(y), \gamma_{D_i}(y)\right)$

$\geq \left(\alpha(x), \beta(x), \theta(x)\right)\}$.

Since $\left(\alpha(x), \beta(x), \theta(x)\right) = \underline{RP_B}(D_i)(x)$,
we have $\alpha(x) = \wedge_{y \in [x]_B} \mu_{D_i}(y)$, $\beta(x) = \wedge_{y \in [x]_B} \eta_{D_i}(y)$, and $\theta(x) = \vee_{y \in [x]_B} \gamma_{D_i}(y)$. So that, for any $x \in U, y \in [x]_B$ then $\mu_{D_i}(y) \geq \alpha(x), \eta_{D_i}(y) \geq \theta(x)$ $\gamma_{D_i}(y) \leq \theta(x)$ and $\eta_{D_i}(y) \geq \theta(x)$. It means that $y \in (D_i)^{\alpha(x),\beta(x)}_{\theta(x)}$, i.e., $[x]_B \subseteq (D_i)^{\alpha(x),\beta(x)}_{\theta(x)}$ $[x]_B \subseteq (D_i)^{\alpha(x),\beta(x)}_{\theta(x)}$

Now, since $\left(\alpha(x), \beta(x), \theta(x)\right) = \underline{RP}_B(D_i)(x) > \underline{RP}_B(D_j)(x)(i \neq j)$then there exists $y \in [x]_B$ such that

$$\left(\mu_{D_i}(y), \eta_{D_i}(y), \gamma_{D_i}(y)\right) < \left(\alpha(x), \beta(x), \theta(x)\right)$$

$\left(\mu_{D_i}(y), \eta_{D_i}(y), \gamma_{D_i}(y)\right) < \left(\alpha(x), \beta(x), \theta(x)\right)$,i.e., or $(\mu_{D_i}(y) < \alpha(x),\ \gamma_{D_i}(y) \geq \theta(x))$ or $(\mu_{D_i}(y) = \alpha(x),\ \gamma_{D_i}(y) > \theta(x))$ or $(\mu_{D_i}(y) = \alpha(x),\ \gamma_{D_i}(y) > \theta(x))$ and $\eta_{D_i}(y) < \beta(x))$. It means that here exists $y \in [x]_B$ such that $\left(\gamma_{D_i}(y), \eta_{D_i}(y), \mu_{D_i}(y)\right) \geq \left(\theta(x), 0, \alpha(x)\right)$, i.e., $y \in (\sim D_j)^{\theta(x),0}_{\alpha(x)}$. So that $[x]_B \cap (\sim D_j)^{\theta(x),0}_{\alpha(x)} \neq \emptyset$.□

Let (U, A, F, D, G) be the standard neutrosophic information system, R_A is the equivalence classes which induced by the condition attribute set A, and the universe is divided by R_A as following: $U/R_A = \{X_1, X_2 \dots, X_k\}$. Then the approximation of the standard neutrosophic decision denoted as, for all $i = 1,2, \dots, k$

$$\underline{RP}_A(D(X_i)) = \left(\underline{RP}_A(D_1(X_i)), \underline{RP}_A(D_2(X_i)), \dots, \underline{RP}_A(D_q(X_i))\right)$$

Example 3.

We consider the standard neutrosophic information system in Table 2. The equivalent classes

$$U/R_A=\{X_1=\{u_1,u_3,u_9\},X_2=\{u_2,u_7,u_{10}\},$$

$$X_3=\{u_4\},X_4=\{u_5,u_8\},X_5=\{u_6\}\}$$

The approximation of the standard neutrosophic decision is as follows:

Table 3: *The approximation of the picture fuzzy decision*

U/R_A	$\underline{RP}_A(D_1(X_i))$	$\underline{RP}_A(D_2(X_i))$	$\underline{RP}_A(D_3(X_i))$
X_1	(0.2,0,0.5)	(0.15,0.05,0.6)	(0.1,0.05,0.5)
X_2	(0.05,0.05,0.9)	(0.3,0.1,0.3)	(0.05,0.1,0.4)
X_3	(0.15, 0.1,0.7)	(0.1,0.05,0.8)	(0.2,0.4,0.3)
X_4	(0.05,0.2,0.7)	(0.2,0.3,0.4)	(0.05,0,0.6)
X_5	(0.1,0.3,0.5)	(0.2,0.3,0.4)	(1,0,0)

Indeed, for $X_1=\{u_1,u_3,u_9\}$. We have $\forall x\in X_1$,

$\mu_{\underline{RP}_A(D_1)}(x)=\wedge_{y\in X_1}\mu_{D_1}(y)=\min\{0.2,0.6,0.45\}=0.2$,

$\eta_{\underline{RP}_A(D_1)}(x)=\wedge_{y\in X_1}\eta_{D_1}(y)=\min\{0.3,0,0.1\}=0$

$\gamma_{\underline{RP}_A(D_1)}(x)=\vee_{y\in X_1}\gamma_{D_1}(y)=\max\{0.5,0.4,0.45\}=0.5$,

$y\in(\sim D_j)^{\beta(x),0}_{\alpha(x)}$, so that $\underline{RP_A}(D_1)(x)=(0.2,0.5,0)$. And

$\mu_{\underline{RP}_A(D_2)}(x)=\wedge_{y\in X_1}\mu_{D_2}(y)=\min\{0.15,0.3,0.25\}=0.15$,

$\eta_{\underline{RP_A}(D_2)}(x)=\wedge_{y\in X_1}\eta_{D_2}(y)=\min\{0.6,0.05,0.4\}=0.05$,

$\gamma_{\underline{RP}_A(D_2)}(x)=\vee_{y\in X_1}\gamma_{D_2}(y)=\max\{0.2,0.6,0.3\}=0.6$ so

$\underline{RP_A}(D_2)(x)=(0.15,0.6,0.05)$ and

$\mu_{\underline{RP_A}(D_3)}(x)=\wedge_{y\in X_1}\mu_{D_3}(y)=\min\{0.4,0.1,0.2\}=0.1$,

$\eta_{\underline{RP}_A(D_3)}(x)=\wedge_{y\in X_1}\eta_{D_3}(y)=\min\{0.05,0.45,0.5\}=0.05$,

$\gamma_{\underline{RP}_A(D_3)}(x)=\vee_{y\in X_1}\gamma_{D_3}(y)=\max\{0.5,0.2,03\}=0.5$

so that $\underline{RP_A}(D_3)(x)=(0.1,0.5,0.05)$.

Hence, for $X_1=\{u_1,u_3,u_9\}$, $\forall x\in X_2$,

$max_{i=\{1,2,3\}}\underline{RP}_A(D_i)(x)=\underline{RP}_A(D_1)(x)=(0.2,0.5,0)$,

$max_{i=\{1,2,3\}}\underline{RP_A}(D_i)(x)=$

and $X_1=\{u_1,u_3,u_9\}\subseteq(D_1)^{0.2,0}_{0.5}=\{u_1,u_2,u_3,u_7,u_9\}$;

For $X_2=\{u_2,u_7,u_{10}\}$. We have $\forall x\in X_2$,

$max_{i=\{1,2,3\}}\underline{RP_A}(D_i)(x)=\underline{RP_A}(D_2)(x)=(0.3,0.3,0.1)$,
and $X_2=\{u_2,u_7,u_{10}\}\subseteq(D_2)^{0.3,0.1}_{0.3}=\{u_2,u_7,u_{10}\}$.

For $X_3=\{u_4\}$, we have $\forall x\in X_2$,

$max_{i=\{1,2,3\}}\underline{RP_A}(D_i)(x)=\underline{RP_A}(D_3)(x)=(0.2,0.3,0.4)$,
and

$X_3=\{u_4\}\subseteq(D_2)^{0.3,0.1}_{0.3}=\{u_4,u_6,u_9\}$
$X_3=\{u_4\}\subseteq(D_2)^{0.3,0.1}_{0.3}=\{u_4,u_6,u_9\}$.

For $X_3=\{u_4\}$, we have $\forall x\in X_2$

$max_{i=\{1,2,3\}}\underline{RP_A}(D_i)(x)=\underline{RP_A}(D_3)(x)=(0.2,0.3,0.4)$
and $X_4=\{u_5,u_8\}\subseteq(D_2)^{0.2,0.3}_{0.4}=\{u_2,u_5,u_8,u_9,u_{10}\}$
$X_4=\{u_5,u_8\}\subseteq(D_2)^{0.2,0.3}_{0.4}=\{u_2,u_5,u_8,u_9,u_{10}\}$.

For $X_3=\{u_4\}$, we have $\forall x\in X_2$,

$max_{i=\{1,2,3\}}\underline{RP_A}(D_i)(x)=\underline{RP_A}(D_3)(x)=(0.2,0.3,0.4)$, and
$X_5=\{u_6\}\subseteq(D_2)^{1,0}_{0}=\{u_6\}$.

VI. THE KNOWLEDGE REDUCTION AND EXTENSION OF STANDARD NEUTROSOPHIC INFORMATION SYSTEMS

Definition 7.

(i) Let (U,A,F) (U,A,F) be the classical information system and $B\subseteq A$. B is called the standard neutrosophic reduction of the classical information system (U,A,F), if B is the minimum set which satisfies the following relations: for any $X\in PFS(U),x\in U$.

$$\underline{RP}_A(X)=\underline{RP}_B(X),\ \overline{RP}_A(X)=\overline{RP}_B(X)$$

(ii) B is called the standard neutrosophic lower approximation reduction of the classical information system (U,A,F), if B is the minimum set which satisfies the following relations: for any $X\in PFS(U),x\in U$

$$\underline{RP_A}(X)=\underline{RP_B}(X),$$

(iii) B is called the standard neutrosophic upper approximation reduction of the classical information system (U,A,F), if B is the minimum set which satisfies the following relations: for any $X\in PFS(U),x\in U$

$$\overline{RP}_A(X)=\overline{RP}_B(X)$$

where

$\underline{RP}_A(X),\underline{RP}_B(X),\ \overline{RP}_A(X),\overline{RP}_B(X)\underline{RP_A}(X),\underline{RP_B}(X),$

$\overline{RP}_A(X), \overline{RP}_B(X)$ are standard neutrosophic lower and standard neutrosophic upper approximation sets of standard neutrosophic set $X \in PFS(U)$ based on $R_A, R_B R_A, R_B$, respectively.

Now, we express the knowledge of the knowledge reduction of standard neutrosophic information system by introducing the discernibility matrix.

Definition 8.

Let (U, A, F, D, G) be the standard neutrosophic information system. Then $M = [D_{ij}]_{k\times k}$ where

$$D_{ij} = \begin{cases} \{a_l \in A : f_l(X_i) \neq f_l(X_j)\}; & g_{X_i}(D_t) \neq g_{X_j}(D_t) \text{ is} \\ A & ; g_{X_i}(D_t) = g_{X_j}(D_t) \end{cases}$$

called the discernibility matrix of (U, A, F, D, G) (where $g_{X_i}(D_k)$ is the maximum of $\underline{RP_A}(D(X_i))$ obtained at $D_t D_k$,

i.e., $g_{X_i}(D_t) = \underline{RP}_A(D_t(X_i))$

=

$\max\{\underline{RP}_A(D_z(X_i)), z = 1, 2, \ldots, q\})$

$g_{X_i}(D_k) = \underline{RP_A}(D_k(X_i)) = \max\{\underline{RP_A}(D_t(X_i)), t = 1,2, \ldots, q\})$.

Definition 9.

Let (U, A, F, D, G) be the standard neutrosophic information system, for any $B \subseteq A$, if the following relations holds, for any $x \in U$:

$\underline{RP}_B(D_i)(x) > \underline{RP}_B(D_j)(x) - \underline{RP}_A(D_i)(x) > \underline{RP}_A(D_j)(x)(i \neq j)$

then B is called the consistent set of A.

Theorem 6.

Let (U, A, F, D, G) be the standard neutrosophic information system. If there exists a subset $B \subseteq A$ such that $B \cap D_{ij} \neq \emptyset$, then B is the consistent set of A.

Definition 10.

Let (U, A, F, D, G) be the standard neutrosophic information system

$$D_{ij}^C = \begin{cases} \{a_l \in A : f_l(X_i) = f_l(X_j)\}; & g_{X_i}(D_t) \neq g_{X_j}(D_t) \\ \emptyset & ; g_{X_i}(D_t) = g_{X_j}(D_t) \end{cases}$$

is called the discernibility matrix of (U, A, F, D, G) (where $g_{X_i}(D_k)$ is the maximum of $\underline{RP_A}(D(X_i))$ obtained at D_k, i.e.,

$g_{X_i}(D_t) = \underline{RP}_A(D_t(X_i)) = \max\{\underline{RP}_A(D_z(X_i)), z = 1, 2, \ldots, q\})$.

$g_{X_i}(D_k) = \underline{RP_A}(D_k(X_i)) = \max\{\underline{RP_A}(D_t(X_i)), t = 1,2, \ldots, q\})$.

Theorem 7.

Let (U, A, F, D, G) be the standard neutrosophic information system. If there exists a subset $B \subseteq A$ such that $B \cap D_{ij}^C = \emptyset$, then B is the consistent set of A.

Proof.

If $B \cap D_{ij}^C = \emptyset$, then $B \subseteq D_{ij}$.

According to Theorem 6, B is the consistent set of A.□

The extension of a standard neutrosophic information system present on the following definition:

Definition 11.

(i) Let (U, A, F) be the classical information system and $A \subseteq B$. B is called the standard neutrosophic extension of the classical information system (U, A, F), if B satisfies the following relations: for any $X \in PFS(U), x \in U$

$$\underline{RP}_A(X) = \underline{RP}_B(X),\ \overline{RP}_A(X) = \overline{RP}_B(X)$$

(ii) B is called the standard neutrosophic lower approximation extension of the classical information system (U, A, F), if $B\, B$ satisfies the following relations: for any $X \in PFS(U), x \in U$

$$\underline{RP}_A(X) = \underline{RP}_B(X)$$

(iii) B is called the standard neutrosophic upper approximation extension of the classical information system (U, A, F), if B satisfies the following relations: for any $X \in PFS(U), x \in U$

$$\overline{RP}_A(X) = \overline{RP}_B(X)$$

Where $\underline{RP_A}(X), \underline{RP_B}(X), \overline{RP}_A(X), \overline{RP}_B(X)$ are picture fuzzy lower and upper approximation sets of standard neutrosophic set $X \in PFS(U)$ based on R_A, R_B, respectively.

We can be easily obtained the following result.

Definition 12.

Let (U, A, F) be the classical information system, for any hyper set B, such that $A \subseteq B$, if A is the standard neutrosophic reduction of the classical information system (U, B, F), then (U, B, F) is the standard neutrosophic extension of (U, A, F), but not conversely necessary.

Example 4.

In the approximation of the standard neutrosophic decision in Table 2, Table 3. Let $B = \{a_1, a_2\}$, then we obtained the family of all equivalent classes of U based on the equivalent relation $R_B = IND(B)$ as follows

$$U/R_B = \{X_1 = \{u_1, u_3, u_9\}, X_2 = \{u_2, u_7, u_{10}\}, X_3 = \{u_4\}, X_4 = \{u_5, u_8\}, X_5 = \{u_6\}\}$$

We can get the approximation value given in Table 4.

Table 4: *The approximation of the standard neutrosophic decision*

U/R_B	$\underline{RP}_B(D_1(X_i))$	$\underline{RP}_B(D_2(X_i))$	$\underline{RP}_B(D_3(X_i))$
X_1	(0.2,0,0.5)	(0.15,0.05,0.6)	(0.1,0.05,0.5)
X_2	(0.05,0.05,0.9)	(0.3,0.1,0.3)	(0.05,0.1,0.4)
X_3	(0.15, 0.1,0.7)	(0.1,0.05,0.8)	(0.2,0.4,0.3)
X_4	(0.05,0.2,0.7)	(0.2,0.3,0.4)	(0.05,0,0.6)
X_5	(0.1,0.3,0.5)	(0.2,0.3,0.4)	(1,0,0)

It is easy to see that B satisfies Definition 7 (ii), i.e., B is the standard neutrosophic lower reduction of the classical information system (U,A,F).

The discernibility matrix of the standard neutrosophic information system (U,A,F,D,G) will be presented in Table 5.

Table 5: *The discernibility matrix of the standard neutrosophic information system*

U/R_B	X_1	X_2	X_3	X_4	X_5
X_1	A				
X_2	A	A			
X_3	$\{a_2\}$	$\{a_1,a_3\}$	A		
X_4	$\{a_1,a_3\}$	A	A	A	
X_5	$\{a_1,a_3\}$	A	A	$\{a_2\}$	A

CONCLUSION

In this paper, we introduce the concept of standard neutrosophic information system, study the knowledge discovery of standard neutrosophic information system based on rough standard neutrosophic sets. We investigate some problems of the knowledge discovery of standard neutrosophic information system: the knowledge reduction and extension of the standard neutrosophic information systems.

ACKNOWLEDGMENT

This research is funded by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 102.01-2017.02.

REFERENCES

[1] Z. Pawlak, Rough sets, International Journal of Computer and Information Sciences, vol. 11, no.5 , pp 341 – 356, 1982.

[2] L. A. Zadeh, Fuzzy Sets, Information and Control, Vol. 8, No. 3 (1965), p 338-353.

[3] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy set and systems, vol.20, pp.87-96, 1986.

[4] B.C. Cuong, V. Kreinovick, Picture fuzzy sets – a new concept for computational intelligence problems, in the proceedings of the third world congress on information and communication technologies WICT'2013, Hanoi, Vietnam, December 15-18, pp 1-6, 2013.

[5] B.C. Cuong, Picture Fuzzy Sets, Journal of Computer Science and Cybernetics, Vol.30, n.4, 2014, 409-420.

[6] B.C. Cuong, P.H.Phong and F. Smarandache, Standard Neutrosophic Soft Theory: Some First Results, Neutrosophic Sets and Systems, vol.12, 2016, pp.80-91.

[7] L.H. Son, DPFCM: A novel distributed picture fuzzy clustering method on picture fuzzy sets, Expert systems with appli-cations 42, pp 51-66, 2015.

[8] P.H. Thong and L.H.Son, Picture Fuzzy Clustering : A New Computational Intelligence Method, Soft Computing, v.20 (9) 3544-3562, 2016.

[9] D. Dubois, H. Prade, Rough fuzzy sets and fuzzy rough sets, International journal of general systems, Vol. 17, p 191-209, 1990.

[10] Y.Y. Yao, Combination of rough and fuzzy sets based on level sets, Rough sets and Data mining: analysis for imprecise data, Kluwer Academic Publisher, Boston, p 301 – 321, 1997.

[11] W. Z. Wu, J. S. Mi, W. X. Zhang, Generalized fuzzy rough sets, Information Sciences 151, p. 263-282, 2003.

[12] W. Z. Wu, Y. H. Xu, On fuzzy topological structures of rough fuzzy sets, Transactions on rough sets XVI, LNCS 7736, Springer – Verlag Berlin Heidelberg, p 125-143, 2013.

[13] Y.H. Xu, W.Z. Wu, Intuitionistic fuzzy topologies in crisp approximation spaces, RSKT 2012, LNAI 7414, © Springer – Verlag Berlin Heidelberg, pp 496-503, 2012.

[14] B. Davvaz, M. Jafarzadeh, Rough intuitionistic fuzzy information systems, Fuzzy information and Engineering, vol.4, pp 445-458, 2013.

[15] N.X. Thao, N.V. Dinh, Rough picture fuzzy set and picture fuzzy topologies, Science computer and Cybernetics, Vol 31, No 3 (2015), pp 245-254.

[16] B. Sun, Z. Gong, Rough fuzzy set in generalized approximation space, Fifth Int. Conf. on Fuzzy Systems and Knowledge Discovery, IEEE computer society 2008, pp 416-420.

[17] F. Smarandache, A unifying field in logics. Neutrosophy: Neutrosophic probability, set and logic, American Re-search Press, Rehoboth, 1998, 1999.

[18] H. Wang, F. Smarandache, Y.Q. Zhang et al., Interval neutrosophic sets and logic: Theory and applications in computing, Hexis, Phoenix, AZ 2005.

[19] H. Wang, F. Smarandache, Y.Q. Zhang, et al., Single valued neutrosophic sets, Multispace and Multistructure 4 (2010), 410-413.

[20] J. Ye, A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets, Jour-nal of Intelligent & Fuzzy Systems 26 (2014) 2459-2466.

[21] P. Majumdar, Neutrosophic sets and its applications to decision making, Computation intelligentce for big data analysis (2015), V.19, pp 97-115.

[22] J. Peng, J. Q. Wang, J. Wang, H. Zhang, X. Chen, Simplified neutrosophic sets and their applica-tions in multi-criteria group decision-making problems, International journal of systems science (2016), V.47, issue 10, pp 2342-2358.

[23] Florentin Smarandache, Degrees of Membership > 1 and < 0 of the Elements With Respect to a Neutrosophic OffSet, Neutrosophic Sets and Systems, vol. 12, 2016, pp. 3-8.

[24] Florentin Smarandache, Degree of Dependence and Independence of the (Sub)Components of Fuzzy Set and Neutrosophic Set, Neutrosophic Sets and Systems, vol. 11, 2016, pp. 95-97; http://fs.gallup.unm.edu/NSS/DegreeOfDependenceAndIndependence.pdf.